



WEYL'S THEOREM FOR ALGEBRAICALLY QUASI- $*$ - A OPERATORS

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ABSTRACT. In the paper, we prove the following assertions: (1) If T is an algebraically quasi- $*$ - A operator, then T is polaroid. (2) If T or T^* is an algebraically quasi- $*$ - A operator, then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. (3) If T^* is an algebraically quasi- $*$ - A operator, then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

1. INTRODUCTION

Let H be an infinite dimensional separable Hilbert space, $B(H)$ and $K(H)$ denote, respectively, the algebra of all bounded linear operators and the ideal of compact operators on H . If $T \in B(H)$, we shall denote the set of all complex numbers by C , and henceforth shorten $T - \lambda I$ to $T - \lambda$. We write $N(T)$ and $R(T)$ for the null space and range space of T ; $\alpha(T) := \dim N(T)$; $\beta(T) := \dim N(T^*)$; $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$ and $\pi(T)$ for the spectrum of T , the approximate point spectrum of T , the point spectrum of T and the set of poles of the resolvent of T . Let $p = p(T)$ be the ascent of T , i.e., the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$, if such an integer does not exist, we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of T , i.e., the smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$, and if such an integer does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then

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$p(T) = q(T)$. Moreover, $0 < p(\lambda - T) = q(\lambda - T) < \infty$ precisely when λ is a pole of the resolvent of T , see Heuser [10, Proposition 50.2].

An operator $T \in B(H)$ is called Fredholm if it has closed range with finite dimension null space and its range of finite co-dimension. The index of a Fredholm operator $T \in B(H)$ is given by

$$i(T) := \alpha(T) - \beta(T).$$

An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. An operator $T \in B(H)$ is called Browder if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined in [9, 8]:

$$\sigma_e(T) := \{\lambda \in C : T - \lambda \text{ is not Fredholm}\};$$

$$w(T) := \{\lambda \in C : T - \lambda \text{ is not Weyl}\};$$

$$\sigma_b(T) := \{\lambda \in C : T - \lambda \text{ is not Browder}\}.$$

Evidently,

$$\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \bigcup \text{acc}\sigma(T),$$

where $\text{acc}K$ denotes the accumulation points of $K \subseteq C$.

We consider the sets

$$\Phi_+(H) := \{T \in B(H) : R(T) \text{ is closed and } \alpha(T) < \infty\};$$

$$\Phi_-(H) := \{T \in B(H) : R(T) \text{ is closed and } \alpha(T^*) < \infty\};$$

$$\Phi_+^-(H) := \{T \in B(H) : T \in \Phi_+(H) \text{ and } i(T) \leq 0\}.$$

On the other hand,

$$\sigma_{ea}(T) := \{\lambda \in C : T - \lambda \notin \Phi_+^-(H)\}$$

is the essential approximate point spectrum and

$$\sigma_{ab}(T) := \cap \{\sigma_a(T + K) : TK = KT \text{ and } K \in K(H)\}$$

is the Browder essential approximate point spectrum.

If we write $\text{iso}K = K \setminus \text{acc}K$, then we let

$$\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

$$\pi_{00}^a(T) := \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We write

$$\sigma(T) \setminus \sigma_b(T) := p_{00}(T).$$

Definition 1.1. Let $T \in B(H)$.

- (1) Weyl's theorem holds for T if $\sigma(T) \setminus w(T) = \pi_{00}(T)$.
- (2) a -Weyl's theorem holds for T if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$.
- (3) a -Browder's theorem holds for T if $\sigma_{ea}(T) = \sigma_{ab}(T)$.

It's known from [3, 8, 15] that if $T \in B(H)$ then we have a -Weyl's theorem $\Rightarrow$ Weyl's theorem; a -Weyl's theorem $\Rightarrow$ a -Browder's theorem.

Recently, some interesting operators were studied in [13], it was also shown in [7, 16] that Weyl's theorem holds for totally $*$ -paranormal operators. In this paper, we extend this result to algebraically quasi- $*$ - A operators.

2. EXAMPLES

Definition 2.1. Let $T \in B(H)$.

- (1) An operator T is said to be hyponormal if $T^*T \geq TT^*$.
- (2) An operator T is said to be class $*$ - A if $|T^2| \geq |T^*|^2$.
- (3) An operator T is said to be $*$ -paranormal if $\|T^2x\| \geq \|T^*x\|^2$ for every unite vector $x \in H$.
- (4) An operator T is said to be quasi- $*$ - A if $T^*|T^2|T \geq T^*|T^*|^2T$.

We say that $T \in B(H)$ is an algebraically quasi- $*$ - A operator if there exists a nonconstant complex polynomial p such that $p(T)$ is a quasi- $*$ - A operator.

From [6, 17], we have the following implications:

hyponormal $\Rightarrow$ class $*$ - $A \Rightarrow$ $*$ -paranormal;

hyponormal $\Rightarrow$ class $*$ - $A \Rightarrow$ quasi- $*$ - $A \Rightarrow$ algebraically quasi- $*$ - A .

By computing, we have the following Lemma 2.2.

Lemma 2.2. Let $K = \bigoplus_{n=1}^{+\infty} H_n$, where $H_n \cong H$. For given positive operators A and B on H , define the operator $T_{A,B}$ on K as follows:

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then the following assertions hold:

- (1) $T_{A,B}$ belongs to hyponormal if and only if $B^2 \geq A^2$.
- (2) $T_{A,B}$ belongs to class $*$ - A if and only if $B^2 \geq A^2$.
- (3) $T_{A,B}$ belongs to quasi- $*$ - A if and only if $AB^2A \geq A^4$.
- (4) $T_{A,B}$ belongs to $*$ -paranormal if and only if $B^4 - 2\lambda A^2 + \lambda^2 \geq 0$ for all $\lambda > 0$.

Remark 2.3. It is meaningless to use this characterization for distinguishing some gaps between hyponormal operators and class $*$ - A operators. However, for $*$ -paranormal operators, quasi- $*$ - A operators, $T_{A,B}$ has a very useful characterization. The following examples show that $*$ -paranormal operators and quasi- $*$ - A operators are independent.

Example 2.4. A non-class $*$ - A , non- $*$ -paranormal and quasi- $*$ - A operator.

Take A and B as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$B^2 - A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \not\geq 0.$$

And hence $T_{A,B}$ is a non-class $*$ - A operator.

On the other hand,

$$A(B^2 - A^2)A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

Thus $T_{A,B}$ is a quasi- $*$ - A operator.

Next we show that $T_{A,B}$ is non- $*$ -paranormal operator.

$$B^4 - 2\lambda A^2 + \lambda^2 = \begin{pmatrix} 8 - 2\lambda + \lambda^2 & 8 \\ 8 & 8 + \lambda^2 \end{pmatrix}.$$

If $\lambda = 1$, then

$$B^4 - 2\lambda A^2 + \lambda^2 \not\geq 0.$$

Hence $T_{A,B}$ is not a $*$ -paranormal operator.

Example 2.5. A non-class $*$ - A , non-quasi- $*$ - A and $*$ -paranormal operator.

Take A and B as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}^{\frac{1}{4}}.$$

Then

$$B^4 - 2\lambda A^2 + \lambda^2 = \begin{pmatrix} (1-\lambda)^2 & 2(1-\lambda) \\ 2(1-\lambda) & \lambda^2 - 4\lambda + 8 \end{pmatrix} \geq 0$$

for every $\lambda > 0$. Thus $T_{A,B}$ is a $*$ -paranormal operator.

On the other hand,

$$A(B^2 - A^2)A = \begin{pmatrix} -0.3359 \dots & -0.2265 \dots \\ -0.2265 \dots & 0.8244 \dots \end{pmatrix} \not\geq 0.$$

Hence $T_{A,B}$ is not a quasi- $*$ - A operator. Therefore $T_{A,B}$ is not a class $*$ - A operator.

The following example provides an operator which is algebraically quasi- $*$ - A but not quasi- $*$ - A operator.

Example 2.6. Let $T = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \in B(l_2 \oplus l_2)$. Then T is an algebraically quasi- $*$ - A but not quasi- $*$ - A operator.

Since $T^* = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, we have

$$(T^{*2}T^2)^{\frac{1}{2}} - TT^* = \begin{pmatrix} 1.1213 \dots & -0.2929 \dots \\ -0.2929 \dots & -1.2929 \dots \end{pmatrix},$$

then

$$T^*((T^{*2}T^2)^{\frac{1}{2}} - TT^*)T = \begin{pmatrix} -0.7574 \dots & -1.5858 \dots \\ -1.5858 \dots & -1.2929 \dots \end{pmatrix} \not\geq 0.$$

Therefore T is not a quasi- $*$ - A operator.

On the other hand, consider the complex polynomial $h(z) = (z - 1)^2$. Then $h(T) = 0$, and hence T is an algebraically quasi- $*$ - A operator.

3. WEYL'S THEOREM FOR ALGEBRAICALLY QUASI- $*$ - A OPERATORS

Before we state main theorems, we need several preliminary results.

Lemma 3.1. [17] *If T is a quasi- $*$ - A operator and T does not have dense range, then*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*),$$

Where $A = T|_{\overline{R(T)}}$ is the restriction of T to $\overline{R(T)}$, and $A \in \text{class } *-A$.

We say that T has the single valued extension property (abbrev. SVEP) if for every open set U of C the only analytic solution $f: U \rightarrow H$ of the equation

$$(T - \lambda)f(\lambda) = 0$$

for all $\lambda \in U$ is the zero function on U .

Lemma 3.2. *If T is an algebraically quasi- $*$ - A operator, then T has SVEP.*

Proof. We first suppose that T is a quasi- $*$ - A operator. We consider the following two cases:

Case I: If the range of T is dense, then T is class $*$ - A , T has SVEP by [6].

Case II: If the range of T is not dense, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*).$$

Suppose $(T - z)f(z) = 0$. Put $f(z) = f_1(z) \oplus f_2(z)$ on $H = \overline{R(T)} \oplus N(T^*)$. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & -z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2f_2(z) \\ -zf_2(z) \end{pmatrix} = 0.$$

Hence $f_2(z) = 0$. Then $(T_1 - z)f_1(z) = 0$. Since T_1 is class $*$ - A , T_1 has SVEP by [6]. Hence $f_1(z) = 0$. Consequently, T has SVEP.

Now suppose that T is an algebraically quasi- $*$ - A operator. Then $p(T)$ is a quasi- $*$ - A operator for some nonconstant complex polynomial p , and hence it follows from the first part of the proof that $p(T)$ has SVEP. Therefore T has SVEP by [11, Theorem 3.3.9]. $\square$

Lemma 3.3. *If T is a quasi- $*$ - A operator with spectrum $\sigma(T) \subseteq \partial D$, where D denotes the unite disc, then T is unitary.*

Proof. If T is quasi- $*$ - A operator with spectrum $\sigma(T) \subseteq \partial D$, then T is invertible and hence class $*$ - A . [6, Proposition 2.6] now implies it is unitary. $\square$

Lemma 3.4. *If T is a quasi- $*$ - A operator, and assume that $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

Proof. We consider the following two cases:

Case I: if $\lambda = 0$, then $A = 0$ in Lemma 3.1, and so $T = 0$.

Case II: if $\lambda \neq 0$, then T is invertible and class $*$ - A . [6, Theorem 2.9] implies $T = \lambda I$. $\square$

An operator $T \in B(H)$ is called polaroid if $\text{iso}\sigma(T) \subset \pi(T)$. In general, if T is polaroid, then it is isoloid, however, the converse isn't true. Consider the following example, let $T \in B(l_2)$ is defined by

$$T(x_1, x_2, x_3 \cdots) = \left(\frac{x_2}{2}, \frac{x_3}{3} \cdots\right).$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, thus T is isoloid, however, since T doesn't have finite ascent, T is not polaroid. In [6] it is showed that every $*$ -paranormal operator is polaroid, we can prove more.

Lemma 3.5. *If T is an algebraically quasi- $*$ - A operator, then T is polaroid.*

Proof. We first show that quasi- $*$ - A operator is polaroid. Suppose T is a quasi- $*$ - A operator. Let $\lambda \in \text{iso}\sigma(T)$. Using the spectral projection $P := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$, where D is a closed disk of center λ which contains no other point of $\sigma(T)$. We can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Since T_1 is a quasi- $*$ - A operator, it follows from Lemma 3.4 that $T_1 - \lambda = 0$, therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda$ has finite ascent and descent, and hence λ is a pole of the resolvent of T , thus $\lambda \in \text{iso}\sigma(T)$ implies $\lambda \in \pi(T)$. Hence T is polaroid.

Next we show that algebraically quasi- $*$ - A operator is polaroid. If T is an algebraically quasi- $*$ - A operator, then $p(T)$ is quasi- $*$ - A operator for some non-constant polynomial p . Hence it follows from the first part of the proof that $p(T)$ is polaroid. Now apply [5, Lemma 3.3] to conclude that $p(T)$ polaroid implies T polaroid. $\square$

Corollary 3.6. *If T is an algebraically quasi- $*$ - A operator, then T is isoloid.*

In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 3.7. *Suppose T or T^* is an algebraically quasi- $*$ - A operator. Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose that T is an algebraically quasi- $*$ - A operator. We first show that Weyl's theorem holds for T . We use the fact [4, Theorem 2.2] that if T is polaroid then Weyl's theorem holds for T if and only if T has *SVEP* at points of $\lambda \in \sigma(T) \setminus w(T)$. We have that T is polaroid by Lemma 3.5 and T has *SVEP* by Lemma 3.2. Hence T satisfies Weyl's theorem.

Next we show that Weyl's theorem holds for $f(T)$. Since T is isoloid, by [12] we have $\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(w(T))$, where the last equality holds since T satisfies Weyl's theorem. Since T has *SVEP*, by [1, Corollary 2.6], we have $f(w(T)) = w(f(T))$. Therefore we have $\sigma(f(T)) \setminus \pi_{00}(f(T)) = w(f(T))$. Hence Weyl's theorem holds for $f(T)$.

Suppose that T^* is an algebraically quasi- $*$ - A operator. We first show that Weyl's theorem holds for T . Since T^* has *SVEP* and is polaroid, T is polaroid.

And T^* has *SVEP* implies T^* satisfies Browder's theorem, then T satisfies Browder's theorem. Therefore Weyl's theorem holds for T . Since T^* has *SVEP*, by [1, Corollary 2.6], we have $f(w(T)) = w(f(T))$. Noting that T is isoloid, as in the proof of the first part, we have that Weyl's theorem holds for $f(T)$. This completes the proof. $\square$

From the proof of Theorem 3.7, we have that the Weyl spectrum obeys the spectral mapping theorem for algebraically quasi- $*$ - A operator.

Corollary 3.8. *Suppose T or T^* is an algebraically quasi- $*$ - A operator. Then for every $f \in H(\sigma(T))$, we have $f(w(T)) = w(f(T))$.*

Corollary 3.9. *Suppose T or T^* is an algebraically quasi- $*$ - A operator. If F is an operator commuting with T and for which there exists a positive integer n such that F^n has a finite rank, then Weyl's theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T or T^* is an algebraically quasi- $*$ - A operator. By Lemma 3.5 and Theorem 3.7, we have that T is isoloid and Weyl's theorem holds for $f(T)$. Observe that if T is isoloid then $f(T)$ is isoloid. The result follows from [13, Theorem 2.4]. $\square$

Theorem 3.10. *Suppose T or T^* is an algebraically quasi- $*$ - A operator. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$.*

Proof. Let $f \in H(\sigma(T))$. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+^-(H)$ and

$$f(T) - \lambda = (T - \lambda_1) \cdots (T - \lambda_k)g(T), \quad (3.1)$$

where $\lambda_1, \lambda_2, \dots, \lambda_k \in C$ and $g(T)$ is invertible. Since the operators on the right side of (3.1) commute, $T - \lambda_i \in \Phi_+(H)$. Suppose T is an algebraically quasi- $*$ - A operator. Then $i(T - \lambda_j) \leq 0$ for each $j = 1, 2, \dots, k$. Therefore $\lambda \notin f(\sigma_{ea}(T))$.

Suppose T^* is an algebraically quasi- $*$ - A operator. It follows by [2, Theorem 2.8] that $i(T - \lambda_j) \geq 0$ for each $j = 1, 2, \dots, k$. Since

$$0 \leq \sum_{j=1}^k i(T - \lambda_j) = i(f(T) - \lambda) \leq 0,$$

$T - \lambda_j$ is Weyl for each $j = 1, 2, \dots, k$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. This completes the proof. $\square$

Theorem 3.11. *Suppose T^* is an algebraically quasi- $*$ - A operator. Then a -Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is an algebraically quasi- $*$ - A operator. We first prove that a -Weyl's theorem holds for T . Since T^* has *SVEP* and T is polaroid, $\overline{\sigma_a(T)} = \overline{\sigma(T)} = \sigma(T^*)$, $\pi_{00}(T^*) = \overline{\pi_{00}^a(T)}$ and $\overline{\sigma_{ea}(T)} = \overline{w(T)} = w(T^*)$. Since T^* satisfies Weyl's theorem, T satisfies a -Weyl's theorem.

Next we prove that a -Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since T satisfies a -Weyl's theorem, we have that a -Browder's theorem holds for T . Hence $\sigma_{ea}(T) = \sigma_{ab}(T)$. Since T^* is an algebraically

quasi- $*$ - A operator, it follows from Theorem 3.10 that $\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$, and hence a -Browder's theorem holds for $f(T)$. We use the fact [3, Theorem 3.8] that if T satisfies a -Browder's theorem then a -Weyl's theorem holds for T if $R(T - \lambda)$ is closed for each $\lambda \in \pi_{00}^a(T)$. Hence it suffices to show that if $\lambda \in \pi_{00}^a(f(T))$, then $R(f(T) - \lambda)$ is closed. Since $f(T^*)$ has SVEP, $\pi_{00}(f(T)) = \pi_{00}^a(f(T))$, and hence $R(f(T) - \lambda)$ is closed for each $\lambda \in \pi_{00}^a(f(T))$. $\square$

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REFERENCES

1. P. Aiena, *Classes of operators satisfying a -Weyl's theorem*, Studia Math. **169** (2005), no. 2, 105–122.
2. P. Aiena and O. Monsalve, *Operators which do not have the single valued extension property*, J. Math. Anal. Appl. **250** (2000), no. 2, 435–448.
3. S.V. Djordjević and Y.M. Han, *Browder's theorems and spectral continuity*, Glasgow Math. J. **42**(2000), no. 3, 479–486.
4. B.P. Duggal, *Polaroid operators satisfying Weyl's theorem*, Linear Algebra Appl. **414** (2006), no. 1, 271–277.
5. B.P. Duggal, *Polaroid operators, SVEP and perturbed Browder, Weyl theorems*, Rend. Circ. Mat. Palermo **56** (2007), no. 3, 317–330.
6. B.P. Duggal, I.H. Jeon and I.H. Kim, *On $*$ -paranormal contractions and properties for $*$ -class A operators*, Linear Algebra Appl. **436** (2012), no. 5, 954–962.
7. Y.M. Han and A.H. Kim, *A note on $*$ -paranormal operators*, Integral Equations Operator Theory **49** (2004), no. 4, 435–444.
8. R.E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Dekker, New York, 1988.
9. R.E. Harte and W.Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), no. 5, 2115–2124.
10. H.G. Heuser, *Functional Analysis*, John Wiley and Sons, 1982.
11. K.B. Laursen and M.M. Neumann, *Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.
12. W.Y. Lee and S.H. Lee, *A spectral mapping theorem for the Weyl spectral*, Glasgow Math. J. **38** (1996), no. 1, 61–64.
13. S. Mecheri, *Bishop's property and Riesz idempotent for k -quasi-paranormal operators*, Banach J. Math. Anal. **6** (2012), no. 1, 147–154.
14. M. Oudghiri, *Weyl's theorem and perturbations*, Integral Equations Operator Theory **53** (2005), no. 4, 535–545.
15. V. Rakočević, *On the essential approximate point spectrum II*, Mat. Vesnik **36** (1984), no. 1, 89–97.
16. E. Ko, H.W. Nam and Y. Yang, *On totally $*$ -paranormal operators*, Czechoslovak Math. J. **56** (2006), no. 4, 1265–1280.
17. J.L. Shen, F. Zuo and C.S. Yang, *On operators satisfying $T^*|T^2|T \geq T^*|T^*|^2T$* , Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 11, 2109–2116.

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