Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://bmathaa.org
Volume 10 Issue 4(2018), Pages 1-13.

ON ∇_2-STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF ORDER α IN RANDOM 2-NORMED SPACE

ÖMER KİŞİ

Abstract. In this present paper, we introduce the notion of ∇_2-statistical convergence of double sequences of order α, ∇_2-statistical Cauchy double sequences of order α in random 2-normed spaces and obtain some results. We display examples which show that our method of convergence is more general in random 2-normed space.

1. Introduction

The idea of the statistical convergence was given by Zygmund [36] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Fast [7] and Steinhaus [34] and then reintroduced by Schoenberg [31] independently. Over the years, statistical convergence has been developed in ([3], [13], [14], [21], [25], [29], [35]) and turned out very useful to resolve many convergence problems arising in Analysis.

Definition 1. *([7]) A number sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_{k\to\infty} x_k = l$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = l$, then $st - \lim x_k = l$. The converse does not hold in general.

In literature, several interesting generalizations of statistical convergence have been appeared. One among these is λ-statistical convergence given by Mursaleen [23] with a non-decreasing sequence $\lambda = (\lambda_n)$ of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

The idea of λ-statistical convergence can be connected to the generalized de la Vallée-Poussin mean. It is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k\in I_n} x_k$$

2000 *Mathematics Subject Classification.* 40A05, 40A35.

Key words and phrases. λ-convergence; 2-norm; 2-normed space.

Submitted September 22, 2018. Published October 10, 2018.
Communicated by F. Basar.
We thank the editor and referees for their careful reading, valuable suggestions and remarks.

where $I_n = [n - \lambda_n + 1, n]$.

Definition 2. *([23]) A sequence $x = (x_k)$ of numbers is said to be λ -statistical convergent to a number l provided that for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - l| \geq \varepsilon\}| = 0.$$

In this case, the number l is called λ-statistical limit of the sequence $x = (x_k)$ and we write $S_\lambda - \lim_{k\to\infty} x_k = l$.

Recently, for $\alpha \in (0, 1]$ Çolak and Bektaş [2] generalized *Definition 2* in terms of λ-statistical convergence of order α and obtained some analogous results.

The concept of probabilistic normed spaces was initially introduced by A. N. Sherstnev [33] in 1962. Menger [22] introduced the notion of probabilistic metric spaces in 1942. The idea of Menger [22] was to use distribution function instead of non-negative real values of a metric. In last few years these spaces are grown up rapidly and many detereministic results of linear normed spaces are obtained for probabilistic normed spaces. For a detailed study on probabilistic functional analysis, we refer ([1], [17], [26], [32]). In 2005, Golet [16] used the concept of 2-norm of Gähler [15] and presented generalized probabilistic normed space which he called random 2-normed space. Gürdal and Pehlivan ([37], [38]) studied statistical convergence in 2-normed spaces and in 2-Banach spaces. Recently, Savaş [39] defined and studied generalized statistical convergence in random 2-normed space. Esi and Özdemir [6] introduced and studied the concept of generalized Δ^m-statistical convergence of sequences in probabilistic normed space. Esi [5], defined and studied the notion of ∇-statistical convergence and ∇-statistical Cauchy sequences using by λ-sequences in random 2-normed spaces, and proved some theorems.

The existing literature on statistical convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed [40] and intutionistic fuzzy normed spaces [18], [20], [27] and [28]. Several authors studied on the sets of fuzzy valued sequences and the characterization of the classes of related matrix transformations ([8], [9], [10], [11], [12]).

Let $\mathbb{R}$ denotes the set of reals and $\mathbb{R}_0^+ = [0, \infty)$. A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. We will denote the set of all distribution functions by $\mathcal{D}$. Also, a distance distribution function is a non decreasing function $\mathcal{F}$ defined on $\mathbb{R}^+ = [0, \infty)$ that satisfies $\mathcal{F}(0) = 0$ and $\mathcal{F}(\infty) = 1$; and is left continuous on $(0, \infty)$. Let $\mathcal{D}^+$ denotes the set of all distance distribution functions.

A triangular norm, briefly t-norm, is a binary operation $*$ on $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c \in [0, 1]$:

(i) $a * 1 = a$,
(ii) $a * b = b * a$,
(iii) $c * d \geq a * b$ if $c \geq a$ and $d \geq b$,
(iv) $(a * b) * c = a * (b * c)$.

The $*$ operations $a * b = \max\{a + b - 1, 0\}$, $a * b = ab$, and $a * b = \min\{a, b\}$ on $[0, 1]$ are t-norms.

In following, we give some useful definitions.

Definition 3. *([15]) Let X be a real vector space of dimension $d>1$ (d may be infinite). A real valued function $\|.,.\|: X^2 \to \mathbb{R}$ satisfying the following conditions:*

(i) $\|x_1,x_2\|=0$, if and only if x_1,x_2 are linearly dependent.
(ii) $\|x_1,x_2\|=\|x_2,x_1\|$ for all $x_1,x_2\in X$,
(iii) $\|\alpha x_1,x_2\|=|\alpha|\,\|x_1,x_2\|$, for any $\alpha\in\mathbb{R}$ and
(iv) $\|x_1+x_2,x_3\|\leq\|x_1,x_3\|+\|x_2,x_3\|$
is called a 2-norm and the pair $(X,\|.,.\|)$ is called a 2-normed space.

Definition 4. *([16]) Let X be a real vector space of dimension $d>1$ (d may be infinite),τ be a triangle function(a binary operation on $\mathcal{D}^+$ which is associative, commutative, nondecreasing and ε_0 as a unit) and $\mathcal{F}: X\times X\to\mathcal{D}^+$ (for $x,y\in X$, $\mathcal{F}(x,y;t)$ is the value of $\mathcal{F}(x,y)$ at $t\in\mathbb{R}$). Then $\mathcal{F}$ is called a probabilistic norm $(X,\mathcal{F},\tau)$ a probabilistic 2-normed space if the following conditions are satisfied:*

(i) $\mathcal{F}(x,y;t)=H_0(t)$, if x,y are linearly dependent, where $H_0(t)=0$ if $t\leq 0$ and $H_0(t)=1$ if $t>0$.
(ii) $\mathcal{F}(x,y;t)\neq H_0(t)$, if x,y are linearly dependent.
(iii) $\mathcal{F}(x,y;t)=\mathcal{F}(y,x;t)$, for all $x,y\in X$,
(iv) $\mathcal{F}(\alpha x,y;t)=\mathcal{F}\left(x,y;\frac{t}{|\alpha|}\right)$ for every $t>0$, $\alpha\neq 0$ and $x,y\in X$,
(v) $\mathcal{F}(x+y,z;t)\geq\tau(\mathcal{F}(x,z;t),\mathcal{F}(y,z;t))$, where $x,y,z\in X$.

If (v) is replaced by $\mathcal{F}(x+y,z;t_1+t_2)\geq\mathcal{F}(x,z;t_1)*\mathcal{F}(y,z;t_2)$ for all $x,y,z\in X$ and $t_1,t_2\in\mathbb{R}_0^+$ then $(X,\mathcal{F},*)$ is called a random 2-normed space.

Example 1. *Let $(X,\|.,.\|)$ be a 2-normed space with $\|x,z\|=|x_1z_2-x_2z_1|$; $x=(x_1,x_2)$, $z=(z_1,z_2)$ and $a*b=ab$ for all $a,b\in[0,1]$. For every $x,y\in X$ and $t>0$ we define $\mathcal{F}(x,y;t)=\frac{t}{t+\|x,y\|}$, then $(X,\mathcal{F},*)$ is a random 2-normed space.*

Definition 5. *([24]) Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x=(x_k)$ is said to be convergent to $x_0\in X$ with respect to norm $\mathcal{F}$ if for every $\varepsilon>0$, $t\in(0,1)$ and non-zero $z\in X$, there exists a positive integer k_0 such that $\mathcal{F}(x_k-x_0,z;\varepsilon)>1-t$ whenever $k\geq k_0$. It is denoted by $\mathcal{F}$-$\lim x_k=x_0$.*

Definition 6. *([24]) Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x=(x_k)$ is said to be statistically convergent S^{R2N} convergent to $x_0\in X$ with respect to norm $\mathcal{F}$ if for every $\varepsilon>0$, $t\in(0,1)$ and non-zero $z\in X$,*

$$\delta(\{k\in\mathbb{N}:\mathcal{F}(x_k-x_0,z;\varepsilon)\leq 1-t\})=0.$$

In this case, we write S^{R2N}-$\lim x_k=x_0$.

Definition 7. *([4]) Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x=(x_k)$ is said to be statistically convergent to l with respect to $\mathcal{F}$ if for every $\varepsilon>0$, $t\in(0,1)$ and non-zero $z\in X$,*

$$\lim_{n\to\infty}\frac{1}{n}\left|\left(\{k\leq n:\mathcal{F}(x_k-l,z;\varepsilon)\leq 1-t\}\right)\right|=0.$$

In this case, we write S^{R2N}-$\lim x_k=l$.

Kişi [19] has recently defined the ∇_2-statistical convergence of double sequences in random 2-normed spaces.

Throughout the paper, we consider a random 2-normed space $(X,\ \mathcal{F},*)$ and $\overline{\lambda}_{r,s}=\lambda_r\mu_s$ be the collection of such sequences $\overline{\lambda}$ will be denoted by Δ_2.

Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two non-decreasing sequences of positive real numbers, each tending to ∞ and such that $\lambda_{r+1} \leq \lambda_r + 1$, $\lambda_1 = 1$; $\mu_{s+1} \leq \mu_s + 1$, $\mu_1 = 1$. Let $I_r = [r - \lambda_r + 1, r]$, $I_s = [s - \mu_s + 1, s]$ and $I_{r,s} = I_r \times I_s$.

For any set $X \subseteq \mathbb{N} \times \mathbb{N}$, the number,

$$\delta_{\overline{\lambda}}(X) = P\text{-}\lim_{r,s\to\infty} \frac{1}{\overline{\lambda}_{r,s}} \left|\{(k,l) \in I_r \times I_s : (k,l) \in X\}\right|;$$

is said to be $\overline{\lambda}$-density of the set X, provided the limit exists, where $\overline{\lambda}_{r,s} = \lambda_r \mu_s$.

2. Main results

In this section, we define ∇_2-statistical convergent double sequence of order $\alpha (0 < \alpha \leq 1)$ in random 2-normed space $(X, \mathcal{F}, *)$. Also, we obtain some basic properties of this notion in random 2-normed space.

Definition 8. *A double sequence $x = (x_{kl})$ in random 2-normed space $(X, \mathcal{F}, *)$ is said to be ∇_2-convergent to $l \in X$ of order α with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $t \in (0,1)$ and for non-zero $z \in X$, there exists an positive integer n_0 such that $\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \varepsilon\right) > 1 - t$ whenewer $k, l \geq n_0$. In this case we write $\mathcal{F}^{\alpha}_{\nabla_2} - \lim_{k,l\to\infty} x_{kl} = l$, and l is called the $\mathcal{F}^{\alpha}_{\nabla_2}$−limit of $x = (x_{kl})$.*

Definition 9. *A double sequence $x = (x_{kl})$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be ∇_2-Cauchy of order α with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $t \in (0,1)$ and for non-zero $z \in X$, there exists positive integers p, q such that*

$$\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} (x_{kl} - x_{mn}), z; \varepsilon\right) < 1 - t,$$

whenever $k, m > p$, $l, n > q$.

Definition 10. *A double sequence $x = (x_{kl})$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be ∇_2-statistical convergent or S_{∇_2}-convergent to l of order $\alpha (0 < \alpha \leq 1)$ with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $t \in (0,1)$ andfor non-zero $z \in X$ such that*

$$\delta_{\nabla_2}\left(\left\{(k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \varepsilon\right) \leq 1 - t\right\}\right) = 0.$$

In other ways we can write

$$\left|\left\{(k,l) \in I_{r,s} : \mathcal{F}\left(\lim_{r,s\to\infty} \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \varepsilon\right) \leq 1 - t\right\}\right| = 0,$$

or, equivalently,

$$\delta_{\nabla_2}\left(\left\{(k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \varepsilon\right) > 1 - t\right\}\right) = 1,$$

i.e.,

$$S^{\alpha}_{\nabla_2} - \lim_{r,s\to\infty} \mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \varepsilon\right) = 1.$$

In this case, we write $S^{\alpha}_{\nabla_2}(R2N)-\lim_{k,l} x_{kl}=l$ or $x_{kl}\to l\left(S^{\alpha}_{\nabla_2}(R2N)\right)$ and

$$S^{\alpha}_{\nabla_2}(R2N)(X)=\left\{x=(x_{kl}):\exists l\in\mathbb{R}, S^{\alpha}_{\nabla_2}(R2N)-\lim_{k,l} x_{kl}=l\right\}.$$

The collection of all ∇_2-statistically convergent double sequences of order α in random 2-normed space is symbolized as $S^{\alpha}_{\nabla_2}(R2N)(X)$.

Definition 11. *A double sequence $x=(x_{kl})$ in a random 2-normed space $(X,\mathcal{F},*)$ is said to be ∇_2 -statistically Cauchy of order α with respect to $\mathcal{F}$ if for every $\varepsilon>0$, $t\in(0,1)$ and for non-zero $z\in X$, there exist positive integers p,q such that for all $k,m>p$, $l,n>q$*

$$\delta_{\nabla_2}\left(\left\{(k,l)\in I_{r,s}:\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}}\sum_{(k,l)\in I_{r,s}}(x_{kl}-x_{mn}),z;\varepsilon\right)\le 1-t\right\}\right)=0,$$

or, equivalently,

$$\delta_{\nabla_2}\left(\left\{(k,l)\in I_{r,s}:\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}}\sum_{(k,l)\in I_{r,s}}(x_{kl}-x_{mn}),z;\varepsilon\right)> 1-t\right\}\right)=1.$$

Definition 11, immediately implies the following Lemma.

Lemma 1. *Let $(X,\mathcal{F},*)$ be a random 2-normed space. If $x=(x_{kl})$ is a double sequence in X, then for every $\varepsilon>0$, $t\in(0,1)$ and for non-zero $z\in X$, the following statetements are equivalent.*

(*i*) $S^{\alpha}_{\nabla_2}-\lim_{k,l} x_{kl}=l$.

(*ii*) $\delta_{\nabla_2}\left(\left\{(k,l)\in I_{r,s}:\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}}\sum_{(k,l)\in I_{r,s}}x_{kl}-l,z;\varepsilon\right)\le 1-t\right\}\right)=0.$

(*iii*) $\delta_{\nabla_2}\left(\left\{(k,l)\in I_{r,s}:\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}}\sum_{(k,l)\in I_{r,s}}x_{kl}-l,z;\varepsilon\right)> 1-t\right\}\right)=1.$

(*iv*) $S^{\alpha}_{\nabla_2}-\lim_{k,l\to\infty}\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}}\sum_{(k,l)\in I_{r,s}}x_{kl}-l,z;\varepsilon\right)=1.$

Theorem 2. *Let $(X,\mathcal{F},*)$ be a random 2-normed space and $\alpha\in(0,1]$ be given. If $x=(x_{kl})$ is a double sequence in X such that $S^{\alpha}_{\nabla_2}(R2N)-\lim_{k,l} x_{kl}=l$ exists, then it is unique.*

Proof. Suppose that $S^{\alpha}_{\nabla_2}(R2N)-\lim_{k,l} x_{kl}=l'$, where $l\ne l'$. Let $\varepsilon>0$ be given. Choose $\nu>0$ such that

$$(1-\nu)*(1-\nu)>1-\varepsilon. \tag{1}$$

Then, for any $t>0$ and for non-zero $z\in X$, we define

$$K_1(\upsilon,t)=\left\{(k,l)\in I_{r,s}:\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}}\sum_{(k,l)\in I_{r,s}}(x_{kl}-l),z;\frac{t}{2}\right)\le 1-\nu\right\};$$

$$K_2(\upsilon,t)=\left\{(k,l)\in I_{r,s}:\mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}}\sum_{(k,l)\in I_{r,s}}(x_{kl}-l'),z;\frac{t}{2}\right)\le 1-\nu\right\}.$$

Since

$$S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} x_{kl} = l \text{ and } S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} x_{kl} = l',$$

we have

$$\delta_{\nabla_2}(K_1(\upsilon,t)) = 0 \text{ and } \delta_{\nabla_2}(K_2(\upsilon,t)) = 0 \text{ for all } t > 0.$$

Let $K(\upsilon,t) = K_1(\upsilon,t) \cup K_2(\upsilon,t)$, then it is easy to observe that $\delta_{\nabla_2}(K(\upsilon,t)) = 0$ which immediately implies $\delta_{\nabla_2}(K^c(\upsilon,t)) = 1$. Let $k \in K^c(\upsilon,t) = K_1^c(\upsilon,t) \cap K_2^c(\upsilon,t)$. Now one can write,

$$\begin{aligned} \mathcal{F}(l-l',z;t) &\geq \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \frac{t}{2}\right) * \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l', z; \frac{t}{2}\right) \\ &> (1-\nu) * (1-\nu). \end{aligned}$$

It follows by (1) that

$$\mathcal{F}(l-l',z;t) > (1-\varepsilon).$$

Since ε is arbitrary, it follows that $\mathcal{F}(l-l',z;t) = 1$, for all $t > 0$ and non-zero $z \in X$. This shows that $l = l'$. ∎

Theorem 3. *Let $(X,\mathcal{F},*)$ be a random 2-normed space and $\alpha \in (0,1]$ be given. Let $x = (x_{kl})$ and $y = (y_{kl})$ be two double sequences in X.*

(*i*) If $S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} x_{kl} = l$ and $0 \neq c \in \mathbb{R}$, then $S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} cx_{kl} = cl$.

(*ii*) If $S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} x_{kl} = l$ and $S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} y_{kl} = l'$, then $S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l}(x_{kl} + y_{kl}) = l + l'$.

Proof. The proof of the theorem is not so hard so is omitted here. ∎

Theorem 4. *Let $(X,\mathcal{F},*)$ be a random 2-normed space and $\alpha \in (0,1]$ be given. If $x = (x_{kl})$ be a double sequence in X such that $\mathcal{F}_{\nabla_2}^{\alpha} - \lim_{k,l\to\infty} x_{kl} = l$, then $S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} x_{kl} = l$. Hovewer the converse need not be true in general.*

Proof. Since $\mathcal{F}_{\nabla_2}^{\alpha} - \lim_{k,l\to\infty} x_{kl} = l$, for every $\varepsilon > 0$, $t > 0$ and for non-zero $z \in X$ there is a positive integer n_0 such that

$$\mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t\right) > 1-\varepsilon, \ \forall k,l > n_0.$$

The set

$$K(\varepsilon,t) = \left\{(k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t\right) \leq 1-\varepsilon\right\}$$

has at most finitely many terms. Also, since every finite subset of $\mathbb{N}$ has δ_{∇_2}-density zero, consequently we have $S_{\nabla_2}^{\alpha}(K(\varepsilon,t)) = 0$. This shows that $S_{\nabla_2}^{\alpha}(R2N) - \lim_{k,l} x_{kl} = l$. We next give the following example which shows that the converse need not be true. ∎

Example 2. *Let $X = \mathbb{R}^2$ with the 2-norm $\|x,z\| = \|x_1z_2 - x_2z_1\|$ where $x = (x_1,x_2)$, $z = (z_1,z_2)$ and $a * b = ab$ for all $a,b \in [0,1]$. Let $\mathcal{F}(x_{kl},z,t) = \frac{t}{t+\|x,z\|}$,* *where each $t > 0$, non-zero $z \in X$, $z_2 > 0$. We define a sequence $x = (x_{kl})$ as follows:*

$$\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} = \begin{cases} (k,l), & \text{if } n-\sqrt{\lambda_n}+1 \le k \le n,\ \ m-\sqrt{\mu_m}+1 \le l \le m, \\ (0,0), & \text{otherwise.} \end{cases}$$

Now for $\varepsilon > 0$, $t \in (0,1)$, write

$$K(\varepsilon,t) = \left\{(k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t\right) \le 1-\varepsilon\right\},$$

where $l = (0,0)$. Then

$$K(\varepsilon,t) = \left\{(k,l) \in I_{r,s} : \frac{t}{t+\left|\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl}\right|} \le 1-\varepsilon\right\}, \theta = (0,0)$$

$$= \left\{(k,l) \in I_{r,s} : \left|\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl}\right| \ge \frac{t\varepsilon}{1-\varepsilon} > 0\right\}$$

$$= \{(k,l) \in I_{r,s} : x_{kl} = (k,l)\}$$

$$= \left\{(k,l) \in I_{r,s} : n-\sqrt{\lambda_n}+1 \le k \le n,\ \ m-\sqrt{\mu_m}+1 \le l \le m\right\},$$

so we get

$$\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} |K(\varepsilon,t)| \le \frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \left|\left\{(k,l) \in I_{r,s} : r-\sqrt{\lambda_r}+1 \le k \le r,\ \ s-\sqrt{\mu_s}+1 \le l \le s\right\}\right| \le \frac{\sqrt{\lambda_{rs}}}{\overline{\lambda}_{r,s}^{\alpha}}$$

Letting limit, r, s as ∞, we get

$$\delta_{\nabla_2}(K(\varepsilon,t)) = \lim_{r,s\to\infty} \frac{1}{\overline{\lambda}_{r,s}^{\alpha}} |K(\varepsilon,t)| \le \lim_{r,s\to\infty} \frac{\sqrt{\lambda_{rs}}}{\overline{\lambda}_{r,s}^{\alpha}} = 0.$$

This shows that $x_{kl} \to 0 \left(S_{\nabla_2}^{\alpha}(R2N)(X)\right)$.

On the other hand the sequence is not $\mathcal{F}_{\nabla_2}^{\alpha}-$convergent to zero as

$$\begin{aligned} \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t\right) &= \frac{t}{t+\left|\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl}\right|} \\ &= \begin{cases} \frac{t}{t+(k+l)z_2}, & \text{if } n-\sqrt{\lambda_n}+1 \le k \le n, \\ & m-\sqrt{\mu_m}+1 \le l \le m, \\ 1, & \text{otherwise.} \end{cases} \\ &\le 1. \end{aligned}$$

Theorem 5. *Let $(X, \mathcal{F}, *)$ be a R2N space and $0 < \alpha \le \beta \le 1$, then $S_{\nabla_2}^{\alpha}(X) \subset S_{\nabla_2}^{\beta}(X)$ and the inclusion is strict for some α and β such that $\alpha < \beta$.*

Proof. If $0 < \alpha \leq \beta \leq 1$, then for every $\varepsilon > 0$ and $t > 0$ and non-zero $z \in X$, we have

$$\frac{1}{\overline{\lambda}^{\beta}_{r,s}} \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) \leq 1-\varepsilon \right\}$$
$$\leq \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) \leq 1-\varepsilon \right\}$$

which immediately implies the inclusion $S^{\alpha}_{\nabla_2}(X) \subset S^{\beta}_{\nabla_2}(X)$. We next give an example that shows the inclusion in $S^{\alpha}_{\nabla_2}(X) \subset S^{\beta}_{\nabla_2}(X)$ is strict for some α and β with $\alpha < \beta$. ∎

Example 3. *Let $X = \mathbb{R}^2$ with the 2-norm $\|x,z\| = \|x_1 z_2 - x_2 z_1\|$ where $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for all $a, b \in [0,1]$. Let $\mathcal{F}(x_{kl}, z, t) = \frac{t}{t+\|x,z\|}$, where each $t > 0$, non-zero $z \in X$, $z_2 > 0$. We define a sequence $x = (x_{kl})$ as follows:*

$$\sum_{(k,l)\in I_{r,s}} x_{kl} = \begin{cases} (1,0), & \text{if } k+l \text{ is even,} \\ (0,0), & \text{if } k+l \text{ is odd.} \end{cases}$$

For $\varepsilon > 0$, $t \in (0,1)$, if we define

$$K(\varepsilon,t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - \theta, z; t \right) \leq 1-\varepsilon \right\}, \theta = (0,0)$$

$$= \left\{ (k,l) \in I_{r,s} : \frac{t}{t + \left\| \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - \theta, z \right\|} \leq 1-\varepsilon \right\}$$

$$= \left\{ (k,l) \in I_{r,s} : \left\| \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - \theta, z \right\| \geq \frac{\varepsilon t}{1-\varepsilon} > 0 \right\}$$

$$= \{(k,l) \in I_{r,s} : (x_{kl}) = (1,0)\} = \{(k,l) \in I_{r,s} : k+l \text{ is even}\};$$

then,

$$\lim_{r,s\to\infty} \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} |K(\varepsilon,t)| = \lim_{r,s\to\infty} \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} |\{(k,l) \in I_{r,s} : k+l \text{ is even}\}| \leq \lim_{r,s\to\infty} \frac{\sqrt{\overline{\lambda}_{r,s}}+1}{2\overline{\lambda}^{\alpha}_{r,s}} = 0$$

for $\alpha > 1$. Similarly, for $\varepsilon > 0$, $t \in (0,1)$, if we define

$$B(\varepsilon,t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}^{\alpha}_{r,s}} \sum_{(k,l)\in I_{r,s}} x_{kl} - x_0, z; t \right) \leq 1-\varepsilon \right\}, \ x_0 = (1,0)$$

then

$$\lim_{r,s\to\infty} \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} |B(\varepsilon,t)| = \lim_{r,s\to\infty} \frac{1}{\overline{\lambda}^{\alpha}_{r,s}} |\{(k,l) \in I_{r,s} : k+l \text{ is odd}\}| \leq \lim_{r,s\to\infty} \frac{\sqrt{\overline{\lambda}_{r,s}}+1}{2\overline{\lambda}^{\alpha}_{r,s}} = 0$$

for $\alpha > 1$. This shows that $S^{\alpha}_{\nabla_2} - \lim_{k,l} x_{kl}$ is not unique and we obtain a contradiction to theorem 1.

Example 4. *Let $(\mathbb{R}^2, \mathcal{F}, *)$ be a R2N space as defined above. We define a sequence $x = (x_{kl})$ as follows:*

$$\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} = \begin{cases} (1,0), & \text{if } r - \sqrt{\lambda_r} + 1 \le k \le r, \quad s - \sqrt{\lambda_s} + 1 \le l \le s, \\ (0,0), & \text{otherwise.} \end{cases}$$

Then one can easily see $S_{\nabla_2}^{\beta} - \lim_{k,l} x_{kl} = 0$, i.e., $x \in S_{\nabla_2}^{\beta}$ for $\frac{1}{2} < \beta \le 1$ but $x \notin S_{\nabla_2}^{\alpha}(X)$ for $0 < \alpha \le \frac{1}{2}$. This shows that the inclusion in $S_{\nabla_2}^{\alpha}(X) \subset S_{\nabla_2}^{\beta}(X)$ is strict.

Theorem 6. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $\alpha \in (0,1]$ be given. If $x = (x_{kl})$ be a sequence in X, then $S_{\nabla_2}^{\alpha} - \lim_{k,l} x_{kl} = l$ if and only if there exists a subset $K = \{k_m : k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\lim_{r,s\to\infty} \frac{1}{\overline{\lambda}_{r,s}^{\alpha}} |K| = 1$ and $\mathcal{F}_{\nabla_2}^{\alpha} - \lim_{k,l\to\infty} x_{kl} = l$.*

Proof. First suppose that $S_{\nabla_2}^{\alpha} - \lim_{k,l} x_{kl} = l$. For $t > 0$ and non-zero $z \in X$ and $s \in \mathbb{N}$, we define

$$A(s,t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) > 1 - \frac{1}{s} \right\}$$

$$K(s,t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) \le 1 - \frac{1}{s} \right\}$$

Since $S_{\nabla_2}^{\alpha} - \lim_{k,l} x_{kl} = l$ it follows that

$$\delta_{\nabla_2}(K(s,t)) = 0$$

Also, for $s = 1, 2, 3, ...$ and for $t > 0$, we observe that

$$A(s,t) \supset A(s+1,t)$$

and

$$\lim_{r,s\to\infty} \frac{1}{\overline{\lambda}_{r,s}^{\alpha}} |A(s,t)| = 1; \text{ i.e., } \delta_{\nabla_2}(A(s,t)) = 1. \tag{2}$$

Now, to prove the result, it is sufficient to prove that $\mathcal{F}_{\nabla_2}^{\alpha} - \lim_{k,l\to\infty} x_{kl} = l$. Suppose that for $k \in A(s,t)$, $x = (x_{kl})$ does not convergent to l with respect to $\mathcal{F}_{\nabla_2}^{\alpha}$. Then, there exists some $u > 0$ such that

$$\left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) \le 1 - u \right\}$$

for infinitely many terms (x_{kl}). Let

$$A(u,t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) > 1 - u \right\}$$

and $u > \frac{1}{s}$ for $s = 1, 2, 3....$ This implies that $\delta_{\nabla_2}(A(s,t)) = 0$, which contradicts (2) as $\delta_{\nabla_2}(A(s,t)) = 1$. Hence $\mathcal{F}_{\nabla_2}^{\alpha} - \lim_{k,l\to\infty} x_{kl} = l$.

Conversely, suppose that there exists a subset

$$K = \{k_m : k_1 < k_2 < ...\}$$

of $\mathbb{N}$ such that $\lim_{r,s\to\infty} \frac{1}{\overline{\lambda}_{r,s}^{\alpha}} |K| = 1$ and $\mathcal{F}^{\alpha}_{\nabla_2} - \lim_{k,l\to\infty} x_{kl} = l$. Then for every $\varepsilon > 0$ and $t > 0$ and non-zero $z \in X$, there exists a positive integer n_0 such that

$$\left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) > 1 - \varepsilon \right\}$$

for all $k, l > n_0$. If we take

$$K(\varepsilon, t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t \right) \leq 1 - \varepsilon \right\}$$

then it is easy to see that

$$K(\varepsilon, t) \subset \mathbb{N} \times \mathbb{N} - \{n_0 + 1, n_0 + 2, ...\}$$

and consequently

$$\delta_{\nabla_2}(K(\varepsilon, t)) \leq 1 - 1 = 0.$$

Hence, $S^{\alpha}_{\nabla_2} - \lim_{k,l} x_{kl} = l$. ∎

Finally, we establish the Cauchy convergence criteria of double sequences of order α in random 2-normed spaces.

Theorem 7. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $\alpha \in (0,1]$ be given. A double sequence $x = (x_{kl})$ is said to be ∇_2-statistical convergent of order α if and only if it is ∇_2-statistical Cauchy of order α.*

Proof. Let $x = (x_{kl})$ be ∇_2-statistical convergent sequence of order α. Suppose that $S^{\alpha}_{\nabla_2} - \lim_{k,l} x_{kl} = l$. For $\varepsilon > 0$, $t > 0$ and non-zero $z \in X$ choose $s > 0$ such that $(1 - s) * (1 - s) > 1 - \varepsilon$. We define

$$A(s, t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \frac{t}{2} \right) \leq (1 - s) \right\};$$

then

$$A^c(s, t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \frac{t}{2} \right) > (1 - s) \right\}.$$

Since $S^{\alpha}_{\nabla_2} - \lim_{k,l} x_{kl} = l$ it follows that $\delta_{\nabla_2}(A(s, t)) = 0$ and consequently $\delta_{\nabla_2}(A^c(s, t)) = 1$. Let $(u, \gamma) \in A^c(s, t)$, then

$$\mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{u\gamma} - l, z; \frac{t}{2} \right) > (1 - s).$$

If we take

$$B(\varepsilon, t) = \left\{ (k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}} \sum_{(k,l)\in I_{r,s}} x_{kl} - x_{u\gamma}, z; t \right) \leq (1 - \varepsilon) \right\},$$

then to prove the first part it is sufficient to prove that $B(\varepsilon,t) \subset A(s,t)$. Let $(k,l) \in B(\varepsilon,t)$, which gives

$$\mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{kl} - x_{u\gamma}, z; t\right) \leq (1-\varepsilon).$$

Suppose $(k,l) \notin A(s,t)$, then

$$\mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; t\right) > (1-s),$$

Also it can be easily seen that

$$\begin{aligned} 1-\varepsilon &\geq \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{kl} - x_{u\gamma}, z; t\right) \\ &\geq \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \frac{t}{2}\right) * \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{u\gamma} - l, z; \frac{t}{2}\right) \\ &\geq (1-s)*(1-s) > 1-\varepsilon. \end{aligned}$$

This contradiction shows that $B(\varepsilon,t) \subset A(s,t)$ and therefore, the theorem is proved.

Conversely, let $x=(x_{kl})$ is ∇_2 -statistical Cauchy double sequence of order α but not double ∇_2-statistical convergent of order α with respect to $\mathcal{F}$. Now

$$\begin{aligned} &\mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{kl} - x_{u\gamma}, z; t\right) \\ &\geq \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{kl} - l, z; \frac{t}{2}\right) * \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{u\gamma} - l, z; \frac{t}{2}\right) \\ &\geq (1-s)*(1-s) > 1-\varepsilon. \end{aligned}$$

since x is not double ∇_2-statistical convergent. Therefore $\delta_{\nabla_2}(B^c(t,\varepsilon)) = 0$, where

$$B(\varepsilon,t) = \left\{(k,l) \in I_{r,s} : \mathcal{F}\left(\frac{1}{\overline{\lambda}_{r,s}^{\alpha}}\sum_{(k,l)\in I_{r,s}} x_{kl} - x_{u\gamma}, z; t\right) \leq 1-\varepsilon\right\}.$$

and so $\delta_{\nabla_2}(B(t,\varepsilon)) = 1$, which is contradiction, since x is ∇_2-statistical Cauchy double sequence. Hence x must be ∇_2-statistical Cauchy. This completes the proof. ∎

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ÖMER KİŞİ, Faculty of Science, Depertmant of Mathematics, Bartin University, Bartin, Turkey
E-mail address: okisi@bartin.edu.tr