THE PROBLEM OF ISOMETRIC EXTENSION IN THE UNIT SPHERE OF THE SPACE $s_p(\alpha, H)$

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Communicated by D. E. Alspach

ABSTRACT. In this paper, we consider the problem of isometric extension in the unit sphere of the space $s_p(\alpha, H)$. We obtain that Lamperti isometric mapping of the unit sphere $S(s_p(\alpha, H))$ into itself can be extended to an isometry on the whole space $s_p(\alpha, H)$.

1. Introduction

Let E and F be normed spaces. A mapping $V : E \to F$ is called an isometry if $||Vx - Vy|| = ||x - y||$ for all $x_1, x_2 \in E$ (see, e.g., [10]). The classical Mazur-Ulam theorem in [11] describes the relation between isometry and linearity and states that every onto isometry V between two normed spaces with $V(0) = 0$ is linear. So far, this has been generalized in several directions (see, e.g., [14]). One of them is the study of the isometric extension problem.

Mankiewicz in [9] showed that an isometry which maps a connected subset of a normed space X onto an open subset of another normed space Y can be extended to an affine isometry from X to Y. In 1987, Tingley [17] posed the problem of extending an isometry between unit spheres as follows.

> Let E and F be two real Banach spaces. Suppose that V_0 is a surjective isometry between the two unit spheres $S_1(E)$ and $S_1(F)$. Is V_0 necessarily a restriction of a linear or affine transformation to $S_1(E)$?

It is very difficult to answer this question, even in two dimensional cases. In the same paper, Tingley proved that if E and F are finite-dimensional Banach spaces and $V_0 : S_1(E) \to S_1(F)$ is a surjective isometry, then $V_0(x) = -V_0(-x)$ for all $x \in S_1(E)$. In [3], Ding gave an affirmative answer to Tingley problem, when E and F are Hilbert spaces. In the case when E and F are some metric vector spaces, the corresponding extension problem was investigated in [1, 5] . See also [6, 7, 8, 12, 15, 17] for some related results.

By $s_p(\alpha)$, $1 \leq p < \infty$, $p \neq 2, \alpha \geq 0$, is denoted the discrete analogue of the classical Sobolev space, which is the linear space of all real or complex sequences $x = (x(k))_{k \in \mathbf{N}}$ for which

$$||x|| = \left(\sum_{k=1}^{\infty} |x(k)|^p + \alpha \sum_{k=1}^{\infty} |x(k+1) - x(k)|^p \right)^{\frac{1}{p}} < \infty. \tag{1.1}$$

It is clear that $s_p(0) = l^p$. If $\alpha > 0$, then since

$$||x||_{l^p} \leq ||x|| \leq (1 + 2^p \alpha)^{1/p} ||x||_{l^p},$$

where $||x||_{l^p}$ is the standard l^p norm, it follows that $s_p(\alpha)$ is isomorphic to l^p, but these spaces are not isometric.

If an isometry on $s_p(\alpha)$ preserves both sums in formula (1.1), then it is called the Lamperti isometry (see [2]). From the definition we see that Lamperti isometry is independent of α for the space $s_p(\alpha)$.

Define $s_p(\alpha, H)$ $(1 \leq p < \infty, p \neq 2, \alpha \geq 0)$ to be the collection of all element sequences in the Hilbert space H such that

$$||x|| = \left(\sum_{k=1}^{\infty} ||x(k)||^p + \alpha \sum_{k=1}^{\infty} ||x(k+1) - x(k)||^p \right)^{\frac{1}{p}} < \infty. \tag{1.2}$$

If an isometry on $s_p(\alpha, H)$ preserves both sums in formula (1.2), then it is called the Lamperti isometry. From the definition we see that Lamperti isometry is independent of α for the space $s_p(\alpha, H)$.

Define $l^p(H)$ $(1 \leq p < \infty, p \neq 2)$ to be the collection of all element sequences in the Hilbert space H such that

$$||x|| = \left(\sum_{k=1}^{\infty} ||x(k)||^p \right)^{\frac{1}{p}} < \infty. \tag{1.3}$$

In [4], Fleming and Jamison proved that for $1 < p < \infty$, $\alpha > 0$, $\alpha \neq 1$, the only surjective isometries on $s_p(\alpha)$ are scalar multiples of the identity. In [2], it is shown that there are no Lamperti isometries of finite codimension that are not surjective, and two distinct examples are given. In this paper, we study the problem of isometric extension in the unit sphere of the space $s_p(\alpha, H)$. We prove that if V_0 is an isometric mapping on the unit sphere of the subspace E of $s_p(\alpha, H)$:

$$suppE = \{k(n) : k(n+1) - k(n) > 1, \ n \in \mathbf{N}\} \tag{1.4}$$

and

$$e_{x(n)} = \underbrace{(o, \cdots, x(n), o, \cdots)}_{n-\text{th place}} \in E, \tag{1.5}$$

then it can be extended to an isometry on the whole space E. Also, we prove that Lamperti isometric mapping on the unit sphere $S(s_p(\alpha, H))$ into itself can be extended to an isometry on the whole space $s_p(\alpha, H)$.

2. MAIN RESULTS

In this section, we give our main results. For this purpose, we need some lemmas that will be used in the proofs of our main results. We begin with the following result.

Lemma 2.1. *If V_0 is an isometric mapping from the unit sphere $S(s_p(\alpha, H))$ into $S(s_p(\alpha, H))$, then*

$$V_0(-x) = -V_0(x), \quad \text{for all } x \in S(s_p(\alpha, H)).$$

Proof. It is easy to see that the space $s_p(\alpha, H)$ is strictly convex , since $s_p(\alpha)$ and the Hilbert space H are strictly convex spaces(in fact, from [4], $s_p(\alpha)$ is uniformly convex and smooth). That is, if $||u + v|| = ||u|| + ||v||$, then $u = cv$ for some $c > 0$. , From this and since

$$||V_0(-x) - V_0(x)|| = || - x - x|| = 2 = ||V_0(-x)|| + || - V_0(x)||,$$

we have that $V_0(-x) = -cV_0(x)$ for some $c > 0$. This implies $|c| = 1$, from which the lemma follows. $\square$

Lemma 2.2. *If $x, y \in l^p(H)$ $(1 \le p < \infty, \ p \ne 2)$, then*

$$||x + y||^p + ||x - y||^p = 2(||x||^p + ||y||^p) \Leftrightarrow supp \ x \cap supp \ y = \emptyset \ ,$$

where $supp \ x = \{n : x(n) \ne 0, \ n \in \mathbf{N}\}$.

Proof. The sufficiency is trivial. Following, we prove the necessity.

Suppose that $x = \{x(n)\}$ and $y = \{y(n)\}$ are elements in $l^p(H)$. Since H is a Hilbert space, we have

$$||x(n) + y(n)||^2 + ||x(n) - y(n)||^2 = 2(||x(n)||^2 + ||y(n)||^2). \tag{2.1}$$

We will divide the proof into two steps.

Firstly, we prove if $0 < p < 2$

$$||x(n) + y(n)||^p + ||x(n) - y(n)||^p \le 2(||x(n)||^p + ||y(n)||^p), \tag{2.2}$$

while if $p > 2$

$$||x(n) + y(n)||^p + ||x(n) - y(n)||^p \ge 2(||x(n)||^p + ||y(n)||^p) \tag{2.3}$$

and the equality holds if and only if $||x(n)||||y(n)|| = 0$.

Indeed, if $0 < p < 2$, the function $\varphi(u) = u^{\frac{p}{2}}(u \geq 0)$ satisfies the following inequality

$$\varphi(\alpha + \beta) \leq \varphi(\alpha) + \varphi(\beta) \tag{2.4}$$

and the equality holds if and only if $\alpha\beta = 0$.

Since φ is concave on $[0, \infty)$, it follows from (2.1) and (2.4) that

$$\begin{aligned}
&\frac{1}{2}\Big(||x(n) + y(n)||^p + ||x(n) - y(n)||^p\Big) \\
&= \frac{1}{2}\Big(\varphi(||x(n) + y(n)||^2) + \varphi(||x(n) - y(n)||^2)\Big) \\
&\leq \varphi\Big(\frac{||x(n) + y(n)||^2 + ||x(n) - y(n)||^2}{2}\Big) \\
&= \varphi\Big(||x(n)||^2 + ||y(n)||^2\Big) \\
&\leq \varphi\Big(||x(n)||^2\Big) + \varphi\Big(||y(n)||^2\Big) \\
&= ||x(n)||^p + ||y(n)||^p.
\end{aligned} \tag{2.5}$$

The equality holds if and only if $||x(n)||\,||y(n)|| = 0$.

For the case $p > 2$, φ is convex on $[0, \infty)$. All the reverse inequalities hold in (2.4) and (2.5).

Then, we can prove the necessity of Lemma 2.2 by the definition of norm in the space $l^p(H)$ and the result from the first step.   □

Lemma 2.3. *Let V_0 be an isometric mapping from the unit sphere $S(l^p(H))$ into $S(l^p(H))$ $(1 < p < \infty,\ p \neq 2)$. Then*

$$(supp\ x) \cap (supp\ y) = \emptyset \Leftrightarrow (supp\ V_0(x)) \cap (supp\ V_0(y)) = \emptyset.$$

Proof. By the assumption of V_0, we have

$$||V_0(x) - V_0(y)|| = ||x - y||. \tag{2.6}$$

Since $l^p(H)$ is strictly convex, it follows from the proof of Lemma 2.1 that,

$$V_0(-x) = -V_0(x)$$

and thus

$$||V_0(x) + V_0(y)|| = ||V_0(x) - V_0(-y)|| = ||x + y||. \tag{2.7}$$

Notice that (2.6) and (2.7) assure that

$$||x + y||^p + ||x - y||^p = 2(||x||^p + ||y||^p)$$

and

$$||V_0(x) + V_0(y)||^p + ||V_0(x) - V_0(y)||^p = 2(||V_0(x)||^p + ||V_0(y)||^p)$$

hold simultaneously. According to Lemma 2.2, we obtain

$$(supp\ x) \cap (supp\ y) = \emptyset \Leftrightarrow (supp\ V_0(x)) \cap (supp\ V_0(y)) = \emptyset.$$

□

The following result is similar to that in [4].

Lemma 2.4. *Let V_0 be an isometric mapping from the unit sphere $S(s_p(\alpha, H))$ into $S(s_p(\alpha, H))$ and R be the right-translation operator, that is,*

$$Rx = (0, x(1), x(2), \cdots, x(n), \cdots), \quad \forall x = \{x(n)\} \in s_p(\alpha, H).$$

If

$$\begin{cases} supp \ x \cap supp \ y = \emptyset \\ supp \ x \cap supp \ R(y) = \emptyset \\ supp \ R(x) \cap supp \ y = \emptyset, \end{cases} \tag{2.8}$$

then

$$\begin{cases} supp \ V_0(x) \cap supp \ V_0(y) = \emptyset \\ supp \ V_0(x) \cap supp \ R(V_0(y)) = \emptyset \\ supp \ R(V_0(x)) \cap supp \ V_0(y) = \emptyset. \end{cases}$$

Remark 2.5. By the definition of the norm in the space $s_p(\alpha, H)$ and the condition of (2.8), it is easy to see that,

$$||x + y||^p = ||x||^p + ||y||^p$$

and

$$||V_0(x) + V_0(y)||^p = ||V_0(x)||^p + ||V_0(y)||^p$$

for every $x, y \in S(E)$. Where, E is a subspace of the space $s_p(\alpha, H)$ in which for every x and y of E if $supp \ x \cap supp \ y = \emptyset$ then above relation (2.8) hold.

Lemma 2.6. *Let E be a subspace of the space $s_p(\alpha, H)$ in which for every x and y of E if $supp \ x \cap supp \ y = \emptyset$ then above relation (2.8) hold. Suppose that V_0 is an isometric mapping on the unit sphere $S(E)$ into itself. If x and y are two disjoint elements of $S(E)$, ξ and η are real numbers and $z = \xi x + \eta y \in S(E)$, then*

$$V_0(z) = \xi V_0(x) + \eta V_0(y).$$

Proof. By the hypothesis of the subspace E of $s_p(\alpha, H)$, we have above relation (2.8). Hence, if we put

$$A = supp \ V_0(x), \quad B = supp \ V_0(y),$$

Remark 2.5 implies that

$$||V_0(z)|_A||^p + ||V_0(z)|_B||^p = ||V_0(z)|_{A \cup B}||^p \leq ||V_0(z)|_\Omega||^p = 1, \tag{2.9}$$

where $\Omega = \bigcup_{x \in E} supp \ x$.

If $\xi = 0$ then $\eta = 1$ or $\eta = -1$, so the relation $V_0(z) = \eta V_0(y)$ follows directly or by Lemma 2.1. The case when $\eta = 0$ is dual.

Now assume that $\xi \neq 0$ and $\eta \neq 0$. From Lemma 2.1, Remark 2.5 and the hypothesis of x and y, we obtain

$$\left\|V_0(z) + \frac{\xi}{|\xi|}V_0(x)\right\|^p \;=\; \left\|z + \frac{\xi}{|\xi|}x\right\|^p$$

$$=\; \left\|\xi x + \frac{\xi}{|\xi|}x\right\|^p + ||\eta y||^p$$

$$=\; \left\|\frac{\xi}{|\xi|}(1+|\xi|)x\right\|^p + ||\eta y||^p$$

$$=\; (1+|\xi|)^p + 1 - |\xi|^p, \tag{2.10}$$

where in the last equality we have used the the following identity

$$1^p = ||z||^p = ||\xi x + \eta y||^p = ||\xi x||^p + ||\eta y||^p = |\xi|^p + |\eta|^p, \tag{2.11}$$

which follows from the assumption $supp\ x \cap supp\ y = \emptyset$ and Remark 2.5.

Similarly, we also have

$$\left\|V_0(z) + \frac{\eta}{|\eta|}V_0(y)\right\|^p = (1+|\eta|)^p + 1 - |\eta|^p. \tag{2.12}$$

On the other hand, we have

$$\left\|V_0(z) + \frac{\xi}{|\xi|}V_0(x)\right\|^p = \left\|\left(V_0(z) + \frac{\xi}{|\xi|}V_0(x)\right)\Big|_A + \left(V_0(z) + \frac{\xi}{|\xi|}V_0(x)\right)\Big|_{\Omega-A}\right\|^p.$$

Notice the hypothesis of the subspace E of $s_p(\alpha, H)$ and the definition of the norm of $s_p(\alpha, H)$, along with Remark 2.5, imply

$$\left\|\left(V_0(z) + \tfrac{\xi}{|\xi|}V_0(x)\right)\Big|_A + \left(V_0(z) + \tfrac{\xi}{|\xi|}V_0(x)\right)\Big|_{\Omega-A}\right\|^p$$

$$=\; \left\|\left(V_0(z) + \tfrac{\xi}{|\xi|}V_0(x)\right)\Big|_A\right\|^p + \left\|\left(V_0(z) + \tfrac{\xi}{|\xi|}V_0(x)\right)\Big|_{\Omega-A}\right\|^p$$

$$\leq\; \left(||V_0(z)|_A|| + \left\|\tfrac{\xi}{|\xi|}V_0(x)\Big|_A\right\|\right)^p + ||V_0(z)|_{\Omega-A}||^p$$

$$=\; (1 + ||V_0(z)|_A||)^p + 1 - ||V_0(z)|_A||^p.$$

That is,

$$\left\|V_0(z) + \frac{\xi}{|\xi|}V_0(x)\right\|^p \leq (1 + ||V_0(z)|_A||)^p + 1 - ||V_0(z)|_A||^p. \tag{2.13}$$

Similarly, we also obtain

$$\left\|V_0(z) + \frac{\eta}{|\eta|}V_0(y)\right\|^p \leq (1 + ||V_0(z)|_B||)^p + 1 - ||V_0(z)|_B||^p. \tag{2.14}$$

Hence, by (2.10) and ((2.13),

$$(1+|\xi|)^p + 1 - |\xi|^p \leq (1 + ||V_0(z)|_A||)^p + 1 - ||V_0(z)|_A||^p \tag{2.15}$$

and also by (2.12) and (2.14) we have

$$(1+|\eta|)^p + 1 - |\eta|^p \leq (1 + ||V_0(z)|_B||)^p + 1 - ||V_0(z)|_B||^p. \tag{2.16}$$

Notice that the real function $\varphi(t) = (1+t)^p + 1 - t^p$ is strictly increasing in $[0, \infty)$, hence (2.15) and (2.16) imply

$$|\xi| \leq ||V_0(z)|_A|| \ \text{ and } \ |\eta| \leq ||V_0(z)|_B||. \tag{2.17}$$

By (2.9), (2.11) and (2.17), we obtain

$$|\xi| = ||V_0(z)|_A|| \text{ and } |\eta| = ||V_0(z)|_B||. \tag{2.18}$$

From (2.9),(2.11) and (2.18), we obtain

$$||V_0(z)|_\Omega||^p = ||V_0(z)|_A||^p + ||V_0(z)|_B||^p.$$

Therefore $supp\ V_0(z)|_{\Omega-A\cup B} = \emptyset$, and so

$$V_0(z) = V_0(z)|_A + V_0(z)|_B. \tag{2.19}$$

By (2.10), (2.11),(2.18) and (2.19), we have

$$\begin{aligned}
&(1 + |\xi|)^p + |\eta|^p \\
&= \left\|V_0(z) + \frac{\xi}{|\xi|}V_0(x)\right\|^p \\
&\leq \left\|V_0(z)|_A + \frac{\xi}{|\xi|}V_0(x)\right\|^p + ||V_0(z)|_B||^p \\
&\leq (1 + ||V_0(z)|_A||)^p + ||V_0(z)|_B||^p \\
&= (1 + |\xi|)^p + |\eta|^p,
\end{aligned}$$

which implies

$$\left\|V_0(z)|_A + \frac{\xi}{|\xi|}V_0(x)\right\| = ||V_0(z)|_A|| + \left\|\frac{\xi}{|\xi|}V_0(x)\right\|.$$

Since $s_p(\alpha, H)$ is strictly convex, we have $V_0(z)|_A = c\frac{\xi}{|\xi|}V_0(x)$ for some $c > 0$, and $c = |\xi|$ because of (2.18). That is, $V_0(z)|_A = \xi V_0(x)$. Similarly, we also have $V_0(z)|_B = \eta V_0(x)$. Thus we obtain $V_0(z) = \xi V_0(x) + \eta V_0(y)$. We have completed the proof. □

Lemma 2.7. *Let E be a subspace of the space $s_p(\alpha, H)$ in which for every x and y of E if supp $x \cap$ supp $y = \emptyset$ then relation (2.8) hold. Suppose that V_0 is an isometric mapping on the unit space $S(E)$ into itself. If $x_1, x_2, \cdots, x_m$ are mutual disjoint elements of $S(E)$ and $\sum_{n=1}^{m} \xi_n x_n \in S(E)$, then*

$$V_0\left(\sum_{n=1}^{m} \xi_n x_n\right) = \sum_{n=1}^{m} \xi_n V_0(x_n).$$

Proof. We prove this by induction. Indeed, for $m = 2$ the result holds by Lemma 2.6. If for $m = m_0 - 1$ the results holds. For $m = m_0$, by using the following equality

$$\sum_{n=1}^{m_0} \xi_n x_n = \xi_1 x_1 + \left\|\sum_{n=2}^{m_0} \xi_n x_n\right\| \cdot \sum_{n=2}^{m_0} \frac{\xi_n}{\left\|\sum_{n=2}^{m_0} \xi_n x_n\right\|} x_n$$

and the hypothesis of the subspace E, where at least one ξ_n for $n \geq 2$ is a non-zero number, we get the desired result. If all ξ_n are zero for $n \geq 2$, then $|\xi_1| = 1$, so $V_0(\xi_1 x_1) = \xi_1 V_0(x_1)$ and we are done. □

Lemma 2.8. *Suppose that E is a subspace of the space $s_p(\alpha, H)$ satisfying (1.4) and (1.5) and that V_0 is an isometric mapping from the unit sphere $S(E)$ onto itself. Then, for any $e_{x(n)} \in S(E)$, $\text{supp}V_0 e_{x(n)}$ is a single point set.*

Proof. It is easy to see that the result of Lemma 2.3 and relation (2.8) hold for the space E. Without loss of generality, we can assume

$$V_0(e_{x(1)}) = e_{x'(k)} + \sum_{i \neq k} e_{x'(i)},$$

where $x'(k) \neq o$. Let

$$y = \frac{e_{x'(k)}}{||e_{x'(k)}||}.$$

Since V_0 is surjective, there exists $u \in S(E)$ such that $V_0 u = y$. By Lemma 2.3,

$$(\text{supp}V_0 u) \bigcap (\text{supp}V_0 e_{x(n)}) = (\text{supp}y) \bigcap (\text{supp}V_0 e_{x(n)})$$

$$\subseteq (\text{supp}V_0 e_{x(1)}) \bigcap (\text{supp}V_0 e_{x(n)}) = \emptyset$$

holds for any $n \neq 1$, where $x(n) \neq 0$.

Applying Lemma 2.3 again, we have

$$(\text{supp}u) \bigcap (\text{supp}e_{x(n)}) = \emptyset \quad (n \neq 1).$$

This means that $u = e_{x(1)}$, and this implies $\text{Supp}V_0(e_{x(1)}) = \{k\}$. $\square$

Lemma 2.9. *Suppose that x_1 and y_1 are elements in the Hilbert space H, that λ and μ are non - zero real numbers, that $||\lambda x_1 \pm \mu y_1|| = ||\lambda x_2 \pm \mu y_2||$, that $||x_1|| = ||x_2||$ and that $||y_1|| = ||y_2||$. Then $||x_1 - y_1|| = ||x_2 - y_2||$.*

Proof. It is easy to prove that by the parallelogram law. $\square$

Now we are in a position to state the main results and proofs in this paper.

Theorem 2.10. *Suppose that E is a subspace of the space $s_p(\alpha, H)$ satisfying (1.4) and (1.5), that $1 \notin \text{Supp}E$ and that V_0 is an isometric mapping from the unit sphere $S(E)$ onto itself. Then V_0 can be extended to an isometry on the whole space E.*

Proof. By Lemma 2.7, we see that for every $x = \{x(n)\}_{n=1}^{n=m} \in S(E)$, we have

$$V_0(x) = \sum_{n=1}^{m} ||e_{x(n)}|| V_0 \left(\frac{e_{x(n)}}{||e_{x(n)}||} \right). \tag{2.20}$$

If $||x(n)|| = 0$, then $V_0 \left(\frac{e_{x(n)}}{||e_{x(n)}||} \right) \overset{def}{=} 0$.

We now define a mapping on the subspace E_0 of E which consists of all elements in which every element only has finitely many non-zero items as follows

$$V_1 \left(\{x(n)\}_{n=1}^{n=m} \right) \overset{def}{=} \sum_{n=1}^{m} ||e_{x(n)}|| V_0 \left(\frac{e_{x(n)}}{||e_{x(n)}||} \right) \tag{2.21}$$

for all $\{x(n)\}_{n=1}^{n=m} \in E_0 \subset E$.

Suppose that $\{x(n)\}_{n=1}^{n=m}$ and $\{y(n)\}_{n=1}^{n=m}$ are elements in E_0. By lemma 2.8, we can assume

$$V_1(\{x(n)\}_{n=1}^{n=m}) = \{x'(k(n))\}_{n=1}^{n=m}$$

and

$$V_1(\{y(n)\}_{n=1}^{n=m}) = \{y'(k(n))\}_{n=1}^{n=m}.$$

To prove

$$\|V_1(\{x(n)\}_{n=1}^{n=m}) - V_1(\{y(n)\}_{n=1}^{n=m})\| = \|\{x(n)\}_{n=1}^{n=m} - \{y(n)\}_{n=1}^{n=m}\|. \qquad (2.22)$$

We proceed as follows

$$\left\|V_0\left(\frac{e_{x(n)}}{\|e_{x(n)}\|}\right) \pm V_0\left(\frac{e_{y(n)}}{\|e_{y(n)}\|}\right)\right\|^p$$

$$= \left\|\frac{e_{x(n)}}{\|e_{x(n)}\|} \pm \frac{e_{y(n)}}{\|e_{y(n)}\|}\right\|^p$$

$$= (1 + 2^p\alpha)\left\|\frac{x(n)}{\|e_{x(n)}\|} \pm \frac{y(n)}{\|e_{y(n)}\|}\right\|^p. \qquad (2.23)$$

On the other hand

$$\left\|V_0\left(\frac{e_{x(n)}}{\|e_{x(n)}\|}\right) \pm V_0\left(\frac{e_{y(n)}}{\|e_{y(n)}\|}\right)\right\|^p$$

$$= \left\|\frac{e_{x'(k(n))}}{\|e_{x'(k(n))}\|} \pm \frac{e_{y'(k(n))}}{\|e_{y'(k(n))}\|}\right\|^p$$

$$= (1 + 2^p\alpha)\left\|\frac{x'(k(n))}{\|e_{x'(k(n))}\|} \pm \frac{y'(k(n))}{\|e_{y'(k(n))}\|}\right\|^p. \qquad (2.24)$$

If $\|x(n)\| = 0$, then $\frac{x(n)}{\|e_{x(n)}\|} \overset{def}{=} 0$ and $\frac{x'(k(n))}{\|e_{x'(k(n))}\|} \overset{def}{=} 0$. It follows from (2.23) and (2.24) that

$$\left\|\frac{x(n)}{\|e_{x(n)}\|} \pm \frac{y(n)}{\|e_{y(n)}\|}\right\| = \left\|\frac{x'(k(n))}{\|e_{x'(k(n))}\|} \pm \frac{y'(k(n))}{\|e_{y'(k(n))}\|}\right\|. \qquad (2.25)$$

Notice that $\|e_{x(n)}\| = \|e_{x'(k(n))}\|$, that $\|e_{y(n)}\| = \|e_{y'(k(n))}\|$ and (2.25), it follows that from Lemma 2.9 that

$$\|x(n) - y(n)\| = \|x'(k(n)) - y'(k(n))\|. \qquad (2.26)$$

Since

$$\|V_1(\{x(n)\}_{n=1}^{n=m}) - V_1(\{y(n)\}_{n=1}^{n=m})\|^p = \sum_{n=1}^{m}(1 + 2^p\alpha)\|x'(k(n)) - y'(k(n))\|^p \quad (2.27)$$

and

$$\|\{x(n)\}_{n=1}^{n=m} - \{y(n)\}_{n=1}^{n=m}\|^p = \sum_{n=1}^{m}(1 + 2^p\alpha)\|x(n) - y(n)\|^p. \qquad (2.28)$$

(2.26), (2.27) and (2.28) assure (2.22) holds. That is we have obtained an isometry on the subspace E_0 of E. It is evident that E_0 is dense in the space E, E is a Banach space and V_1 is isometric on E_0. Hence V_1 has a unique linear isometric extension V on the whole space E. Thus V is the desired extension of V_0 and the proof is complete. □

Theorem 2.11. *Let V_0 be a Lamperti isometric mapping on the unit sphere $S(s_p(\alpha, H))$ into itself. Then V_0 can be extended to an isometry on the whole space $s_p(\alpha, H)$.*

Proof. We notice that, if V_0 is a Lamperti isometric mapping on the subset of the space $s_p(\alpha, H)$, then V_0 is also a isometric mapping on the same subset of the space $l_p(H)$. By Lemmas 2.1 and 2.3, we can obtain the simple conclusion like Lemma 2.6 as follows (and much easy because, for every disjoint x and y in $l^p(H)$, we have that $||x + y||^p = ||x||^p + ||y||^p$).

Similarly, for every such element $\{x(n)\}_{n=1}^{n=m} \in S(s_p(\alpha, H))$, we have

$$V_0\Big(\{x(n)\}_{n=1}^{n=m}\Big) = \sum_{n=1}^{m} ||e_{x(n)}||V_0\Big(\frac{e_{x(n)}}{||e_{x(n)}||}\Big). \tag{2.29}$$

If $||x(n)|| = 0$, then $V_0\Big(\frac{e_{x(n)}}{||e_{x(n)}||}\Big) \stackrel{def}{=} 0$. Now we define a mapping on the whole space $s_p(\alpha, H)$ as follows

$$Vx = \begin{cases} ||x||V_0\Big(\frac{x}{||x||}\Big), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then it is evident that

$$||V(x)|| = ||x||, \quad \forall x \in s_p(\alpha, H). \tag{2.30}$$

By (2.29) and (2.30) for every such element $x = \{x(n)\}_{n=1}^{n=m} \in s_p(\alpha, H)$, we have that

$$\begin{aligned} V\Big(\{x(n)\}_{n=1}^{n=m}\Big) &= ||x||V_0\Big(\frac{x}{||x||}\Big) \\ &= \sum_{n=1}^{m} ||e_{x(n)}||V_0\Big(\frac{e_{x(n)}}{||e_{x(n)}||}\Big). \end{aligned} \tag{2.31}$$

That is, we obtain an isometry on the subspace E of $s_p(\alpha, H)$ in which every element only has finite many non-zero items since (2.30) and (2.31) hold. Since E is dense in $s_p(\alpha, H)$ and V is continuous mapping on $s_p(\alpha, H)$, we see that V is an isometry on $s_p(\alpha, H)$ and it is the extension of V_0. $\qquad\square$

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