



THE BEST LOWER BOUND FOR JENSEN'S INEQUALITY WITH THREE FIXED ORDERED VARIABLES

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ABSTRACT. In this paper we establish the best lower bound for the weighted Jensen's discrete inequality with ordered variables applied to a convex function f , in the case when the bound depends on f , weights and three fixed variables. Some applications for particular cases of interest are provided.

1. INTRODUCTION

Let $\mathbb{I}$ be an interval in $\mathbb{R}$, let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{I}^n$, and let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be a positive n -tuple such that $p_1 + p_2 + \dots + p_n = 1$. If $f : \mathbb{I} \rightarrow \mathbb{R}$ is a convex function, then the well-known discrete Jensen's inequality [2] states that

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) \geq 0,$$

where the functional

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) = p_1 f(x_1) + p_2 f(x_2) + \dots + p_n f(x_n) - f(p_1 x_1 + p_2 x_2 + \dots + p_n x_n)$$

is so called Jensen's difference.

In [1], we presented the theorem below which establishes the best lower bound $L_{\mathbf{p},f}(x_i, x_k)$ of Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ for

$$x_1 \leq \dots \leq x_i \leq \dots \leq x_k \leq \dots \leq x_n$$

and fixed x_i and x_k .

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Theorem 1.1. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \dots, x_n \in \mathbb{I}$ ($n \geq 3$) such that*

$$x_1 \leq x_2 \leq \dots \leq x_n.$$

For fixed x_i and x_k ($i < k$), Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ is minimal when

$$x_1 = x_2 = \dots = x_{i-1} = x_i, \quad x_n = x_{n-1} = \dots = x_{k+1} = x_k,$$

$$x_{i+1} = x_{i+2} = \dots = x_{k-1} = \frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}},$$

where

$$P_{1,i} = p_1 + p_2 + \dots + p_i, \quad P_{k,n} = p_k + p_{k+1} + \dots + p_n;$$

that is

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) \geq L_{p,f}(x_i, x_k),$$

where

$$L_{p,f}(x_i, x_k) = P_{1,i}f(x_i) + P_{k,n}f(x_k) - (P_{1,i} + P_{k,n})f\left(\frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}}\right).$$

Towards proving Theorem 1.1, we have used the following three lemmas.

Lemma 1.2. *Let p, q be nonnegative real numbers, and let f be a convex function on $\mathbb{I}$. If $a, b, c, d \in \mathbb{I}$ are such that $c, d \in [a, b]$, and*

$$pa + qb = pc + qd,$$

then

$$pf(a) + qf(b) \geq pf(c) + qf(d).$$

Lemma 1.3. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \dots, x_n \in \mathbb{I}$ ($n \geq 3$) be such that*

$$x_1 \leq x_2 \leq \dots \leq x_n.$$

For fixed $x_i, x_{i+1}, \dots, x_n$, where $i \in \{2, 3, \dots, n\}$, Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ is minimal when

$$x_1 = x_2 = \dots = x_{i-1} = x_i.$$

Lemma 1.4. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \dots, x_n \in \mathbb{I}$ ($n \geq 3$) be such that*

$$x_1 \leq x_2 \leq \dots \leq x_n.$$

For fixed $x_1, x_2, \dots, x_k$, where $k \in \{1, 2, \dots, n-1\}$, Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ is minimal when

$$x_k = x_{k+1} = \dots = x_{n-1} = x_n.$$

In this paper, we will use these lemmas to establish the best lower bound of weighted Jensen's difference for three fixed variables. In addition, we will use the following lemma.

Lemma 1.5. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \dots, x_n \in \mathbb{I}$ ($n \geq 4$) be such that*

$$x_1 \leq \dots \leq x_i \leq \dots \leq x_k \leq \dots \leq x_n,$$

where $1 \leq i < i+1 < k \leq n$. We have

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) \geq \Delta_n(f, \mathbf{p}, \mathbf{y}),$$

where

$$\begin{aligned} y_1 &= x_1, \quad y_2 = x_2, \quad \dots, \quad y_i = x_i, \\ y_{i+1} &= \dots = y_{k-1} = \frac{p_{i+1}x_{i+1} + \dots + p_{k-1}x_{k-1}}{p_{i+1} + \dots + p_{k-1}}, \\ y_k &= x_k, \quad y_{k+1} = x_{k+1}, \quad \dots, \quad y_n = x_n. \end{aligned}$$

Note that the proof of this lemma follows immediately from Lemma 1.3, Lemma 1.4 and Jensen's inequality

$$p_{i+1}f(x_{i+1}) + \dots + p_{k-1}f(x_{k-1}) \geq (p_{i+1} + \dots + p_{k-1})f\left(\frac{p_{i+1}x_{i+1} + \dots + p_{k-1}x_{k-1}}{p_{i+1} + \dots + p_{k-1}}\right).$$

2. MAIN RESULT

We will establish the best lower bound $L_{\mathbf{p},f}(x_i, x_j, x_k)$ of Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ for $x_1 \leq x_2 \leq \dots \leq x_n$ and fixed x_i, x_j, x_k such that

$$1 \leq i < j < k \leq n.$$

To do this, we need Lemmas 1.2, 1.3, 1.4, 1.5 and Lemma 2.1 below.

Lemma 2.1. *Let f be a convex function on $\mathbb{I}$, let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{I}$ be such that*

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5,$$

let r_1, r_2, r_3, r_4, r_5 be positive weights satisfying $r_1 + r_2 + r_3 + r_4 + r_5 = 1$, and let

$$\begin{aligned} A_1 &= \frac{r_1a_1 + (r_2 + r_3)a_3 + r_5a_5}{r_1 + r_2 + r_3 + r_5} \\ A_2 &= \frac{r_1a_1 + (r_3 + r_4)a_3 + r_5a_5}{r_1 + r_3 + r_4 + r_5}. \end{aligned}$$

For fixed a_1, a_3, a_5 , the best lower bound of Jensen's difference $\Delta_5(f, \mathbf{r}, \mathbf{a})$ is

$$\Lambda_{\mathbf{r},f}(a_1, a_3, a_5) = \begin{cases} \Lambda_1, & \text{for } a_3 \leq \frac{r_1a_1 + r_5a_5}{r_1 + r_5} \\ \Lambda_2, & \text{for } a_3 \geq \frac{r_1a_1 + r_5a_5}{r_1 + r_5} \end{cases},$$

where

$$\Lambda_1 = r_1f(a_1) + (r_2 + r_3)f(a_3) + r_5f(a_5) - (r_1 + r_2 + r_3 + r_5)f(A_1).$$

$$\Lambda_2 = r_1f(a_1) + (r_3 + r_4)f(a_3) + r_5f(a_5) - (r_1 + r_3 + r_4 + r_5)f(A_2).$$

In addition, we have

$$\Delta_5(f, \mathbf{r}, \mathbf{a}) = \Lambda_1$$

for $a_2 = a_3$ and $a_4 = A_1$, and

$$\Delta_5(f, \mathbf{r}, \mathbf{a}) = \Lambda_2$$

for $a_2 = A_2$ and $a_4 = a_3$.

For any $1 \leq i < j \leq n$, we introduce the notation

$$P_{i,j} = p_i + p_{i+1} + \cdots + p_j.$$

In addition, let us denote

$$X_1 = \frac{P_{1,i}x_i + P_{i+1,j}x_j + P_{k,n}x_k}{P_{1,i} + P_{i+1,j} + P_{k,n}},$$

$$X_2 = \frac{P_{1,i}x_i + P_{j,k-1}x_j + P_{k,n}x_k}{P_{1,i} + P_{j,k-1} + P_{k,n}},$$

and

$$L_1 = P_{1,i}f(x_i) + P_{i+1,j}f(x_j) + P_{k,n}f(x_k) - (P_{1,i} + P_{i+1,j} + P_{k,n})f(X_1),$$

$$L_2 = P_{1,i}f(x_i) + P_{j,k-1}f(x_j) + P_{k,n}f(x_k) - (P_{1,i} + P_{j,k-1} + P_{k,n})f(X_2).$$

Our main result is given by the following theorem.

Theorem 2.2. *Let f be a convex function on $\mathbb{I}$, let $x_1, x_2, \dots, x_n \in \mathbb{I}$ ($n \geq 4$) be such that*

$$x_1 \leq x_2 \leq \cdots \leq x_n,$$

and let $p_1, p_2, \dots, p_n$ be positive weights satisfying $p_1 + p_2 + \cdots + p_n = 1$. For fixed x_i, x_j, x_k ($1 \leq i < j < k \leq n$), the best lower bound of Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ is

$$L_{\mathbf{p},f}(x_i, x_j, x_k) = \begin{cases} L_1, & \text{for } x_j \leq \frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}} \\ L_2, & \text{for } x_j \geq \frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}} \end{cases}.$$

In addition, we have $\Delta_n(f, \mathbf{p}, \mathbf{x}) = L_1$ for

$$x_1 = x_2 = \cdots = x_{i-1} = x_i,$$

$$x_{i+1} = x_{i+2} = \cdots = x_{j-1} = x_j,$$

$$x_{j+1} = x_{j+2} = \cdots = x_{k-1} = X_1,$$

$$x_k = x_{k+1} = \cdots = x_{n-1} = x_n,$$

and $\Delta_n(f, \mathbf{p}, \mathbf{x}) = L_2$ for

$$x_1 = x_2 = \cdots = x_{i-1} = x_i,$$

$$x_{i+1} = x_{i+2} = \cdots = x_{j-1} = X_2,$$

$$x_j = x_{j+1} = \cdots = x_{k-2} = x_{k-1},$$

$$x_k = x_{k+1} = \cdots = x_{n-1} = x_n.$$

From Theorem 2.2, for the particular case

$$x_j = \frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}},$$

which implies

$$X_1 = X_2 = x_j$$

and

$$L_1 = L_2 = P_{1,i}f(x_i) + P_{k,n}f(x_k) - (P_{1,i} + P_{k,n})f\left(\frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}}\right),$$

we get Theorem 1.1.

On the other hand, according to Jensen's inequality, we have

$$\begin{aligned} L_2 - L_1 &= (P_{j,k-1} - P_{i+1,j})f(x_j) + (P_{1,i} + P_{i+1,j} + P_{k,n})f(X_1) \\ &\quad - (P_{1,i} + P_{j,k-1} + P_{k,n})f(X_2) \geq 0 \end{aligned}$$

for $P_{i+1,j} \leq P_{j,k-1}$, and

$$\begin{aligned} L_1 - L_2 &= (P_{i+1,j} - P_{j,k-1})f(x_j) + (P_{1,i} + P_{j,k-1} + P_{k,n})f(X_2) \\ &\quad - (P_{1,i} + P_{i+1,j} + P_{k,n})f(X_1) \geq 0 \end{aligned}$$

for $P_{i+1,j} \geq P_{j,k-1}$. Thus, from Theorem 2.2, we obtain the following proposition.

Proposition 2.3. *Let f be a convex function on $\mathbb{I}$, let $x_1, x_2, \dots, x_n \in \mathbb{I}$ ($n \geq 4$) such that*

$$x_1 \leq \dots \leq x_i \leq \dots \leq x_j \leq \dots \leq x_k \leq \dots \leq x_n,$$

and let $p_1, p_2, \dots, p_n$ be positive weights satisfying $p_1 + p_2 + \dots + p_n = 1$.

(a) *If $P_{i+1,j} \leq P_{j,k-1}$, then*

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) \geq L_1;$$

(b) *If $P_{i+1,j} \geq P_{j,k-1}$, then*

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) \geq L_2.$$

Applying Theorem 2.2 and Proposition 2.3 for $f(x) = e^x$ and using the substitutions $a_1 = e^{x_1}$, $a_2 = e^{x_2}$, $\dots$, $a_n = e^{x_n}$, we obtain

Corollary 2.4. *Let $p_1, p_2, \dots, p_n$ ($n \geq 4$) be positive real numbers such that $p_1 + p_2 + \dots + p_n = 1$, and let*

$$0 < a_1 \leq \dots \leq a_i \leq \dots \leq a_j \leq \dots \leq a_k \leq \dots \leq a_n.$$

(a) *If $a_j^{P_{1,i}+P_{k,n}} \leq a_i^{P_{1,i}} a_k^{P_{k,n}}$ or $P_{i+1,j} \leq P_{j,k-1}$, then*

$$\begin{aligned} p_1 a_1 + p_2 a_2 + \dots + p_n a_n - a_1^{p_1} a_2^{p_2} \dots a_n^{p_n} &\geq P_{1,i} a_i + P_{i+1,j} a_j + P_{k,n} a_k \\ &\quad - (P_{1,i} + P_{i+1,j} + P_{k,n}) \left(a_i^{P_{1,i}} a_j^{P_{i+1,j}} a_k^{P_{k,n}} \right)^{\frac{1}{P_{1,i} + P_{i+1,j} + P_{k,n}}}, \end{aligned}$$

with equality for

$$\begin{aligned}
a_1 &= a_2 = \cdots = a_{i-1} = a_i, \\
a_{i+1} &= a_{i+2} = \cdots = a_{j-1} = a_j, \\
a_{j+1} &= a_{j+2} = \cdots = a_{k-1} = \left(a_i^{P_{1,i}} a_j^{P_{i+1,j}} a_k^{P_{k,n}} \right)^{\frac{1}{P_{1,i} + P_{i+1,j} + P_{k,n}}}, \\
a_n &= a_{n-1} = \cdots = a_{k+1} = a_k; \\
(b) \text{ If } a_j^{P_{1,i} + P_{k,n}} &\geq a_i^{P_{1,i}} a_k^{P_{k,n}} \text{ or } P_{i+1,j} \geq P_{j,k-1}, \text{ then} \\
p_1 a_1 + p_2 a_2 + \cdots + p_n a_n - a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} &\geq P_{1,i} a_i + P_{j,k-1} a_j + P_{k,n} a_k \\
&\quad - (P_{1,i} + P_{j,k-1} + P_{k,n}) \left(a_i^{P_{1,i}} a_j^{P_{j,k-1}} a_k^{P_{k,n}} \right)^{\frac{1}{P_{1,i} + P_{j,k-1} + P_{k,n}}},
\end{aligned}$$

with equality for

$$\begin{aligned}
a_1 &= a_2 = \cdots = a_{i-1} = a_i, \\
a_{i+1} &= a_{i+2} = \cdots = a_{j-1} = \left(a_i^{P_{1,i}} a_j^{P_{j,k-1}} a_k^{P_{k,n}} \right)^{\frac{1}{P_{1,i} + P_{j,k-1} + P_{k,n}}}, \\
a_{k-1} &= a_{k-2} = \cdots = a_{j+1} = a_j, \\
a_n &= a_{n-1} = \cdots = a_{k+1} = a_k.
\end{aligned}$$

Applying Theorem 2.2 and Proposition 2.3 for $f(x) = -\ln x$, we obtain

Corollary 2.5. Let $p_1, p_2, \dots, p_n$ ($n \geq 4$) be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$, let

$$0 < x_1 \leq \cdots \leq x_i \leq \cdots \leq x_j \leq \cdots \leq x_k \leq \cdots \leq x_n,$$

and let

$$X_1 = \frac{P_{1,i} x_i + P_{i+1,j} x_j + P_{k,n} x_k}{P_{1,i} + P_{i+1,j} + P_{k,n}}, \quad X_2 = \frac{P_{1,i} x_i + P_{j,k-1} x_j + P_{k,n} x_k}{P_{1,i} + P_{j,k-1} + P_{k,n}}.$$

(a) If $x_j \leq \frac{P_{1,i} x_i + P_{k,n} x_k}{P_{1,i} + P_{k,n}}$ or $P_{i+1,j} \leq P_{j,k-1}$, then

$$\frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}} \geq \frac{X_1^{P_{1,i} + P_{i+1,j} + P_{k,n}}}{x_i^{P_{1,i}} x_j^{P_{i+1,j}} x_k^{P_{k,n}}},$$

with equality for

$$\begin{aligned}
x_1 &= x_2 = \cdots = x_{i-1} = x_i, \\
x_{i+1} &= x_{i+2} = \cdots = x_{j-1} = x_j, \\
x_{j+1} &= x_{j+2} = \cdots = x_{k-1} = X_1, \\
x_n &= x_{n-1} = \cdots = x_{k+1} = x_k;
\end{aligned}$$

(b) If $x_j \geq \frac{P_{1,i} x_i + P_{k,n} x_k}{P_{1,i} + P_{k,n}}$ or $P_{i+1,j} \geq P_{j,k-1}$, then

$$\frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}} \geq \frac{X_2^{P_{1,i} + P_{j,k-1} + P_{k,n}}}{x_i^{P_{1,i}} x_j^{P_{j,k-1}} x_k^{P_{k,n}}},$$

with equality for

$$\begin{aligned} x_1 &= x_2 = \cdots = x_{i-1} = x_i, \\ x_{i+1} &= x_{i+2} = \cdots = x_{j-1} = X_2, \\ x_{k-1} &= x_{k-2} = \cdots = x_{j+1} = x_j, \\ x_n &= x_{n-1} = \cdots = x_{k+1} = x_k. \end{aligned}$$

3. PROOF OF LEMMA 2.1

Let us denote

$$\begin{aligned} R_1 &= r_1 + r_2 + r_3 + r_5, & B_1 &= \frac{r_1 a_1 + r_2 a_2 + r_3 a_3 + r_5 a_5}{r_1 + r_2 + r_3 + r_5}, \\ R_2 &= r_1 + r_3 + r_4 + r_5, & B_2 &= \frac{r_1 a_1 + r_3 a_3 + r_4 a_4 + r_5 a_5}{r_1 + r_3 + r_4 + r_5}. \end{aligned}$$

We have two cases to consider.

Case 1: $(r_1 + r_5)a_3 \leq r_1 a_1 + r_5 a_5$. We need to show that $\Delta_5(f, \mathbf{r}, \mathbf{a}) \geq \Lambda_1$, that is,

$$r_2 f(a_2) + r_4 f(a_4) + R_1 f(A_1) \geq r_2 f(a_3) + f(r_1 a_1 + r_2 a_2 + r_3 a_3 + r_4 a_4 + r_5 a_5).$$

By Jensen's inequality, we have

$$r_4 f(a_4) + R_1 f(B_1) \geq f(r_1 a_1 + r_2 a_2 + r_3 a_3 + r_4 a_4 + r_5 a_5).$$

Thus, it suffices to show that

$$r_2 f(a_2) + R_1 f(A_1) \geq r_2 f(a_3) + R_1 f(B_1). \quad (3.1)$$

From $(r_1 + r_5)a_3 \leq r_1 a_1 + r_5 a_5$, it is easy to prove that

$$a_3, B_1 \in [a_2, A_1].$$

In addition, we have

$$r_2 a_2 + R_1 A_1 = r_2 a_3 + R_1 B_1.$$

Therefore, (3.1) is true according to Lemma 1.2.

Case 2: $(r_1 + r_5)a_3 \geq r_1 a_1 + r_5 a_5$. We can write the desired inequality as

$$r_2 f(a_2) + r_4 f(a_4) + R_2 f(A_2) \geq r_4 f(a_3) + f(r_1 a_1 + r_2 a_2 + r_3 a_3 + r_4 a_4 + r_5 a_5).$$

Using Jensen's inequality

$$r_2 f(a_2) + R_2 f(B_2) \geq f(r_1 a_1 + r_2 a_2 + r_3 a_3 + r_4 a_4 + r_5 a_5),$$

it suffices to show that

$$r_4 f(a_4) + R_2 f(A_2) \geq r_4 f(a_3) + R_2 f(B_2). \quad (3.2)$$

From $(r_1 + r_5)a_3 \geq r_1 a_1 + r_5 a_5$, we get

$$a_3, B_2 \in [A_2, a_4].$$

Since

$$r_4 a_4 + R_2 A_2 = r_4 a_3 + R_2 B_2,$$

(3.2) follows by Lemma 1.2. Thus, the proof of Lemma 1.5 is completed.

4. PROOF OF THEOREM 2.2

Let us denote

$$X_{ij} = \frac{p_{i+1}x_{i+1} + \cdots + p_{j-1}x_{j-1}}{P_{i+1,j-1}},$$

$$Y_{jk} = \frac{p_{j+1}x_{j+1} + \cdots + p_{k-1}x_{k-1}}{P_{j+1,k-1}}.$$

According to Lemmas 1.3, 1.4 and 1.5, we have

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) \geq \Delta_n(f, \mathbf{p}, \mathbf{y}), \quad (4.1)$$

where

$$\begin{aligned} y_1 &= y_2 = \cdots = y_i = x_i, \\ y_{i+1} &= \cdots = y_{j-1} = X_{ij}, \\ y_j &= x_j, \\ y_{j+1} &= \cdots = y_{k-1} = Y_{jk}, \\ y_k &= y_{k+1} = \cdots = y_n = x_k, \end{aligned}$$

and hence

$$\begin{aligned} \Delta_n(f, \mathbf{p}, \mathbf{y}) &= P_{1,i}f(x_i) + P_{i+1,j-1}f(X_{ij}) + p_jf(x_j) + P_{j+1,k-1}f(Y_{jk}) + P_{k,n}f(x_k) \\ &\quad - f(P_{1,i}x_i + P_{i+1,j-1}X_{ij} + p_jx_j + P_{j+1,k-1}Y_{jk} + P_{k,n}x_k). \end{aligned}$$

Case 1: $(P_{1,i} + P_{k,n})x_j \leq P_{1,i}x_i + P_{k,n}x_k$. According to (4.1), it suffices to prove that $\Delta_n(f, \mathbf{p}, \mathbf{y}) \geq L_1$, which is equivalent to

$$\begin{aligned} &P_{i+1,j-1}f(X_{ij}) + P_{j+1,k-1}f(Y_{jk}) + (P_{1,i} + P_{i+1,j-1} + p_j + P_{k,n})f(X_1) \geq \\ &\geq P_{i+1,j-1}f(x_j) + f(P_{1,i}x_i + P_{i+1,j-1}X_{ij} + p_jx_j + P_{j+1,k-1}Y_{jk} + P_{k,n}x_k). \end{aligned} \quad (4.2)$$

Using the substitutions

$$\begin{aligned} r_1 &= P_{1,i}, \quad r_2 = P_{i+1,j-1}, \quad r_3 = p_j, \quad r_4 = P_{j+1,k-1}, \quad r_5 = P_{k,n}, \\ a_1 &= x_i, \quad a_2 = X_{ij}, \quad a_3 = x_j, \quad a_4 = Y_{jk}, \quad a_5 = x_k, \end{aligned}$$

the condition $(P_{1,i} + P_{k,n})x_j \leq P_{1,i}x_i + P_{k,n}x_k$ becomes $(r_1 + r_5)a_3 \leq r_1a_1 + r_5a_5$, while the inequality (4.2) turns into

$$\begin{aligned} &r_2f(a_2) + r_4f(a_4) + (r_1 + r_2 + r_3 + r_5)f(A_1) \geq \\ &\geq r_2f(a_3) + f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5), \end{aligned} \quad (4.3)$$

where

$$A_1 = \frac{r_1a_1 + (r_2 + r_3)a_3 + r_5a_5}{r_1 + r_2 + r_3 + r_5}.$$

The inequality (4.3) is equivalent to $\Delta_5(f, \mathbf{r}, \mathbf{a}) \geq \Lambda_1$ in Lemma 2.1.

Case 2: $(P_{1,i} + P_{k,n})x_j \geq P_{1,i}x_i + P_{k,n}x_k$. According to (4.1), it suffices to prove that $\Delta_n(f, \mathbf{p}, \mathbf{y}) \geq L_2$, which is equivalent to

$$\begin{aligned} &P_{i+1,j-1}f(X_{ij}) + P_{j+1,k-1}f(Y_{jk}) + (P_{1,i} + p_j + P_{j+1,k-1} + P_{k,n})f(X_2) \geq \\ &\geq P_{j+1,k-1}f(x_j) + f(P_{1,i}x_i + P_{i+1,j-1}X_{ij} + p_jx_j + P_{j+1,k-1}Y_{jk} + P_{k,n}x_k). \end{aligned} \quad (4.4)$$

Using the same substitutions as the ones from the case 1, the condition $(P_{1,i} + P_{k,n})x_j \geq P_{1,i}x_i + P_{k,n}x_k$ becomes $(r_1 + r_5)a_3 \geq r_1a_1 + r_5a_5$, while the inequality (4.4) turns into

$$\begin{aligned} & r_2f(a_2) + r_4f(a_4) + (r_1 + r_3 + r_4 + r_5)f(A_2) \geq \\ & \geq r_4f(a_3) + f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5), \end{aligned} \quad (4.5)$$

where

$$A_2 = \frac{r_1a_1 + (r_3 + r_4)a_3 + r_5a_5}{r_1 + r_3 + r_4 + r_5}.$$

Since (4.5) is equivalent to the inequality $\Delta_5(f, \mathbf{r}, \mathbf{a}) \geq \Lambda_2$ in Lemma 2.1, the proof is completed.

5. APPLICATIONS

Proposition 5.1. *If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive real numbers such that*

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

then (see [3])

$$\begin{aligned} (a) \quad & a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1a_2 \cdots a_n} \geq \frac{1}{3}(2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n})^2; \\ (b) \quad & a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1a_2 \cdots a_n} \geq \frac{1}{2}(2\sqrt{a_2} - \sqrt{a_{n-1}} - \sqrt{a_n})^2. \end{aligned}$$

Proof. (a) In the case $n \geq 4$, we apply Corollary 2.4 for $p_1 = p_2 = \dots = p_n = \frac{1}{n}$, $i = 1$, $j = n - 1$ and $k = n$. We have

$$P_{1,i} = \frac{1}{n}, \quad P_{i+1,j} = \frac{n-2}{n}, \quad P_{j,k-1} = \frac{1}{n}, \quad P_{k,n} = \frac{1}{n}.$$

Since $P_{i+1,j} > P_{j,k-1}$, by Corollary 2.4 we have

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1a_2 \cdots a_n} \geq a_1 + a_{n-1} + a_n - 3\sqrt[3]{a_1a_{n-1}a_n}.$$

Notice that this inequality is also true (as identity) for $n = 3$. Therefore, it suffices to prove that

$$a_1 + a_{n-1} + a_n - 3\sqrt[3]{a_1a_{n-1}a_n} \geq \frac{1}{3}(2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n})^2,$$

which is equivalent to

$$2(a_{n-1} + a_n) + 4\sqrt{a_1}(\sqrt{a_{n-1}} + \sqrt{a_n}) - 2\sqrt{a_{n-1}a_n} - 9\sqrt[3]{a_1a_{n-1}a_n} - a_1 \geq 0.$$

Taking into account that $a_{n-1} + a_n \geq 2\sqrt{a_{n-1}a_n}$ and $\sqrt{a_{n-1}} + \sqrt{a_n} \geq 2\sqrt[4]{a_{n-1}a_n}$, it is enough to show that

$$2\sqrt{a_{n-1}a_n} + 8\sqrt[4]{a_1^2a_{n-1}a_n} - 9\sqrt[3]{a_1a_{n-1}a_n} - a_1 \geq 0.$$

Since this inequality is homogeneous in a_1 , a_{n-1} and a_n , without loss of generality, assume that $a_1 = 1$, $a_n \geq a_{n-1} \geq 1$. In addition, using the notation $x = \sqrt[12]{a_{n-1}a_n}$, $x \geq 1$, we can write the inequality as

$$2x^6 - 9x^4 + 8x^3 - 1 \geq 0.$$

This is true since

$$2x^6 - 9x^4 + 8x^3 - 1 = (x-1)^3(2x^3 + 6x^2 + 3x + 1) \geq 0.$$

(b) If $n = 3$, then the inequality is equivalent to

$$2a_1 + a_2 + a_3 + 2\sqrt{a_2a_3} \geq 6\sqrt[3]{a_1a_2a_3},$$

which is a consequence of the AM-GM inequality.

Consider now that $n \geq 4$, and apply Corollary 2.4 for $p_1 = p_2 = \dots = p_n = \frac{1}{n}$, $i = 2$, $j = n - 1$ and $k = n$. We have

$$P_{1,i} = \frac{2}{n}, \quad P_{i+1,j} = \frac{n-3}{n}, \quad P_{j,k-1} = \frac{1}{n}, \quad P_{k,n} = \frac{1}{n}.$$

By Corollary 2.4, since $P_{i+1,j} \geq P_{j,k-1}$, we have

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1a_2 \dots a_n} \geq 2a_2 + a_{n-1} + a_n - 4\sqrt[4]{a_2^2a_{n-1}a_n}.$$

Therefore, it suffices to prove that

$$2a_2 + a_{n-1} + a_n - 4\sqrt[4]{a_2^2a_{n-1}a_n} \geq \frac{1}{2}(2\sqrt{a_2} - \sqrt{a_{n-1}} - \sqrt{a_n})^2,$$

which is equivalent to the obvious inequality

$$(\sqrt{a_{n-1}} - \sqrt{a_n})^2 + 4\sqrt{a_2}(\sqrt[4]{a_{n-1}} - \sqrt[4]{a_n})^2 \geq 0.$$

Both inequalities in (a) and (b) become equalities if and only if $a_1 = a_2 = \dots = a_n$. □

Proposition 5.2. *If $a_1, a_2, \dots, a_n$ ($n \geq 3$) are positive real numbers such that*

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

then

$$(a) \quad a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1a_2 \dots a_n} \geq \frac{1}{4}(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n})^2;$$

$$(b) \quad a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1a_2 \dots a_n} \geq \frac{1}{2}(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_{n-1}})^2.$$

Proof. (a) In the case $n \geq 4$, we apply Corollary 2.4 for $p_1 = p_2 = \dots = p_n = \frac{1}{n}$, $i = 1$, $j = 2$ and $k = n$. We have

$$P_{1,i} = \frac{1}{n}, \quad P_{i+1,j} = \frac{1}{n}, \quad P_{j,k-1} = \frac{n-2}{n}, \quad P_{k,n} = \frac{1}{n}.$$

Since $P_{i+1,j} < P_{j,k-1}$, by Corollary 2.4 we have

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1a_2 \dots a_n} \geq a_1 + a_2 + a_n - 3\sqrt[3]{a_1a_2a_n}.$$

Clearly, this inequality is also true (as identity) for $n = 3$. Then, it suffices to prove that

$$a_1 + a_2 + a_n - 3\sqrt[3]{a_1a_2a_n} \geq \frac{1}{4}(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n})^2,$$

which is equivalent to

$$3(a_1 + a_2) + 4\sqrt{a_n}(\sqrt{a_1} + \sqrt{a_2}) - 2\sqrt{a_1 a_2} - 12\sqrt[3]{a_1 a_2 a_n} \geq 0.$$

Taking into account that $a_1 + a_2 \geq 2\sqrt{a_1 a_2}$ and $\sqrt{a_1} + \sqrt{a_2} \geq 2\sqrt[4]{a_1 a_2}$, it is enough to show that

$$\sqrt{a_1 a_2} + 2\sqrt[4]{a_1 a_2 a_n^2} - 3\sqrt[3]{a_1 a_2 a_n} \geq 0,$$

which follows by the AM-GM inequality.

(b) For $n = 3$, the inequality is equivalent to

$$a_1 + a_2 + 2a_3 + 2\sqrt{a_1 a_2} \geq 6\sqrt[3]{a_1 a_2 a_3},$$

which is a consequence of the AM-GM inequality.

For $n \geq 4$, we apply Corollary 2.4 for $p_1 = p_2 = \dots = p_n = \frac{1}{n}$, $i = 1$, $j = 2$ and $k = n - 1$. We have

$$P_{1,i} = \frac{1}{n}, \quad P_{i+1,j} = \frac{1}{n}, \quad P_{j,k-1} = \frac{n-3}{n}, \quad P_{k,n} = \frac{2}{n}.$$

By Corollary 2.4, since $P_{i+1,j} \leq P_{j,k-1}$, we have

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n} \geq a_1 + a_2 + 2a_{n-1} - 4\sqrt[4]{a_1 a_2 a_{n-1}^2}.$$

Therefore, it suffices to prove that

$$a_1 + a_2 + 2a_{n-1} - 4\sqrt[4]{a_1 a_2 a_{n-1}^2} \geq \frac{1}{2}(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_{n-1}})^2,$$

which is equivalent to the obvious inequality

$$(\sqrt{a_1} - \sqrt{a_2})^2 + 4\sqrt{a_{n-1}}(\sqrt[4]{a_1} - \sqrt[4]{a_2})^2 \geq 0.$$

Both inequalities in (a) and (b) become equalities if and only if $a_1 = a_2 = \dots = a_n$. □

Proposition 5.3. *If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers such that*

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

then (see [3])

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n} \geq 2\left(1 - \frac{1}{n}\right)(\sqrt{a_1} - 2\sqrt{a_2} + \sqrt{a_3})^2.$$

Proof. Apply Corollary 2.4 for $p_1 = p_2 = \dots = p_n = \frac{1}{n}$, $i = 1$, $j = 2$ and $k = 3$.

We have

$$P_{1,i} = \frac{1}{n}, \quad P_{i+1,j} = \frac{1}{n}, \quad P_{j,k-1} = \frac{1}{n}, \quad P_{k,n} = \frac{n-2}{n}.$$

By Corollary 2.4, we have

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n} \geq a_1 + a_2 + (n-2)a_3 - n\sqrt[n]{a_1 a_2 a_3^{n-2}}. \quad (5.1)$$

Therefore, it suffices to prove that

$$a_1 + a_2 + (n-2)a_3 - n\sqrt[n]{a_1 a_2 a_3^{n-2}} \geq 2\left(1 - \frac{1}{n}\right)(\sqrt{a_1} - 2\sqrt{a_2} + \sqrt{a_3})^2.$$

Write this inequality as $f(x) \geq 0$, where $0 < x \leq a \leq b$ and

$$f(x) = x + a + (n-2)b - n\sqrt[n]{ab^{n-2}x} - 2\left(1 - \frac{1}{n}\right)(\sqrt{x} - 2\sqrt{a} + \sqrt{b})^2.$$

We have

$$f'(x) = 1 - \sqrt[n]{\frac{ab^{n-2}}{x^{n-1}}} + 2\left(1 - \frac{1}{n}\right)\left(\frac{2\sqrt{a} - \sqrt{b}}{\sqrt{x}} - 1\right).$$

Clearly, $f'(x)$ increases when b decreases. Therefore, replacing b with a , we have

$$f'(x) \leq 1 - \sqrt[n]{\frac{a^{n-1}}{x^{n-1}}} + 2\left(1 - \frac{1}{n}\right)\left(\sqrt{\frac{a}{x}} - 1\right).$$

Substituting

$$\frac{a}{x} = t^{2n}, \quad t \geq 1,$$

we get $f'(x) \leq g(t)$, where

$$g(t) = 1 - t^{2n-2} + 2\left(1 - \frac{1}{n}\right)(t^n - 1).$$

Since

$$g'(t) = 2(n-1)t^{n-1}(1 - t^{n-2}) \leq 0,$$

$g(t)$ is decreasing, $g(t) \leq g(1) = 0$, $f'(x) \leq g(t) \leq 0$, $f(x)$ is decreasing, $f(x) \geq f(a)$. Thus, to show that $f(x) \geq 0$ for $0 < x \leq a \leq b$, we only need to show that $f(a) \geq 0$; that is,

$$2a + (n-2)b - n\sqrt[n]{a^2b^{n-2}} - 2\left(1 - \frac{1}{n}\right)(\sqrt{b} - \sqrt{a})^2 \geq 0.$$

Due to homogeneity, we may set $a = 1$. In addition, substituting $b = t^{2n}$, $t \geq 1$, we need to prove that $h(t) \geq 0$, where

$$h(t) = 2 + (n-2)t^{2n} - nt^{2n-4} - 2\left(1 - \frac{1}{n}\right)(t^n - 1)^2.$$

We have

$$h'(t) = 2t^{n-1}h_1(t), \quad h_1(t) = (n^2 - 4n + 2)t^n - n(n-2)t^{n-4} + 2(n-1).$$

For $n = 4$, we have $h_1(t) = 2(t^4 - 1) \geq 0$, and for $n > 4$, we have

$$\begin{aligned} h_1'(t) &= nt^{n-5}[(n^2 - 4n + 2)t^4 - (n-2)(n-4)] \\ &\geq nt^{n-5}[(n^2 - 4n + 2) - (n-2)(n-4)] = 2n(n-3)t^{n-5} > 0, \end{aligned}$$

$h_1(t)$ is increasing, $h_1(t) \geq h_1(1) = 0$. Thus, $h'(t) \geq 0$ for $n \geq 4$, $h(t)$ is increasing, $h(t) \geq h(1) = 0$. This completes the proof. Equality occurs if and only if $a_1 = a_2 = \dots = a_n$.

□

Proposition 5.4. *If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers such that*

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

then (see [3])

$$a_1 + a_2 + \dots + a_n - n \sqrt[n]{a_1 a_2 \dots a_n} \geq \frac{2}{9} \left(1 - \frac{1}{n}\right) (3\sqrt{a_1} - \sqrt{a_2} - 2\sqrt{a_3})^2.$$

Proof. According to (5.1), it suffices to prove that

$$a_1 + a_2 + (n-2)a_3 - n \sqrt[n]{a_1 a_2 a_3^{n-2}} \geq \frac{2}{9} \left(1 - \frac{1}{n}\right) (3\sqrt{a_1} - \sqrt{a_2} - 2\sqrt{a_3})^2.$$

Write this inequality as $f(x) \geq 0$, where $0 < a \leq b \leq x$ and

$$f(x) = a + b + (n-2)x - n \sqrt[n]{abx^{n-2}} - \frac{2}{9} \left(1 - \frac{1}{n}\right) (3\sqrt{a} - \sqrt{b} - 2\sqrt{x})^2.$$

We have

$$f'(x) = (n-2) \left(1 - \sqrt[n]{\frac{ab}{x^2}}\right) + \frac{4}{9} \left(1 - \frac{1}{n}\right) \left(\frac{3\sqrt{a} - \sqrt{b}}{\sqrt{x}} - 2\right).$$

Clearly, $f'(x)$ decreases when b increases. Therefore, replacing b with x , we have

$$\begin{aligned} f'(x) &\geq (n-2) \left(1 - \sqrt[n]{\frac{a}{x}}\right) + \frac{4}{9} \left(1 - \frac{1}{n}\right) \left(\frac{3\sqrt{a} - \sqrt{x}}{\sqrt{x}} - 2\right) \\ &= (n-2) \left(1 - \sqrt[n]{\frac{a}{x}}\right) - \frac{4}{3} \left(1 - \frac{1}{n}\right) \left(1 - \sqrt{\frac{a}{x}}\right). \end{aligned}$$

Substituting

$$\frac{a}{x} = t^{2n}, \quad 0 < t \leq 1,$$

we get $f'(x) \geq g(t)$, where

$$g(t) = (n-2)(1-t^2) - \frac{4}{3} \left(1 - \frac{1}{n}\right) (1-t^n).$$

Since

$$g'(t) = 2t \left[2 - n + \frac{2}{3}(n-1)t^{n-2}\right] \leq 2t \left[2 - n + \frac{2}{3}(n-1)\right] = \frac{2(4-n)t}{3} \leq 0,$$

$g(t)$ is decreasing, $g(t) \geq g(1) = 0$, $f'(x) \geq g(t) \geq 0$, $f(x)$ is increasing, $f(x) \geq f(b)$. Thus, to show that $f(x) \geq 0$ for $x \geq b$, we only need to show that $f(b) \geq 0$; that is,

$$a + (n-1)b - n \sqrt[n]{ab^{n-1}} - 2 \left(1 - \frac{1}{n}\right) (\sqrt{a} - \sqrt{b})^2 \geq 0.$$

Due to homogeneity, we may set $a = 1$. In addition, substituting $b = t^{2n}$, $t \geq 1$, we need to prove that $h(t) \geq 0$, where

$$h(t) = 1 + (n-1)t^{2n} - nt^{2n-2} - 2 \left(1 - \frac{1}{n}\right) (1-t^n)^2.$$

We have

$$h'(t) = 2(n-1)t^{n-1}h_1(t), \quad h_1(t) = (n-2)t^n - nt^{n-2} + 2.$$

Since

$$h'_1(t) = n(n-2)t^{n-3}(t^2 - 1) \geq 0,$$

$h_1(t)$ is increasing, $h_1(t) \geq h_1(1) = 0$. Thus, $h'(t) \geq 0$, $h(t)$ is increasing, $h(t) \geq h(1) = 0$. This completes the proof. Equality occurs if and only if $a_1 = a_2 = \dots = a_n$.

□

Proposition 5.5. *If $a_1, a_2, \dots, a_n$ ($n \geq 4$) are positive real numbers such that*

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

then (see [4])

$$a_1 + a_2 + \dots + a_n - n\sqrt[n]{a_1 a_2 \dots a_n} \geq \frac{4}{9} \left(1 - \frac{2}{n}\right) (\sqrt{a_1} + 2\sqrt{a_2} - 3\sqrt{a_3})^2.$$

Proof. According to (5.1), it suffices to prove that

$$a_1 + a_2 + (n-2)a_3 - n\sqrt[n]{a_1 a_2 a_3^{n-2}} \geq \frac{4}{9} \left(1 - \frac{2}{n}\right) (\sqrt{a_1} + 2\sqrt{a_2} - 3\sqrt{a_3})^2.$$

Write this inequality as $f(x) \geq 0$, where $0 < x \leq a \leq b$ and

$$f(x) = x + a + (n-2)b - n\sqrt[n]{ab^{n-2}x} - \frac{4}{9} \left(1 - \frac{2}{n}\right) (\sqrt{x} + 2\sqrt{a} - 3\sqrt{b})^2.$$

We have

$$f'(x) = 1 - \sqrt[n]{\frac{ab^{n-2}}{x^{n-1}}} - \frac{4}{9} \left(1 - \frac{2}{n}\right) \left(1 + \frac{2\sqrt{a} - 3\sqrt{b}}{\sqrt{x}}\right).$$

Clearly, $f'(x)$ increases when b decreases. Therefore, replacing b with a , we have

$$f'(x) \leq 1 - \sqrt[n]{\frac{a^{n-1}}{x^{n-1}}} - \frac{4}{9} \left(1 - \frac{2}{n}\right) \left(1 - \sqrt{\frac{a}{x}}\right).$$

Substituting

$$\frac{a}{x} = t^{2n}, \quad t \geq 1,$$

we get

$$\begin{aligned} f'(x) &\leq 1 - t^{2n-2} + \frac{4}{9} \left(1 - \frac{2}{n}\right) (t^n - 1) \leq \frac{2n-2}{n} (t^n - 1) - (t^{2n-2} - 1) \\ &= 2(n-1)(t-1) \left(\frac{t^{n-1} + t^{n-2} + \dots + 1}{n} - \frac{t^{2n-3} + t^{2n-4} + \dots + 1}{2n-2} \right) \leq 0. \end{aligned}$$

Therefore, $f(x)$ is decreasing, and hence $f(x) \geq f(a)$. To show that $f(x) \geq 0$ for $0 < x \leq a \leq b$, we only need to show that $f(a) \geq 0$; that is,

$$2a + (n-2)b - n\sqrt[n]{a^2 b^{n-2}} - 4 \left(1 - \frac{2}{n}\right) (\sqrt{a} - \sqrt{b})^2 \geq 0.$$

Due to homogeneity, we may set $a = 1$. In addition, substituting $b = t^{2n}$, $t \geq 1$, we need to prove that $h(t) \geq 0$, where

$$h(t) = 2 + (n-2)t^{2n} - nt^{2n-4} - 4 \left(1 - \frac{2}{n}\right) (t^n - 1)^2.$$

We have

$$h'(t) = 2(n-2)t^{n-1}h_1(t), \quad h_1(t) = (n-4)t^n - nt^{n-4} + 4.$$

Since

$$h'_1(t) = n(n-4)t^{n-5}(t^4 - 1) \geq 0,$$

$h_1(t)$ is increasing, $h_1(t) \geq h_1(1) = 0$. Thus, $h'(t) \geq 0$, $h(t)$ is increasing, $h(t) \geq h(1) = 0$. This completes the proof. For $n > 4$, equality occurs if and only if $a_1 = a_2 = \dots = a_n$. If $n = 4$, then equality holds for $a_1 = a_2$ and $a_3 = a_4$. □

Proposition 5.6. *Let $a_1, a_2, \dots, a_n$ ($n \geq 3$) and m be positive real numbers such that*

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

and

$$a_1 + a_2 + \dots + a_n = m \sqrt[n]{a_1 a_2 \dots a_n}.$$

(a) If

$$n \leq m \leq \left(n - \frac{i}{2}\right) \sqrt[n]{2^i}, \quad i \in \{2, 3, \dots, n-1\},$$

then a_{i-1}, a_i, a_{i+1} are the side-lengths of a degenerate or non-degenerate triangle (see [5]);

(b) If

$$n \leq m \leq \max_{i \in \{2, 3, \dots, n-1\}} \left(n - \frac{i}{2}\right) \sqrt[n]{2^i},$$

then among the numbers $a_1, a_2, \dots, a_n$ there exist three which are the side-lengths of a degenerate or non-degenerate triangle.

Proof. (a) The condition $m \geq n$ follows by the AM-GM Inequality

$$a_1 + a_2 + \dots + a_n \geq n \sqrt[n]{a_1 a_2 \dots a_n}.$$

For the sake of contradiction, assume that a_{i-1}, a_i, a_{i+1} are not the side-lengths of a triangle; that is, $a_{i-1} + a_i < a_{i+1}$. Setting $p_1 = p_2 = \dots = p_n = \frac{1}{n}$ in Corollary 2.5 and replacing then i, j, k by $i-1, i, i+1$, respectively, we have

$$\frac{a_1 + a_2 + \dots + a_n}{\sqrt[n]{a_1 a_2 \dots a_n}} \geq \frac{(i-1)a_{i-1} + a_i + (n-i)a_{i+1}}{\sqrt[n]{a_{i-1}^{i-1} a_i a_{i+1}^{n-i}}},$$

and hence

$$m \geq g(a_{i-1}, a_i, a_{i+1}),$$

where

$$g(a_{i-1}, a_i, a_{i+1}) = \frac{(i-1)a_{i-1} + a_i + (n-i)a_{i+1}}{\sqrt[n]{a_{i-1}^{i-1} a_i a_{i+1}^{n-i}}}$$

Since $a_{i+1} > a_{i-1} + a_i$ and

$$\frac{\partial g}{\partial a_{i+1}} = \frac{(n-i)[(i-1)(a_{i+1} - a_{i-1}) + a_{i+1} - a_i]}{na_{i+1} \sqrt[n]{a_{i-1}^{i-1} a_i a_{i+1}^{n-i}}} > 0,$$

we get

$$m > g(a_{i-1}, a_i, a_{i-1} + a_i) = \frac{(n-1)a_{i-1} + (n-i+1)a_i}{\sqrt[n]{a_{i-1}^{i-1} a_i (a_{i-1} + a_i)^{n-i}}}.$$

Due to homogeneity, we consider $a_{i-1} = 1$. Denoting $a_i = x$, $x \geq 1$, we have $m > h(x)$, where

$$h(x) = \frac{n-1 + (n-i+1)x}{\sqrt[n]{x(1+x)^{n-i}}}.$$

Since

$$\begin{aligned} n \sqrt[n]{x^{n-1}(1+x)^i} h'(x) &= (i-1)(n-i+1)x^2 - n + 1 \\ &\geq (i-1)(n-i+1) - n + 1 = (i-2)(n-i) \geq 0, \end{aligned}$$

$h(x)$ is increasing, $h(x) \geq h(1) = (n - \frac{i}{2}) \sqrt[n]{2^i}$, and hence $m > (n - \frac{i}{2}) \sqrt[n]{2^i}$, which is false.

(b) The conclusion follows immediately from (a).

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