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SOME RESULTS ON $(LCS)_{2n+1}$-MANIFOLDS

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. In this paper we study Lorentzian Concircular Structure manifolds (briefly $(LCS)_{2n+1}$-manifold) and obtain some interesting results.

1. Introduction

An $(2n+1)$-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature (-,+,.....,+), where T_pM denotes the tangent space of M at p and R is the real number space.

In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X),$$

for any vector field $X \in \chi(M)$ is said to be a concircular vector field[8] if

$$(\nabla_X A)(Y) = \alpha[g(X, Y) + \omega(X)A(Y)],$$

where α is a non-zero scalar function, A is a 1-form and ω is a closed 1-form.
Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{1.1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X) \tag{1.2}$$

the equation of the following form holds:

$$(\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)] \quad (\alpha \neq 0) \tag{1.3}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is a non-zero scalar function satisfying

$$\nabla_X \alpha = (X\alpha) = \rho\eta(X), \tag{1.4}$$

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ρ being a certain scalar function. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{1.5}$$

then from (1.3) and (1.5) we have

$$\phi^2 X = X + \eta(X)\xi, \tag{1.6}$$

from which if follows that ϕ is a symmetric $(1,1)$ tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_{2n+1} - manifold$)[9]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [6].

A Riemannian manifold M is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifold the notion of semi-symmetric manifold was defined by

$$(R(X,Y)\cdot R)(U,V,W) = 0, \quad X, \quad W \quad \in \chi(M)$$

and studied by many authors (e.g. [7], [11]). A complete intrinsic classification of these spaces was given by Z.I. Szabo [11]. The notion of semi-symmetry was weakened by R. Deszcz and his coauthors ([4]- [5]) and introduced the notion of pseudosymmetric and Ricci-Pseudosymmetric manifolds.

We define endomorphisms $R(X,Y)$ and $X \wedge Y$ by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{1.7}$$

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{1.8}$$

respectively, where $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M.

The present paper deals with the study of $(LCS)_{2n+1}$-manifold satisfying certian conditions. After preliminaries, in section 3 we show that $(LCS)_{2n+1}$-manifold satisfying the condition $R(X,Y).\overline{P} = 0$, where $\overline{P}$ is the pseudo projective curvature tensor and $R(X,Y)$ is considred as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y is an pseudo projectively flat manifold. In section 4 we study pseudo projectively flat $(LCS)_{2n+1}$-manifold and proved that it is an η-Einstein manifold. Section 5 is devoted to the study of pseudo projectively recurrent $(LCS)_{2n+1}$-manifold. In the last section we study partially Ricci-pseudosymmetric $(LCS)_{2n+1}$-manifolds.

2. Preliminaries

A differentiable manifold M of dimension $(2n+1)$ is called $(LCS)_{2n+1}$-manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\xi) = \eta(X), \tag{2.4}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.5}$$

for all $X, Y \in TM$.

Also in a $(LCS)_{2n+1}$-manifold M the following relations are satisfied([10]):

$$\eta(R(X,Y)Z) = (\alpha^2-\rho)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)], \tag{2.6}$$
$$R(\xi,X)Y = (\alpha^2-\rho)(g(X,Y)\xi-\eta(Y)X), \tag{2.7}$$
$$R(X,Y)\xi = (\alpha^2-\rho)[\eta(Y)X-\eta(X)Y], \tag{2.8}$$
$$R(\xi,X)\xi = (\alpha^2-\rho)[\eta(X)\xi+X)], \tag{2.9}$$
$$(\nabla_X\phi)(Y) = \alpha[g(X,Y)\xi+2\eta(X)\eta(Y)\xi+\eta(Y)X], \tag{2.10}$$
$$S(X,\xi) = 2n(\alpha^2-\rho)\eta(X), \tag{2.11}$$
$$S(\phi X,\phi Y) = S(X,Y)+2n(\alpha^2-\rho)\eta(X)\eta(Y), \tag{2.12}$$

where S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

3. $(LCS)_{2n+1}$-Manifolds Satisfying $R(X,Y).\overline{P}=0$

The Pseudo projective curvature tensor $\overline{P}$ is defined as [2]

$$\begin{aligned}\overline{P}(X,Y)Z &= aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y] \\ &\quad -\frac{r}{(2n+1)}\left[\frac{a}{2n}+b\right][g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{3.1}$$

where a and b are constants such that $a,\ b\neq 0$, R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

In view of (2.4) and (2.8), we get

$$\begin{aligned}\eta\left(\overline{P}(X,Y)Z\right) &= \left[a(\alpha^2-\rho)-\frac{r}{(2n+1)}\left(\frac{a}{2n}+b\right)\right][g(Y,Z)X-g(X,Z)Y] \\ &\quad +b[S(Y,Z)X-S(X,Z)Y].\end{aligned} \tag{3.2}$$

Putting $Z=\xi$ in (3.2), we get

$$\eta\left(\overline{P}(X,Y)\xi\right)=0.$$

Again taking $X=\xi$ in (3.2), we have

$$\begin{aligned}\eta\left(\overline{P}(\xi,Y)Z\right) &= -\left[a(\alpha^2-\rho)-\frac{r}{(2n+1)}\left(\frac{a}{2n}+b\right)\right]g(Y,Z) \\ &\quad -\left[(a+2nb)(\alpha^2-\rho)-\frac{r}{(2n+1)}\left(\frac{a}{2n}+b\right)\right]\eta(Y)\eta(Z) \\ &\quad -bS(Y,Z),\end{aligned} \tag{3.3}$$

where (2.4) and (2.11) are used.

Now,

$$\begin{aligned}\left(R(X,Y)\overline{P}\right)(U,V)Z &= R(X,Y).\overline{P}(U,V)Z-\overline{P}(R(X,Y)U,V)Z \\ &\quad -\overline{P}(U,R(X,Y)V)Z-\overline{P}(U,V)R(X,Y)Z.\end{aligned}$$

Let $R(X,Y).\overline{P}=0$. Then we have

$$R(X,Y).\overline{P}(U,V)Z-\overline{P}(R(X,Y)U,V)Z-\overline{P}(U,R(X,Y)V)Z-\overline{P}(U,V)R(X,Y)Z=0.$$

Therefore,

$$g[R(\xi,Y).\overline{P}(U,V)Z,\xi] - g[\overline{P}(R(\xi,Y)U,V)Z,\xi] - g[\overline{P}(U,R(\xi,Y)V)Z,\xi] - g[\overline{P}(U,V)R(\xi,Y)Z,\xi] = 0.$$

From this, it follows that,

$$\begin{aligned}(\alpha^2-\rho)[-\grave{\overline{P}}(U,V,Z,Y) &- \eta(Y)\eta(\overline{P}(U,V)Z) + \eta(U)\ \eta(\overline{P}(Y,V)Z) \\ &- g(Y,U)\eta(\overline{P}(\xi,V)Z) + \eta(V)\eta(\overline{P}(U,Y)Z) \\ &- g(Y,V)\eta(\overline{P}(U,\xi)Z) + \eta(Z)\eta(\overline{P}(U,V)Y)] = 0,\end{aligned} \tag{3.4}$$

where $\grave{\overline{P}}(U,V,Z,Y) = g(\overline{P}(U,V)Z,Y)$.

Putting $Y = U$ in (3.4), we get

$$\begin{aligned}(\alpha^2-\rho)[-\grave{\overline{P}}(U,V,Z,U) &- g(U,U)\eta(\overline{P}(\xi,V)Z) + \eta(V)\eta(\overline{P}(U,U)Z) \\ &- g(U,V)\eta(\overline{P}(U,\xi)Z) + \eta(Z)\eta(\overline{P}(U,V)U)] = 0.\end{aligned} \tag{3.5}$$

Let $\{e_i\}$, $i = 1,2,\ldots,(2n+1)$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le 2n+1$ of the relation (3.5) for $U = e_i$, yields

$$\begin{aligned}\eta(\overline{P}(\xi,V)Z) &= -\left[\frac{a}{2n+1} + b\right] S(V,Z) \\ &+ \left[\frac{(a+2nb)r}{2n(2n+1)} - \frac{a}{2n+1}(\alpha^2-\rho)\right] g(V,Z) \\ &+ \left[\frac{ar}{2n(2n+1)} - a(\alpha^2-\rho)\right]\eta(V)\eta(Z).\end{aligned} \tag{3.6}$$

From (3.3) and (3.6) we have

$$S(V,Z) = 2n(\alpha^2-\rho)g(V,Z) + \frac{b}{a}\left[2n(2n+1) - r\right]\eta(V)\eta(Z). \tag{3.7}$$

Taking $Z = \xi$ in (3.7) and using (2.11) we obtain

$$r = 2n(2n+1)(\alpha^2-\rho). \tag{3.8}$$

Now using (3.2), (3.3), (3.7) and (3.8) in (3.4), we get

$$-\grave{\overline{P}}(U,V,Z,Y) = 0. \tag{3.9}$$

From (3.9) it follows that

$$\overline{P}(U,V)Z = 0.$$

Hence the $(LCS)_{2n+1}$-manifold is pseudo projectively flat. Therefore, we can state

Theorem 3.1. *If in a $(LCS)_{2n+1}$-manifold M of dimension $2n+1$, $n > 0$, the relation $R(X,Y).\overline{P} = 0$ holds then the manifold is pseudo projectively flat.*

4. Pseudo Projectively Flat $(LCS)_{2n+1}$-Manifolds

In this section we assume that $\overline{P} = 0$. Then form (3.1) we get

$$aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{(2n+1)}\left[\frac{a}{2n} + b\right][g(Y,Z)X - g(X,Z)Y] = 0. \tag{4.1}$$

From (4.1), we get

$$\begin{aligned} a\grave{}R(X,Y,Z,W) &+ b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ &- \frac{r}{(2n+1)}\left[\frac{a}{2n}+b\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0, \end{aligned} \tag{4.2}$$

where $\grave{}R(X,Y,Z,W) = g(R(X,Y,Z),W)$.

Putting $X = W = \xi$ in (4.2), we get

$$\begin{aligned} S(Y,Z) &= \left\{\frac{r}{(2n+1)}\left(\frac{a}{2nb}+1\right) - \frac{a}{b}(\alpha^2-\rho)\right\} g(Y,Z) \\ &+ \left\{\frac{r}{(2n+1)}\left(\frac{a}{2nb}+1\right) - \frac{a}{b}(\alpha^2-\rho) - 2n(\alpha^2-\rho)\right\} \eta(Y)\ \eta(Z). \end{aligned} \tag{4.3}$$

Therefore, the manifold is η-Einstein. Hence we can state

Theorem 4.1. *A pseudo projectively flat $(LCS)_{2n+1}$-manifold is an η-Einstein manifold.*

5. Pseudo projectively recurrent $(LCS)_{2n+1}$-Manifolds

A non-flat Riemannian manifold M is said to be pseudo projectively recurrent if the pseudo-projective curvature tensor $\overline{P}$ satisfies the condition $\nabla\overline{P} = A \otimes \overline{P}$, where A is an everywhere non-zero 1-form. We now define a function f on M by $f^2 = g(\overline{P}, \overline{P})$, where the metric g is extended to the inner product between the tensor fields in the standard fashion.

Then we know that $f(Yf) = f^2 A(Y)$. From this we have

$$Yf = fA(Y) \quad (because f \neq 0). \tag{5.1}$$

From (5.1) we have

$$X(Yf) = \frac{1}{f}(Xf)(Yf) + (XA(Y))f.$$

Hence

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f.$$

Therefore we get

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})f = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M by our assumption, we obtain

$$dA(X,Y) = 0. \tag{5.2}$$

that is the 1-form A is closed.

Now, from $(\nabla_X\overline{P})(U,V)Z = A(X)\overline{P}(U,V)Z$, we get

$$(\nabla_U\nabla_V\overline{P})(X,Y)Z = \{UA(V) + A(U)A(V)\}\overline{P}(X,Y)Z.$$

Hence from (5.2), we get

$$(R(X,Y).\overline{P})(U,V)Z = [2dA(X,Y)]\overline{P}(U,V)Z = 0. \tag{5.3}$$

Therefore, for a pseudo projectively recurrent manifold, we have

$$R(X,Y)\overline{P} = 0 \ \ for \ \ all \ \ X,Y. \tag{5.4}$$

Thus, we can state the following:

Theorem 5.1. *A pseudo projectively recurrent $(LCS)_{2n+1}$-manifold M is an η-Einstein manifold.*

6. Partially Ricci-pseudosymmetric $(LCS)_{2n+1}$-Manifolds

A $(LCS)_{2n+1}$-manifold M is said to be a partially Ricci-pseudosymmetric if it satisfies.

$$(R(X,Y)\cdot S)(U,V) = L_C[((X\wedge Y)\cdot S(U,V)], \tag{6.1}$$

where

$$L_C \in C^\infty(M),$$

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V),$$

and

$$((X\wedge Y)\cdot S)(U,V) = -S((X\wedge Y)U,V) - S(U,(X\wedge Y)V).$$

Thus (6.1) has the following more developed form

$$\begin{aligned} & S(R(X,Y)U,V) + S(U,R(X,Y)V) \\ = \;& L_C[S((X\wedge Y)U,V) + S(U,(X\wedge Y)V)]. \end{aligned} \tag{6.2}$$

We want to investigate partially pseudo-Ricci-symmetric $(LCS)_{2n+1}$-manifolds which satisfy (6.1) with the restriction $Y = V = \xi$. So we have

$$\begin{aligned} & S(R(X,\xi)U,\xi) + S(U,R(X,\xi)\xi) \\ = \;& L_C[S((X\wedge\xi)U,\xi) + S(U,(X\wedge\xi)\xi)]. \end{aligned} \tag{6.3}$$

Applying (1.8), (2.7) and (2.11), we obtain

$$\begin{aligned} & 2n(\alpha^2-\rho)\eta(R(X,\xi)U) - (\alpha^2-\rho)S(U,X) - (\alpha^2-\rho)S(U,\xi)\eta(X) \\ = \;& L_C[\eta(U)S(X,\xi) - g(X,U)S(\xi,\xi) - S(U,X) - \eta(X)S(U,\xi)]. \end{aligned}$$

Using (2.1) and (2.11) in above relation, this becomes

$$-(\alpha^2-\rho)[S(X,U)-2n(\alpha^2-\rho)g(X,U)] = -L_C[S(X,U)-2n(\alpha^2-\rho)g(X,U)]. \tag{6.4}$$

This can be written as

$$[L_C - (\alpha^2-\rho)][S(X,U) - 2n(\alpha^2-\rho)g(X,U)] = 0. \tag{6.5}$$

This can be hold only if either (a) $L_C = (\alpha^2-\rho)$ or (b) $S(X,U) = 2n(\alpha^2-\rho)g(X,U)$. However (b) means that M is an Einstein manifold. Hence the we can state

Theorem 6.1. *A partially pseudo-Ricci symmetric $(LCS)_{2n+1}$-manifold with never vanishing function $[L_C - (\alpha^2-\rho)]$ is an Einstein manifold.*

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