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WIJSMAN $\mathcal{I}$-INVARIANT CONVERGENCE OF SEQUENCES OF SETS

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Abstract. In this paper, we study the concepts of Wijsman $\mathcal{I}$-invariant convergence ($\mathcal{I}_\sigma^W$), Wijsman $\mathcal{I}^*$-invariant convergence ($\mathcal{I}_\sigma^{*W}$), Wijsman p-strongly invariant convergence ($[WV_\sigma]_p$) of sequences of sets and investigate the relationships between Wijsman invariant convergence, $[WV_\sigma]_p$, $\mathcal{I}_\sigma^W$ and $\mathcal{I}_\sigma^{*W}$. Also, we introduce the concepts of $\mathcal{I}_\sigma^W$-Cauchy sequence and $\mathcal{I}_\sigma^{*W}$-Cauchy sequence of sets.

1. Introduction

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11], Schoenberg [28] and studied by many authors. Nuray and Ruckle [20] indepedently introduced the same with another name generalized statistical convergence. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [13] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$.

Nuray and Rhoades [19] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [34] defined the Wijsman lacunary statistical convergence of set sequences and considered its relation with Wiijsman statistical convergence defined by Nuray and Rhoades. Kişi and Nuray [12] introduced a new convergence notion, for sequence of sets called Wijsman $\mathcal{I}$-convergence. The concept of convergence of sequence of numbers has been extended by several authors to convergence of set sequences (see, [4–6, 29, 33, 36, 37]).

Several authors including Raimi [26], Schaefer [27], Mursaleen [17], Savaş [30], Pancaroğlu and Nuray [24] and some authors have studied invariant convergent sequences. Nuray et al. [22] defined the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. The concept of strongly σ-convergence was defined by Mursaleen [16]. Savaş and Nuray [32]

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introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Recently, the concept of strong σ-convergence was generalized by Savaş [30]. Nuray and Ulusu [23] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers.

In this paper, we study the concepts of Wijsman $\mathcal{I}$-invariant convergence ($\mathcal{I}_\sigma^W$), Wijsman $\mathcal{I}^*$-invariant convergence (${\mathcal{I}^*}_\sigma^W$), Wijsman p-strongly invariant convergence ($[WV_\sigma]_p$) and investigate the relationships between Wijsman invariant convergence, $[WV_\sigma]_p$, $\mathcal{I}_\sigma^W$ and ${\mathcal{I}^*}_\sigma^W$. Also, we introduce the concepts of $\mathcal{I}_\sigma$-Cauchy sequence and $\mathcal{I}_\sigma^*$-Cauchy sequence of sets.

2. Definitions and Notations

Now, we recall the ideal convergence, invariant convergence, sequence of sets and basic definitions and concepts (See [1–3, 8–10, 13, 15, 19, 21–27, 35–37]).

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is called a filter if and only if

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$. If $x = (x_k)$ is $\mathcal{I}$-convergent to L, then we write $\mathcal{I} - \lim x = L$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$, and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [14].

It can be shown [31] that

$$V_\sigma = \Big\{x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n\Big\}.$$

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |x_{\sigma^k(m)} - L| = 0, \text{ uniformly in } m$$

and in this case, we write $x_k \to L[V_\sigma]$. By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences.
In the case, $\sigma(n) = n + 1$, the space $[V_\sigma]$ is reduced to the space $[\hat{c}]$ of strongly almost convergent sequences.

The concept of strong σ-convergence was generalized by Savaş [30] as below:

$$[V_\sigma]_p = \Big\{x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0 \text{ uniformly in n}\Big\},$$

where $0 < p < \infty$. If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.
A sequence $x = (x_k)$ is σ-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \Big| \big\{k \le m : |x_{\sigma^k(n)} - L| \ge \varepsilon\big\} \Big| = 0, \text{ uniformly in n.}$$

In this case, we write $S_\sigma - \lim x = L$ or $x_k \to L(S_\sigma)$.

Nuray et al. [22] introduced the concepts of σ-uniform density and $\mathcal{I}_\sigma$-convergence.

Let $A \subseteq \mathbb{N}$ and

$$s_n = \min_m \big| A \cap \big\{\sigma(m), \sigma^2(m), ..., \sigma^n(m)\big\} \big|$$

and

$$S_n = \max_m \big| A \cap \big\{\sigma(m), \sigma^2(m), ..., \sigma^n(m)\big\} \big|.$$

If the following limits exists

$$\underline{V}(A) = \lim_{n\to\infty} \frac{s_n}{n}, \qquad \overline{V}(A) = \lim_{n\to\infty} \frac{S_n}{n}$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

A sequence (x_k) is said to be $\mathcal{I}_\sigma$-convergent to the number L if for every $\varepsilon > 0$,

$$A_\varepsilon = \big\{k : |x_k - L| \ge \varepsilon\big\} \in \mathcal{I}_\sigma,$$

that is, $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim x_k = L$.

Throughout the paper, we suppose that (X, ρ) is a metric space, $\mathcal{I} \subset 2^{\mathbb{N}}$ is an admissible ideal and A, A_k are any non-empty closed subsets of X.

For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a\in A} \rho(x, a).$$

A sequence $\{A_k\}$ is Wijsman convergent to A if $\lim_{k\to\infty} d(x, A_k) = d(x, A)$, for each $x \in X$. In this case, we write $W - \lim A_k = A$.

A sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$, for each $x \in X$. L_∞ denotes the set of bounded sequences of sets.

A sequence $\{A_k\}$ is said to be Wijsman invariant convergent to A if for each $x \in X$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x, A_{\sigma^k(m)}) = d(x, A), \text{ uniformly in } m.$$

A sequence $\{A_k\}$ is said to be Wijsman strongly invariant convergent to A, if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x, A_{\sigma^k(m)}) - d(x, A)| = 0, \text{ uniformly in } m.$$

A sequence $\{A_k\}$ is said to be Wijsman invariant statistical convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}|\{0 \le k \le n : |d(x, A_{\sigma^k(m)}) - d(x, A)| \ge \varepsilon\}| = 0, \text{ uniformly in } m.$$

In this case, we write $A_k \to A(WS_\sigma)$ and the set of all Wijsman invariant statistical convergent sequences of sets will be denoted WS_σ.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-convergent to A if for every $\varepsilon > 0$ $A(\varepsilon, x) = \{k : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \in \mathcal{I}$.

Let (X, ρ) be a separable metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. A sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-convergent to A if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that for each $x \in X$, $\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A)$.

A sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-Cauchy sequence if for each $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon)$ such that $\{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \ge \varepsilon\} \in \mathcal{I}$.

A sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-Cauchy sequence if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that the subsequence $A_M = \{A_{m_k}\}$ is Wijsman Cauchy in X that is, $\lim_{k,p\to\infty} |d(x, A_{m_k}) - d(x, A_{m_p})| = 0$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{E_1, E_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{F_1, F_2, \cdots\}$ such that $E_j \Delta F_j$ is a finite set for $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}$.

3. Main Results

Definition 3.1. *A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant convergent or $\mathcal{I}_\sigma^W$-convergent to A if for every $\varepsilon > 0$, the set*

$$A(\varepsilon, x) = \{k : |d(x, A_k) - d(x, A)| \ge \varepsilon\}$$

belongs to $\mathcal{I}_\sigma$, that is, $V(A(\varepsilon, x)) = 0$. In this case, we write $A_k \to A(\mathcal{I}_\sigma^W)$ and denote the set of all Wijsman $\mathcal{I}$-invariant convergent sequences of sets by $\mathcal{I}_\sigma^W$.

Theorem 3.1. *Let $\{A_k\}$ be a bounded sequence. If $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A, then $\{A_k\}$ is Wijsman invariant convergent to A.*

Proof. Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. For each $x \in X$, we estimate

$$t(m,n,x) := \left| \frac{d(x, A_{\sigma(m)}) + d(x, A_{\sigma^2(m)}) + \cdots + d(x, A_{\sigma^n(m)})}{n} - d(x,A) \right|.$$

Then, for each $x \in X$ we have

$$t(m,n,x) \leq t^1(m,n,x) + t^2(m,n,x),$$

where

$$t^1(m,n,x) := \frac{1}{n} \sum_{\substack{j=1 \\ |d(x,A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon}}^{n} |d(x, A_{\sigma^j(m)}) - d(x,A)|$$

and

$$t^2(m,n,x) := \frac{1}{n} \sum_{\substack{j=1 \\ |d(x,A_{\sigma^j(m)}) - d(x,A)| < \varepsilon}}^{n} |d(x, A_{\sigma^j(m)}) - d(x,A)|.$$

Therefore, we have $t^2(m,n,x) < \varepsilon$, for each $x \in X$ and for every $m \in \mathbb{N}$. The boundedness of $\{A_k\}$ implies that there exist $L > 0$ such that for each $x \in X$,

$$|d(x, A_{\sigma^j(m)}) - d(x,A)| \leq L, \quad (j, m \in \mathbb{N}),$$

then this implies that

$$\begin{aligned} t^1(m,n,x) &\leq \frac{L}{n} \left| \{1 \leq j \leq n : |d(x, A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon\} \right| \\ &\leq L. \frac{\max_m \left| \{1 \leq j \leq n : |d(x, A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon\} \right|}{n} \\ &= L. \frac{S_n}{n}. \end{aligned}$$

Hence, $\{A_k\}$ is Wijsman invariant convergent to A. □

Definition 3.2. *Let (X, ρ) be a separable metric space. The sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-invariant convergent or $\mathcal{I}_\sigma^{*W}$-convergent to A if there exists a set $M = \{m_1 < \cdots < m_k < \cdots\} \in F(\mathcal{I}_\sigma)$ such that for each $x \in X$,*

$$\lim_{k \to \infty} d(x, A_{m_k}) = d(x, A).$$

Theorem 3.2. *If a sequence $\{A_k\}$ is $\mathcal{I}_\sigma^{*W}$-convergent to A, then this sequence is $\mathcal{I}_\sigma^W$-convergent to A.*

Proof. By assumption, there exists a set $H \in \mathcal{I}_\sigma$ such that for $M = \mathbb{N} \setminus H = \{m_1 < \cdots < m_k < \cdots\}$ we have

$$\lim_{k \to \infty} d(x, A_{m_k}) = d(x, A), \tag{3.1}$$

for each $x \in X$. Let $\varepsilon > 0$ by (3.1), there exists $k_0 \in \mathbb{N}$ such that for each $x \in X$,

$$|d(x, A_{m_k}) - d(x, A)| < \varepsilon,$$

for each $k > k_0$. Then, obviously

$$\{k \in \mathbb{N} : |d(x, A_k) - d(x,A)| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\}. \tag{3.2}$$

Since $\mathcal{I}_\sigma$ is admissible, the set on the right-hand side of (3.2) belongs to $\mathcal{I}_\sigma$. So $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A. □

Theorem 3.3. *Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal with the property (AP). If $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A, then $\{A_k\}$ is $\mathcal{I}_\sigma^{*W}$-convergent to A.*

Proof. Suppose that $\mathcal{I}_\sigma$ satisfies the property (AP). Let $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A. Then, for $\varepsilon > 0$ and for each $x \in X$

$$\{k : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_\sigma.$$

Put

$$E_1 = \{k : |d(x, A_k) - d(x, A)| \geq 1\} \ \text{and} \ E_n = \Big\{k : \frac{1}{n} \leq |d(x, A_k) - d(x, A)| < \frac{1}{n-1}\Big\},$$

for $n \geq 2$ and for each $x \in X$. Obviously $E_i \cap E_j = \emptyset$, for $i \neq j$. By the property (AP) there exists a sequence of $\{F_n\}_{n\in\mathbb{N}}$ such that $E_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and $F = (\bigcup_{j=1}^{\infty} F_j) \in \mathcal{I}_\sigma$. It is sufficient to prove that for $M = \mathbb{N} \setminus F$ and for each $x \in X$, we have

$$\lim_{k\to\infty} d(x, A_k) = d(x, A), \quad k \in M. \tag{3.3}$$

Let $\lambda > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \lambda$. Then, for each $x \in X$,

$$\{k : |d(x, A_k) - d(x, A)| \geq \lambda\} \subset \bigcup_{j=1}^{n+1} E_j.$$

Since $E_j \Delta F_j$, $j = 1, 2, \ldots, n+1$ are finite sets, there exists $k_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{j=1}^{n+1} F_j\Big) \cap \{k : k > k_0\} = \Big(\bigcup_{j=1}^{n+1} E_j\Big) \cap \{k : k > k_0\}. \tag{3.4}$$

If $k > k_0$ and $k \notin F$, then $k \notin \bigcup_{j=1}^{n+1} F_j$ and by (3.4) $k \notin \bigcup_{j=1}^{n+1} E_j$. But then

$$|d(x, A_k) - d(x, A)| < \frac{1}{n+1} < \lambda$$

so (3.3) holds and $\{A_k\}$ is $\mathcal{I}_\sigma^{*W}$-convergent to A. □

Now, we define the concepts of Wijsman $\mathcal{I}$-invariant Cauchy sequence and Wijsman $\mathcal{I}^*$-invariant Cauchy sequence of sets.

Definition 3.3. *A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant Cauchy sequence or $\mathcal{I}_\sigma^W$-Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon, x) \in \mathbb{N}$ such that*

$$A(\varepsilon, x) = \big\{k : |d(x, A_k) - d(x, A_N)| \geq \varepsilon\big\} \in \mathcal{I}_\sigma,$$

that is, $V\big(A(\varepsilon, x)\big) = 0$.

Definition 3.4. *A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}^*$-invariant Cauchy sequence or ${\mathcal{I}^*}_\sigma^W$-Cauchy sequence if there exists a set $M = \{m_1 < \cdots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that*

$$\lim_{k,p\to\infty} |d(x, A_{m_k}) - d(x, A_{m_p})| = 0,$$

for each $x \in X$.

We give following theorems which show relationships between $\mathcal{I}_\sigma^W$-convergence, $\mathcal{I}_\sigma^W$-Cauchy sequence and ${\mathcal{I}^*}_\sigma^W$-Cauchy sequence. Their proof are similar to the proof of Theorems in [7, 18], so we give them without proof.

Theorem 3.4. *If a sequence $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent, then $\{A_k\}$ is an $\mathcal{I}_\sigma^W$-Cauchy sequence.*

Theorem 3.5. *If a sequence $\{A_k\}$ is ${\mathcal{I}^*}_\sigma^W$-Cauchy sequence, then $\{A_k\}$ is $\mathcal{I}_\sigma^W$-Cauchy sequence.*

Theorem 3.6. *Let $\mathcal{I}_\sigma$ has the property (AP). Then the concepts $\mathcal{I}_\sigma^W$-Cauchy sequence and ${\mathcal{I}^*}_\sigma^W$-Cauchy sequence coincides.*

Definition 3.5. *The sequence $\{A_k\}$ is said to be Wijsman p-strongly invariant convergent to A, if for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{\sigma^k(m)}) - d(x, A)|^p = 0, \text{ uniformly in } m,$$

where $0 < p < \infty$. In this case, we write $A_k \to A[WV_\sigma]_p$ and denote the set of all Wijsman p-strongly invariant convergent sequences of sets by $[WV_\sigma]_p$.

Theorem 3.7. *Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal and $0 < p < \infty$.*

(i) *If $A_k \to A([WV_\sigma]_p)$, then $A_k \to A(\mathcal{I}_\sigma^W)$.*
(ii) *If $\{A_k\} \in L_\infty$ and $A_k \to A(\mathcal{I}_\sigma^W)$, then $A_k \to A([WV_\sigma]_p)$.*
(iii) *If $\{A_k\} \in L_\infty$, then $\{A_k\}$ is $\mathcal{I}_\sigma^W$-convergent to A if and only if $A_k \to A([WV_\sigma]_p)$.*

Proof. **(i)** If $A_k \to A([WV_\sigma]_p)$, then for $\varepsilon > 0$ and for each $x \in X$ we can write

$$\begin{aligned}\sum_{j=1}^{n} |d(x, A_{\sigma^j(m)}) - d(x, A)|^p &\geq \sum_{\substack{j=1 \\ |d(x,A_{\sigma^j(m)})-d(x,A)|\geq\varepsilon}}^{n} |d(x, A_{\sigma^j(m)}) - d(x, A)|^p \\ &\geq \varepsilon^p |\{j \leq n : |d(x, A_{\sigma^j(m)}) - d(x, A)| \geq \varepsilon\}| \\ &\geq \varepsilon^p \max_m |\{j \leq n : |d(x, A_{\sigma^j(m)}) - d(x, A)| \geq \varepsilon\}|\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n} |d(x, A_{\sigma^j(m)}) - d(x, A)|^p &\geq \varepsilon^p \cdot \frac{\max_m |\{1 \leq j \leq n : |d(x, A_{\sigma^j(m)}) - d(x, A)| \geq \varepsilon\}|}{n} \\ &= \varepsilon^p \frac{S_n}{n}\end{aligned}$$

for every $m \in \mathbb{N}$. This implies $\lim_{n\to\infty} \frac{S_n}{n} = 0$ and so $\{A_k\}$ is $(\mathcal{I}_\sigma^W)$-convergent to A.

(ii) Suppose that $\{A_k\} \in L_\infty$ and $A_k \to A(\mathcal{I}_\sigma^W)$. Let $\varepsilon > 0$. By assumption we have $V(A_\varepsilon) = 0$. Since $\{A_k\}$ is bounded, there exists $L > 0$ such that for each $x \in X$,

$$|d(x, A_{\sigma^j(m)}) - d(x, A)| \leq L,$$

for all j and m. Then, we have

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}|d(x,A_{\sigma^j(m)})-d(x,A)|^p &= \frac{1}{n}\sum_{\substack{j=1\\ |d(x,A_{\sigma^j(m)})-d(x,A)|\geq\varepsilon}}^{n}|d(x,A_{\sigma^j(m)})-d(x,A)|^p\\ &+ \frac{1}{n}\sum_{\substack{j=1\\ |d(x,A_{\sigma^j(m)})-d(x,A)|<\varepsilon}}^{n}|d(x,A_{\sigma^j(m)})-d(x,A)|^p\\ &\leq L.\frac{\max_m|\{1\leq j\leq n : |d(x,A_{\sigma^j(m)})-d(x,A)|\geq\varepsilon\}|}{n}+\varepsilon^p\\ &\leq L.\frac{S_n}{n}+\varepsilon^p,\end{aligned}$$

for each $x\in X$. Hence, for each $x\in X$ we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|d(x,A_{\sigma^j(m)})-d(x,A)|^p=0, \text{ uniformly in } m.$$

(iii) This is immediate consequence of Parts (i) and (ii). □

Now, we may state the theorem related to the relationships between WS_σ and $\mathcal{I}_\sigma^W$ without proof.

Theorem 3.8. *A sequence $\{A_k\}$ is WS_σ-convergent to A if and only if it is $\mathcal{I}_\sigma^W$-convergent to A.*

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References

[1] M. Arslan, E. Dündar, *On $\mathcal{I}$-Convergence of sequences of functions in 2-normed spaces,* Southeast Asian Bull. Math. **42** (2018), 491–502.
[2] M. Arslan, E. Dündar, *Rough convergence in 2-normed spaces*, Bull. Math. Anal. Appl. **10**(3) (2018), 1–9.
[3] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, e-books, Monographs, Istanbul, 2012.
[4] M. Baronti, and P. Papini, *Convergence of sequences of sets, In: Methods of functional analysis in approximation theory (pp. 133-155),* ISNM 76, Birkhauser, Basel (1986).
[5] G. Beer, *On convergence of closed sets in a metric space and distance functions,* Bull. Aust. Math. Soc. **31** (1985), 421–432.
[6] G. Beer, *Wijsman convergence: A survey,* Set-Valued Anal. **2** (1994), 77–94.
[7] K. Dems, *On $\mathcal{I}$-Cauchy sequences*, Real Anal. Exchange, **30** (2004/2005), 123–128.
[8] E. Dündar, B. Altay, *$\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences*, Acta Math. Sci. **34**B(2) (2014), 343–353.
[9] E. Dündar, B. Altay, *$\mathcal{I}_2$-uniform convergence of double sequences of functions,* Filomat, **30**(5) (2016), 1273–1281.
[10] E. Dündar, B. Altay, *Multipliers for bounded $\mathcal{I}_2$-convergent of double sequences*, Math. Comput. Modelling, **55**(3-4) (2012), 1193–1198.
[11] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
[12] Ö. Kişi, and F. Nuray, *A new convergence for sequences of sets*, Abstr. Appl. Anal. vol. 2013, Article ID 852796, 6 pages. http://dx.doi.org/10.-1155/2013/852796.

[13] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-Convergence*, Real Anal. Exchange, **26**(2) (2000), 669–686.

[14] G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[15] M. Mursaleen, O. H. H. Edely, *On the invariant mean and statistical convergence*, Appl. Math. Lett. **22** (2009), 1700–1704.

[16] M. Mursaleen, *Matrix transformation between some new sequence spaces*, Houston J. Math. **9** (1983), 505–509.

[17] M. Mursaleen, *On finite matrices and invariant means*, Indian J. Pure Appl. Math. **10** (1979), 457–460.

[18] A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathcal{I}$-Cauchy sequences*, Taiwanese J. Math. **11**(2) (2007), 569–5764.

[19] F. Nuray, B. E. Rhoades, *Statistical convergence of sequences of sets*, Fasc. Math. **49** (2012), 87–99.

[20] F. Nuray, W.H. Ruckle, *Generalized statistical convergence and convergence free spaces,* J. Math. Anal. Appl. **245** (2000), 513–527.

[21] F. Nuray, E. Savaş, I*nvariant statistical convergence and A-invariant statistical convergence,* Indian J. Pure Appl. Math. **10** (1994), 267–274.

[22] F. Nuray, H. Gök, U. Ulusu, *$\mathcal{I}_\sigma$-convergence,* Math. Commun. **16** (2011), 531–538.

[23] U. Ulusu, F. Nuray, *Lacunary $\mathcal{I}_\sigma$-convergence*, (Under Comminication).

[24] N. Pancaroğlu, F. Nuray, *Statistical lacunary invariant summability,* Theoret. Math. Appl. **3**(2) (2013), 71–78.

[25] N. Pancaroğlu, F. Nuray, *On Invariant Statistically Convergence and Lacunary Invariant Statistically Convergence of Sequences of Sets*, Prog. Appl. Math. **5**(2) (2013), 23–29.

[26] R. A. Raimi, *Invariant means and invariant matrix methods of summability,* Duke Math. J. **30** (1963), 81–94.

[27] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. **36** (1972), 104–110.

[28] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.

[29] Y. Sever, U. Ulusu and E. Dündar, *On Strongly $\mathcal{I}$ and $\mathcal{I}^*$-Lacunary Convergence of Sequences of Sets*, AIP Conference Proceedings, 1611, 357 (2014); doi: 10.1063/1.4893860, 7 pages.

[30] E. Savaş, *Some sequence spaces involving invariant means*, Indian J. Pure Appl. Math. **31** (1989), 1–8.

[31] E. Savaş, *Strong σ-convergent sequences*, Bull. Calcutta Math. Soc. **81** (1989), 295–300.

[32] E. Savaş, F. Nuray, *On σ-statistically convergence and lacunary σ-statistically convergence*, Math. Slovaca, **43**(3) (1993), 309–315.

[33] Ö. Talo, Y. Sever, F. Başar, *On statistically convergent sequences of closed set*, Filomat, **30**(6) (2016), 1497–1509.

[34] U. Ulusu, F. Nuray, *Lacunary statistical convergence of sequence of sets*, Prog. Appl. Math. **4**(2) (2012), 99–109.

[35] U. Ulusu and E. Dündar, *$\mathcal{I}$-Lacunary Statistical Convergence of Sequences of Sets*, Filomat, **28**(8) (2013), 1567–1574.

[36] R. A. Wijsman, *Convergence of sequences of convex sets, cones and functions*, Bull. Amer. Math. Soc. **70** (1964), 186–188.

[37] R. A. Wijsman, *Convergence of Sequences of Convex sets, Cones and Functions II*, Trans. Amer. Math. Soc. **123**(1) (1966), 32–45.

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