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KENMOTSU PSEUDO-METRIC FINSLER STRUCTURES

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Abstract. The objective of this article is to introduce (almost) Kenmotsu pseudo-metric Finsler structures and obtain some integrability(normality) conditions for these structures. Also, many significant results for the curvatures of indefinite Kenmotsu Finsler manifolds are acquired. Finally, Kenmotsu structures on indefinite Finsler manifolds are compared with Riemannian case.

1. Introduction

As well known, after the publication of Finsler's dissertation about surfaces and curves, a lot of articles have been dedicated to Finsler manifolds. Indefinite Kenmotsu manifolds have been investigated by Massamba [10],[11] and Aktan [1]. Also, Prasad and Pandey [15] have analyzed the topic in "An Indefinite Kenmotsu Manifold Endowed with Quarter Symmetric metric connection." Miron [13] presented a complicated approach about the research of Finsler geometry of vector bundles. Almost Kenmotsu pseudo-metric manifolds have been studied by Wang Y. Liu X. [19]. On the other hand, there are few papers dealing with the indefinite Finsler manifolds, see, for example, [3], [4], [8], [9]. To the best of our knowledge, especially, Kenmotsu structures on indefinite Finsler manifolds have not been studied before in the literature. So, in this paper we introduce (almost)Kenmotsu structures on indefinite Finsler manifolds and obtain some results for these structures.
The paper is put in order as following: after introduction, in second section, we give some preliminaries. Let M be a $(2n+1)$-dimensional manifold, then we define $F^{2n+1} = (M, M^0, F^*)$ indefinite Finsler manifold with fundamental function F^* on $M^0 = TM \setminus \theta(M)$ vector bundle, where F^* is described as $g_{ij} = \frac{1}{2}\frac{\partial^2 F^*}{\partial y^i \partial y^j}$ quadratic form g with index of q is a pseudo-Finsler metric. Besides, horizontal vector bundle $(TM^0)^H$ (non-linear connection) and vertical vector bundle $(TM^0)^V$ of F^{2n+1} are determined. Then, Finsler connection, tensor field, the operator of h-covariant derivation and v-covariant derivation and differential form on M^0 are given. Finally, Finsler curvatures are described.
In third section, we use distribution of $M^0 = (M^0)^h \oplus (M^0)^v$. By this way, we describe $(\phi^H, \xi^H, \eta^H, G^H)$ almost Kenmotsu pseudo-metric Finsler structure on

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$(M^0)^h$ horizontal vector bundle and $(\phi^V, \xi^V, \eta^V, G^V)$ almost Kenmotsu pseudo-metric Finsler structure on $(M^0)^v$ vertical Finsler vector bundle. Hence, $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are called almost Kenmotsu indefinite Finsler manifolds. Pseudo-Riemannian metric with index of $2q$ on M^0 is defined as below,

$$G = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*} \delta y^i \otimes \delta y^j = G^H + G^V$$

and it is named as Sasaki Finsler metric. g^{F^*} is a pseudo-Finsler metric with index of q of indefinite Finsler manifold. Shortly, g^{F^*} can be considered as a pseudo Riemannian metric on the Finsler vector bundles $(TM^0)^H$ and $(TM^0)^V$. Then, we give the normality conditions of $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ almost Kenmotsu pseudo-metric Finsler structures on $(M^0)^h$ and $(M^0)^v$ Finsler vector bundles, respectively. For $d\Omega = \eta \wedge \Omega$ second fundamental form, $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ Kenmotsu pseudo-metric Finsler structures are described as below:

$$(\nabla_X^H \phi) Y^H = \frac{1}{2}\{\varepsilon G^H(\phi X^H, Y^H)\xi^H - \eta^H(Y^H)\phi X^H\},$$

$$(\nabla_X^V \phi) Y^V = \frac{1}{2}\{\varepsilon G^V(\phi X^V, Y^V)\xi^V - \eta^V(Y^V)\phi X^V\}.$$

Then, $\nabla_X^H \xi^H = -\frac{1}{2}\phi^2 X^H$, $\nabla_X^V \xi^V = -\frac{1}{2}\phi^2 X^V$, $(\nabla_X^H \eta^H) Y^H = \Omega(\phi X^H, Y^H)$, $(\nabla_X^V \eta^V) Y^V = \Omega(\phi X^V, Y^V)$.
In fourth section,we find curvatures of indefinite Kenmotsu Finsler manifolds. Locally symmetric $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ indefinite Kenmotsu Finsler manifolds have constant curvatures $-\frac{\varepsilon}{4}$. If ξ^H and ξ^V are time-like, then the curvature is $\frac{1}{4}$. (If ξ^H and ξ^V are space-like, then the curvature is $-\frac{1}{4}$.) Finally, we find S^H horizontal Ricci tensor and S^V vertical Ricci tensor of indefinite Kenmotsu Finsler manifold. As a conclusion, we compare indefinite Kenmotsu structures with Riemannian case.

2. Some Preliminaries

We recall brief information about indefinite Finsler manifolds in this section.

2.1. **Indefinite Finsler Manifolds.** Let M be a real smooth manifold with $(2n+1)$- dimensional and TM be the tangent bundle of M. A coordinate system in M is referred by $\{(U, \varphi) : x^1, ..., x^{2n+1}\}$, where U is an open subset of M; for any $x \in U$ $\varphi : U \to \mathbb{R}^{2n+1}$ is a diffeomorphism of U to $\varphi(U)$ and $(x^1, ..., x^{2n+1}) = \varphi(x)$. Denote by π the canonical projection of TM on M and by T_xM the fibre at $x \in M$, $T_xM = \pi^{-1}(x)$. The coordinate system $\{(U, \varphi) : x^i\}$ in M describes a coordinate system $\{(U^*, \Phi) : x^1, ..., x^{2n+1}, y^1, ..., y^{2n+1}\} = \{(U^*, \Phi) : x^i, y^i\}$ in TM, where $U^* = \pi^{-1}(U)$ and for any $x \in U$ and $y_x \in T_xM$ $\Phi : U^* \to \mathbb{R}^{4n+2}$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^{2n+1}$ and $(x^1, ..., x^{2n+1}, y^1, ..., y^{2n+1}) = \Phi(y_x)$, [9]. Let M^0 be a nonempty open submanifold of TM such that $\pi(M^0) = M$ and $\theta(M) \cap M^0 = \emptyset$, where θ is the zero section of TM. Assume that for any $k > 0$ and $y \in M_x^0$,$M_x^0 = T_xM \cap M^0$ is a positive conic set, we have $k_y \in M_x^0$. Obviously, the largest M^0 holding the above circumstances is $TM \setminus \theta(M)$, ordinarily taken for the description of a Finsler manifold. Now, we deal a smooth function $F^* : M^0 \to \mathbb{R}$ and be $F^* = F^2$. Also, assume that $\{(U^0, \Phi^0) : x^i, y^i\}$ in M^0 the following conditions

are satisfied for any coordinate system.
$(F1^*)$ F^* is positively homogeneous of degree two with respect to $(y^1,...,y^{2n+1})$, we get,

$$F^*(x^1,...,x^{2n+1},ky^1,...,ky^{2n+1}) = k^2F^*(x^1,...,x^{2n+1},y^1,...,y^{2n+1}) \tag{2.1}$$

for all $(x,y)\in\Phi^0(U^0)$ and $k>0$.
$(F2^*)$ At all point $(x,y)\in\Phi^0(U^0)$,

$$g_{ij} = \frac{1}{2}\frac{\partial^2F^*}{\partial y^i\partial y^j}, i,j\in\{1,2,...,2n+1\} \tag{2.2}$$

are the components of a quadratic form on $\mathbb{R}^{2n+1}$ with $(2n+1)-q$ positive eigenvalues and q negative eigenvalues for $0<q<2n+1$. In this state, $F^{2n+1}=(M,M^0,F^*)$ is called an indefinite Finsler manifold of index q. Particularly, if $q=1$, F^{2n+1} is called a Finsler manifold with Lorentzian signature [9].

2.2. **Vectorial Finsler Connections and Curvatures.** Consider the structure of $F^{2n+1}=(M,M^0,F^*)$ an indefinite Finsler manifold of index q. Then, the tangent mapping $\pi_*:TM^0\to TM$ of the submersion $\pi:M^0\to M$ and describe the vector bundle $(TM^0)^V = ker\pi_*$. As locally $\pi^i(x,y)=x^i$, we obtain $\pi^i_*(\frac{\partial}{\partial x^j})=\delta^i_j$ and $\pi^i_*(\frac{\partial}{\partial y^j})=0$, on a coordinate neighbourhood $U^0\subset M^0$. Thus, $\{\frac{\partial}{\partial y^i}\}$ is a basis of $\Gamma(TM^0|_{U^0})^V$. We call $(TM^0)^V$ the vertical vector bundle of F^{2n+1}.
Locally, we have $X^V=X^i(x,y)\frac{\partial}{\partial y^i}$ where X^i are smooth functions on U^0 on a coordinate neighbourhood $U^0\subset M^0$. Afterwards, we note by $(T^*M^0)^V$ the dual vector bundle of $(TM^0)^V$. Thus, a Finsler 1-form is a smooth section of $(T^*M^0)^V$. Assume $\{\delta y^1,...,\delta y^{2n+1}\}$ is a dual basis to $\{\frac{\partial}{\partial y^1},...,\frac{\partial}{\partial y^{2n+1}}\}$, i.e, $\delta y^i(\frac{\partial}{\partial y^j})=\delta^i_j$. Then each $w\in\Gamma(T^*M^0)^V$ is locally showed as $w^V=w_i(x,y)\delta y^i$ where $w_i(x,y)=w(\frac{\partial}{\partial y^i})$ [9].
A complementary distribution $(TM^0)^H$ to $(TM^0)^V$ in TM^0 is said a non-linear connection or a horizontal distribution on M^0. Thus we can write

$$TM^0=(TM^0)^H\oplus(TM^0)^V. \tag{2.3}$$

The set of the local vector fields $\{\frac{\delta}{\delta x^1},...,\frac{\delta}{\delta x^{2n+1}}\}$ is a basis in $\Gamma(TM^0|_{U^0})^H$. Thus we have

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j_i\frac{\partial}{\partial y^j}. \tag{2.4}$$

Let X be a vector field on M^0. Then,

$$X=X^i(x,y)\frac{\delta}{\delta x^i}+\tilde{X}^i(x,y)\frac{\partial}{\partial y^i}, 1\le i\le 2n+1 \tag{2.5}$$

is obtained. Clearly, for $\tilde{X}^i(x,y)=0$, we get the subbundle of $(M^0)^h\subset M^0$ and for $X^i(x,y)=0$, we obtain the subbundle of $(M^0)^v\subset M^0$. Suppose $\{dx^1,...,dx^{2n+1}\}$ is a dual basis to $\{\frac{\delta}{\delta x^1},...,\frac{\delta}{\delta x^{2n+1}}\}$, i.e, $dx^i(\frac{\delta}{\delta x^j})=\delta^i_j$. Then each $w\in\Gamma(T^*M^0)^H$ is locally written as $w^H=\tilde{w}_i(x,y)dx^i$ where $\tilde{w}_i(x,y)=w(dx^i)$ and $\tilde{w}_i=w_i-N^j_iw_j$. Thus we can write

$$\delta y^i=dy^i+N^i_j(x,y)dx^j. \tag{2.6}$$

Consider a w 1-form, then

$$w = \tilde{w}_i(x,y)dx^i + w_i(x,y)\delta y^i. \tag{2.7}$$

Then $w^H(X^V) = 0$, $w^V(X^H) = 0$ where $w = w^H + w^V$. The Finsler tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ has the following local form on M^0

$$T = T^{i_1 \dots i_p, a_1 \dots a_r}_{j_{i \dots}, j_q, b_1 \dots b_s}(x,y)\frac{\delta}{\delta x^{i_1}} \otimes \dots \otimes \frac{\delta}{\delta x^{i_p}} \otimes dx^{a_1} \otimes \dots \otimes dx^{a_r} \otimes \frac{\partial}{\partial y^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{j_q}} \otimes \delta y^{b_1} \otimes \dots \otimes \delta y^{b_s} \tag{2.8}$$

[9].

Definition 2.1. *A Finsler connection is called a linear connection $\nabla = F\Gamma$ with the horizontal (vertical) linear space $(T_{(x,y)}M^0)^H, (x,y) \in M^0$ $((T_{(x,y)}M^0)^V, (x,y) \in M^0)$ of the distribution N parallel to ∇* [12].

A linear connection ∇ on M^0 is a Finsler connection if and only if

$$(\nabla_X Y^H)^V = 0, (\nabla_X Y^V)^H = 0,$$

$$\nabla_X Y = (\nabla_X Y^H)^H + (\nabla_X Y^V)^V \tag{2.9}$$

for all $X, Y \in T_{(x,y)}M^0$.

$$\nabla_X w = (\nabla_X w^H)^H + (\nabla_X w^V)^V,$$

for all $w \in T^*_{(x,y)}M^0$.

Remark. *Let ∇ on M^0 is a Finsler connection. We have directly following statements.*

$$Y \in (T_{(x,y)}M^0)^V \Rightarrow \forall X \in T_{(x,y)}M^0; \nabla_X Y \in (T_{(x,y)}M^0)^V,$$
$$Y \in (T_{(x,y)}M^0)^H \Rightarrow \forall X \in T_{(x,y)}M^0; \nabla_X Y \in (T_{(x,y)}M^0)^H. \tag{2.10}$$

Definition 2.2. *There is an associated pair of operators called h- and v- covariant derivation in the of Finsler tensor fields algebra for a Finsler connection ∇ on M^0. For every $X \in T_{(x,y)}M^0$,*

$$\nabla^H_X Y = \nabla_{X^H} Y, \nabla^H_X f = X^H(f), \forall Y \in T_{(x,y)}M^0, \forall f \in \Im(M^0). \tag{2.11}$$

If $w \in T^*_{(x,y)}M^0$, we define

$$(\nabla^H_X w)(Y) = X^H(w(Y)) - w(\nabla^H_X Y), \forall Y \in T_{(x,y)}M^0. \tag{2.12}$$

Then, it is called the operator of h-covariant derivation.
Similarly,

$$\nabla^V_X Y = \nabla_{X^V} Y, \nabla^V_X f = X^V(f), \forall Y \in T_{(x,y)}M^0, \forall f \in \Im(M^0) \tag{2.13}$$

for each vector field $X \in T_{(x,y)}M^0$.
If $w \in T^*_{(x,y)}M^0$ we define

$$(\nabla^V_X w)(Y) = X^V(w(Y)) - w(\nabla^V_X Y), \forall Y \in T_{(x,y)}M^0. \tag{2.14}$$

We extend the action of ∇^V_X to any Finsler tensor field in a equivalent way, as for ∇^H_X. Then, on M^0 we obtain an operator on the algebra of Finsler tensor fields; it is noted by ∇^V_X and called the operator of v- covariant derivation [2].

Definition 2.3. *Consider $w \in T^*_{(x,y)}M^0$ a differential q-form, ∇ is a linear connection on M^0 and T is the torsion of ∇. Then, its exterior differential dw is also described as*

$$\begin{aligned} dw(X_1, ...X_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} (\nabla_{X_i} w)(X_1, ...\hat{X}_i, ...X_{q+1}) \\ - \sum_{1\leq i\leq j\leq q+1} &(-1)^{i+j} w(T(X_i, X_j), X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{q+1}) \end{aligned} \tag{2.15}$$

where $\forall X_i \in T_{(x,y)}M^0$ [16].

Proposition 2.4. *If ∇ is a Finsler connection and $w \in T^*_{(x,y)}M^0$ is a 1-form on M^0, then its exterior differential is written by*

$$dw(X^H, Y^H) = (\nabla^H_X w)(Y^H) - (\nabla^H_Y w)(X^H) + w(T(X^H, Y^H)) \tag{2.16}$$

$$dw(X^V, Y^V) = (\nabla^V_X w)(Y^V) - (\nabla^V_Y w)(X^V) + w(T(X^V, Y^V)) \tag{2.17}$$

where $\forall X, Y \in T_{(x,y)}M^0$.

The Curvature of a Finsler Connection ∇ is referred with the following equations,

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R^H(X,Y)Z^H + R^V(X,Y)Z^V \tag{2.18}$$

where $\forall X, Y, Z \in T_{(x,y)}M^0$ [2].

Theorem 2.5. *The curvature of a Finsler connection ∇ on $T_{(x,y)}M^0$ is totally given with the following Finsler tensor fields equations ,*

$$\begin{aligned} R(X^V, Y^V)Z^V &= \nabla^V_X\nabla^V_Y Z^V - \nabla^V_Y\nabla^V_X Z^V - \nabla_{[X^V,Y^V]}Z^V \\ R(X^H, Y^H)Z^H &= \nabla^H_X\nabla^H_Y Z^H - \nabla^H_Y\nabla^H_X Z^H - \nabla_{[X^H,Y^H]}Z^H. \end{aligned} \tag{2.19}$$

Let $(\mathbb{R}, F_1)$ and (N^{2n}, F_2) be indefinite Finsler manifolds with their Cartan connections ∇^1 and ∇^2 and let $f : \mathbb{R} \to \mathbb{R}^+$ be a smooth function. Let $p_1 : \mathbb{R}\times N^{2n} \to \mathbb{R}$ and $p_2 : \mathbb{R} \times N^{2n} \to N^{2n}$. We consider the product manifold $\mathbb{R} \times N^{2n} = M^{2n+1}$ endowed with the pseudo-Riemannian metric $F^* : \mathbb{R}^0 \times (N^0)^{2n} \to \mathbb{R}$, $F^*(v_1, v_2) = F_1^2(v_1) + f^2(\pi_1(v_1))F_2^2(v_2)$, where $\mathbb{R}^0 = T\mathbb{R} \setminus \theta$, $(N^0)^{2n} = TN^{2n} \setminus \theta$, we denote this warped product by $\mathbb{R} \times_f N^{2n}$, we show that $(\mathbb{R} \times_f N^{2n}, F^*)$ is an indefinite Finsler manifold. The canonical projection π_1 gives rise to the vertical bundle $(V_1, d\pi_1, T\mathbb{R})$, where $V_1 = ker(d\pi_1)$ and $d\pi_1 : TT\mathbb{R} \to T\mathbb{R}$. The canonical projection π_2 gives rise to the vertical bundle $(V_2, d\pi_2, TN^{2n})$, where $V_2 = ker(d\pi_2)$ and $d\pi_2 : TTN^{2n} \to TN^{2n}$.
Now we have that

$$d\pi_1 \times d\pi_2 = d(\pi_1 \times \pi_2) : TT\mathbb{R} \times TTN^{2n} = TT\mathbb{R} \times TN^{2n}$$

and

$$Kerd(\pi_1 \times \pi_2) = kerd\pi_1 \oplus kerd\pi_2.$$

It follows that the vertical space of the manifold $\mathbb{R}\times N^{2n} = M^{2n+1}$, $V = V_1 \oplus V_2$, so the pseudo-Riemannian metrics g_1 and g_2, defined on V_1 and V_2, that is,

$$G^V = g_1^{V_1} + f^2(\pi_1(v))g_2^{V_2}$$

$$G(X^V, Y^V)_{(v,w)} = f^2(\pi_1(v))g_2(X^V_w, Y^V_w)$$

where $v \in T\mathbb{R}$, $w \in TN^{2n}$ and $\pi_1(v) \in \mathbb{R}$. This term is constant on leaves. Thus,

$$ZG(X^V, Y^V)_{(v,w)} = 2fZ(f(\pi_1(v)))g_2(X^V_w, Y^V_w) = 2fZf(\pi_1(v))\frac{1}{f^2(\pi_1(v))}G(X^V, Y^V)_{(v,w)}.$$

From these relations, we have that

$$Z^V G(X^V, Y^V) = 2(\frac{Z^V f}{f})G(X^V, Y^V).$$

Now, let H_1 and H_2 be the horizontal space with respect to the Cartan connections ∇^1 and ∇^2 on the Finsler manifolds $(\mathbb{R}, F_1)$ and (N^{2n}, F_2), resp. We get the direct-sum decomposition

$$TT(\mathbb{R} \times N^{2n}) = TTM^{2n+1} = TT\mathbb{R} \oplus TTN^{2n} = V_1 \oplus H_1 \oplus V_2 \oplus H_2.$$

The Finsler metrics, F_1, F_2 on the manifolds $\mathbb{R}$ and N^{2n}, resp., generate the Riemannian metrics g_1 and g_2 on the horizontal spaces H_1 and H_2, resp. Finally, these Riemannian metrics generates a pseudo-Riemannian metric on $T(T\mathbb{R}\times TN^{2n})$. It follows that we work mostly on the direct sum $H_1 \oplus H_2$. The direct sum of the liftings of H^1 and H^2 to the $TT\mathbb{R} \times TTN$ that is,

$$G(X^H, Y^H)_{(v,w)} = f^2(\pi_1(v))g_2(X^H, Y^H).$$

This term is constant on leaves. Thus,

$$Z^H G(X^H, Y^H)_{(v,w)} = 2fZ^H(f(\pi_1(v)))g_2(X^H_w, Y^H_w) = 2(\frac{Z^H f}{f})G(X^H, Y^H).$$

Proposition 2.6. *Consider $F^{2n+1} = (M, M^0, F^*)$ an indefinite Finsler manifold with the warped product space $M^{2n+1} = \mathbb{R} \times_f N^{2n}$. We assume that $(N^0)^{2n} = TN^{2n} \setminus \theta$ is a Kahlerian pseudo-metric manifold and $f(t) = ce^{\frac{t}{2}}$. For the almost Kenmotsu pseudo-metric Finsler structures $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ on $(M^0)^h$ and $(M^0)^v$, resp., the 1-forms η^H, η^V and the second fundamental forms Ω^H, Ω^V hold the following conditions:*

$$d\eta^H = d\eta^V = 0, d\Omega^H = \eta^H \wedge \Omega^H, d\Omega^V = \eta^V \wedge \Omega^V, d\Omega = \eta \wedge \Omega, d\eta = 0.$$

Proof. Let $G = G^H + G^V : TTM \times TTM \to \Im(TM)$ and $G_2 : TTN \times TTN \to \Im(TN)$ are Sasaki metrics.

$$G(X, \phi Y) = G(X^H + X^V, \phi Y^H + \phi Y^V) = f^2(\pi_1(v))G_2(X, \phi Y)$$
$$= f^2(\pi_1(v))G_2(X^H + X^V, \phi Y^H + \phi Y^V),$$

$$G(X^H, \phi Y^H) + G(X^V, \phi Y^V) = f^2(\pi_1(v))[G_2(X^H, \phi Y^V) + G_2(X^V, \phi Y^H)],$$

$$\Omega(X^H, Y^H) + \Omega(X^V, Y^V) = f^2(\pi_1(v))[\Omega^*(X^H, Y^V) + \Omega^*(X^V, Y^H)]$$

where Ω^* is the second fundamental form with respect to Kahlerian metric G_2 on Kahler vector bundle TN. That is $d\Omega^* = 0$. Thus we have

$$d\Omega(X^H, Y^H) + d\Omega(X^V, Y^V) = 2f(t)f'(t)dt \wedge [\Omega^*(X^H, Y^V) + \Omega^*(X^V, Y^H)]$$

$$d\Omega^H + d\Omega^V = (\frac{2f'}{f})dt \wedge [\Omega^H + \Omega^V]$$

$$f(t) = e^{\frac{t}{2}} \to \frac{2f'(t)}{f(t)} = 1$$

and

$$dt = \eta, d\eta^H = d\eta^V = 0, d\Omega^H = \eta^H \wedge \Omega^H, d\Omega^V = \eta^V \wedge \Omega^V, d\Omega = \eta \wedge \Omega, d\eta = 0$$

□

3. Kenmotsu Pseudo-metric Finsler Structures

In our main results, we present almost Kenmotsu and Kenmotsu structures on indefinite Finsler manifolds. Then, we acquire some integrability or normality conditions for these structures.

3.1. **Almost Contact Finsler Structures.** Consider tensor field ϕ, 1-form η and vector field ξ on M^0, as follows:

$$\phi = \phi^H + \phi^V = \phi^i_j(x,y)\frac{\delta}{\delta x^i} \otimes dx^j + \tilde{\phi}^i_j(x,y)\frac{\partial}{\partial y^i} \otimes \delta y^j \tag{3.1}$$

$$\eta = \eta^H + \eta^V = \eta_i(x,y)dx^i + \tilde{\eta}_i(x,y)\delta y^i$$

$$\xi = \xi^H + \xi^V = \xi^i(x,y)\frac{\delta}{\delta x^i} + \tilde{\xi}^i(x,y)\frac{\partial}{\partial y^i} \tag{3.2}$$

Definition 3.1. *Assume that ϕ, η and ξ are given by* (3.1) *and* (3.2) *on M^0 such that*

$$(\phi^H)^2 = -I^H + \eta^H \otimes \xi^H, (\phi^V)^2 = -I^V + \eta^V \otimes \xi^V, \tag{3.3}$$

$$\eta^H(\xi^H) = \eta^V(\xi^V) = 1. \tag{3.4}$$

Then, (ϕ^H, η^H, ξ^H) and (ϕ^V, η^V, ξ^V) are called the almost contact Finsler structures on $(M^0)^h$ and $(M^0)^v$, respectively, where $M^0 = (M^0)^h \oplus (M^0)^v$ is a Finsler vector bundle.

Theorem 3.2. *Suppose that horizontal and vertical Finsler vector bundles $(M^0)^h$ and $(M^0)^v$ have the almost contact Finsler structures, then*

$$\phi^H(\xi^H) = \phi^V(\xi^V) = 0, \eta^H \circ \phi^H = \eta^V \circ \phi^V = 0. \tag{3.5}$$

Proof. By (3.3) we get $(\phi^H)^2(\xi^H) = -\xi^H + \eta^H(\xi^H)\xi^H$. Then $\phi^H(\xi^H) = 0$ or $\phi^H(\xi^H)$ is a nontrivial eigenvector of ϕ^H corresponding to eigenvalue 0. Using (3.3), we obtain

$$0 = (\phi^H)^2(\phi^H(\xi^H)) = -\phi^H(\xi^H) + \eta^H(\phi^H(\xi^H))\xi^H$$

or

$$\phi^H(\xi^H) = \eta^H(\phi(\xi^H))\xi^H.$$

Now, if $\phi^H(\xi^H)$ is a nontrivial eigenvector of the eigenvalue 0, then $\eta^H(\phi^H(\xi^H)) \neq 0$. Thus, we have

$$0 = (\phi^H)^2(\xi^H) = \eta^H(\phi^H(\xi^H))\phi^H(\xi^H) = \eta^H(\phi^H(\xi^H))^2\xi^H \neq 0$$

which is a contradiction. Therefore $\phi^H(\xi^H) = 0$. Similarly by (3.3) we get $\phi^V(\xi^V) = 0$. On the other hand, since $\phi^H(\xi^H) = 0$, then we get for all $X^H \in (TM^0)^H$

$$\eta^H(\phi(X^H))\xi^H = (\phi^H)^3(X^H) + \phi^H(X^H) = 0$$

and for all $X^V \in (TM^0)^V$

$$\eta^V(\phi^V(X^V)\xi^V) = 0.$$

Hence $\eta^H \circ \phi^H = 0$ and $\eta^V \circ \phi^V = 0$. □

Remark. *We deduce that (ϕ^H, ξ^H, η^H) and (ϕ^V, ξ^V, η^V) are almost contact structures on subbundles $(M^0)^h$ and $(M^0)^v$, where $(M^0)^h$ and $(M^0)^v$ have $(2n+1)-$dimensional. We call that $((M^0)^h, \phi^H, \xi^H, \eta^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V)$ are almost contact Finsler manifolds.*

3.2. **Almost Kenmotsu Pseudo-Metric Structures on Indefinite Finsler Manifolds.** Let $F^{2n+1} = (M, M^0, F^*)$ be an indefinite Finsler manifold. We define

$$g^{F^*} : \Gamma(TM^0)^V \times \Gamma(TM^0)^V \to \Im(M^0),$$

$$g^{F^*}_{(ij)}(x, y) = g^{F^*}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})(x, y). \tag{3.6}$$

Obviously, g^{F^*} is a symmetric Finsler tensor field. g^{F^*} is called the pseudo-Finsler metric of F^{2n+1}. Thus, g^{F^*} is thought of as a pseudo-Riemannian metric on the Finsler vector bundle $(TM^0)^V$. Similarly, we define

$$g^{F^*} : \Gamma(TM^0)^H \times \Gamma(TM^0)^H \to \Im(M^0),$$

$$g^{F^*}_{(ij)}(x, y) = g^{F^*}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})(x, y) \tag{3.7}$$

where $g^{F^*}_{ij}$ are functions given by (2.2). On the Finsler vector bundle $(TM^0)^H$, g^{F^*} is thought of as a pseudo-Riemannian metric. A Finsler vector has a casual character described with following statements:
$X \in (T_{(x,y)}M^0)^V$ $(X \in (T_{(x,y)}M^0)^H)$ is called

$$g^{F^*}_{(x,y)}(X, X) > 0 or X = 0 \Rightarrow Space - like,$$

$$g^{F^*}_{(x,y)}(X, X) < 0 \Rightarrow time - like, \tag{3.8}$$

$$g^{F^*}_{(x,y)}(X, X) = 0, X \neq 0 \Rightarrow light - like(null)$$

where $(x, y) \in M^0$. The Finsler norm (length) of X is a non-negative number $\|X\|$ described by

$$\|X\| = |g^{F^*}_{(x,y)}(X, X)|^{\frac{1}{2}}. \tag{3.9}$$

If $g^{F^*}_{(x,y)}(X,X)=1$, X is called a unit spacelike or $g^{F^*}_{(x,y)}(X,X)=-1$, then it is called a unit timelike Finsler vector. If X is a unit Finsler vector then $\varepsilon = g^{F^*}_{(x,y)}(X,X)$ is demonstrated the signature of X. We describe,

$$G:\Gamma(TM^0)\times\Gamma(TM^0)\to\Im(M^0),$$

$$G(X,Y)=G^H(X,Y)+G^V(X,Y),\forall X,Y\in\Gamma(TM^0). \tag{3.10}$$

Evidently, on M^0, G is a symmetric tensor field of type $(0,2)$. Furthermore, it is non-degenerate with a constant index. G is a pseudo-Riemannian metric on M^0 of index $2q$(q is the index of the pseudo-Finsler metric g^{F^*}). G is called the Sasaki Finsler metric on M^0. Then, we remark ,

$$G=g^{F^*}_{ij}dx^i\otimes dx^j+g^{F^*}_{ij}\delta y^i\otimes\delta y^j=G^H+G^V \tag{3.11}$$

[9].

Definition 3.3. *Suppose that (ϕ^H,ξ^H,η^H) and (ϕ^V,ξ^V,η^V) are almost contact structures on horizontal and vertical Finsler vector bundles $(M^0)^h$ and $(M^0)^v$. If the metric structures G^H and G^V satisfy the following equations;*

$$G^H(\phi X^H,\phi Y^H)=G^H(X^H,Y^H)-\varepsilon\eta^H(X^H)\eta^H(Y^H)$$

$$G^V(\phi X^V,\phi Y^V)=G^V(X^V,Y^V)-\varepsilon\eta^V(X^V)\eta^V(Y^V) \tag{3.12}$$

$$\eta^H(X^H)=\varepsilon G^H(X^H,\xi^H),\eta^V(X^V)=\varepsilon G^V(X^V,\xi^V) \tag{3.13}$$

where $\varepsilon=\pm1$, then $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ are called almost contact pseudo-metric Finsler structures, respectively, on $(M^0)^h$ and $(M^0)^v$.

Result 3.4. *Let $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ are the almost contact pseudo-metric Finsler structures on $(M^0)^h$ and $(M^0)^v$, respectively. Then from* (3.12) *and* (3.13), *we get [17],*

$$\begin{aligned}G^H(\phi X^H,Y^H)&=-G^H(X^H,\phi Y^H),\\ G^V(\phi X^V,Y^V)&=-G^V(X^V,\phi Y^V)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}G^H(\phi X^H,\phi Y^H)&=-G^H(\phi^2X^H,Y^H),\\ G^V(\phi X^V,\phi Y^V)&=-G^V(\phi^2X^V,Y^V).\end{aligned} \tag{3.15}$$

Now, we define fundamental 2-form.

$$\begin{aligned}\Omega(X^H,Y^H)&=G^H(X^H,\phi Y^H),\\ \Omega(X^V,Y^V)&=G^V(X^V,\phi Y^V).\end{aligned} \tag{3.16}$$

Proposition 3.5. *The fundamental 2-form, given above, hold* [17]

$$\Omega(\phi X^H,\phi Y^H)=\Omega(X^H,Y^H),\Omega(\phi X^V,\phi Y^V)=\Omega(X^V,Y^V), \tag{3.17}$$

$$\Omega(X^H,Y^H)=-\Omega(Y^H,X^H),\Omega(X^V,Y^V)=-\Omega(Y^V,X^V). \tag{3.18}$$

Definition 3.6. *∇ be a Finsler connection on M^0 and η be the fundamental 1-form which satisfies $d\eta(X,Y)=0$,then there exists a function f on M^0 such that $\eta = df$,[17]*

$$(\nabla^H_X \eta^H)(Y^H) - (\nabla^H_Y \eta^H)(X^H) + \eta^H(T(X^H,Y^H)) = 0,$$

$$(\nabla^V_X \eta^V)(Y^V) - (\nabla^V_Y \eta^V)(X^V) + \eta^V(T(X^V,Y^V)) = 0. \tag{3.19}$$

Then, the almost contact pseudo-metric Finsler structure is called an almost Kenmotsu pseudo-metric Finsler structure and the Finsler connection ∇ satisfying (3.19) *is called an almost Kenmotsu Finsler connection on M^0. Thus $((M^0)^h,\phi^H,\xi^H,\eta^H,G^H)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are called the almost Kenmotsu pseudo-metric Finsler manifolds or almost ε-Kenmotsu Finsler manifolds.*

Theorem 3.7. *If the almost Kenmotsu Finsler connection ∇ is torsion free, then we have the following equations for $X^H,Y^H \in (TM^0)^H$ and $X^V,Y^V \in (TM^0)^V$*

$$(\nabla^H_X \eta^H)(Y^H) - (\nabla^H_Y \eta^H)(X^H) = 0, (\nabla^V_X \eta^V)(Y^V) - (\nabla^V_Y \eta^V)(X^V) = 0 \tag{3.20}$$

[17].

3.3. **Integrability Tensor Field of the Almost Kenmotsu Pseudo-Metric Finsler Manifolds.** Let $F^{2n+1} = (M, M^0, F^*)$ be an indefinite Finsler manifold. The integrability tensor field of the almost Kenmotsu pseudo-metric Finsler structures (ϕ^H,ξ^H,η^H) and (ϕ^V,ξ^V,η^V) on $(M^0)^h$ and $(M^0)^v$ are given by:

$$N^H_\phi(X^H,Y^H) = [\phi X^H,\phi Y^H] - \phi[\phi X^H,Y^H] - \phi[X^H,\phi Y^H] + \phi^2[X^H,Y^H],$$

$$N^V_\phi(X^V,Y^V) = [\phi X^V,\phi Y^V] - \phi[\phi X^V,Y^V] - \phi[X^V,\phi Y^V] + \phi^2[X^V,Y^V],$$

$\forall X^H,Y^H \in (TM^0)^H$ and $\forall X^V,Y^V \in (TM^0)^V$.

We determine four tensors $N^{(1)}$,$N^{(2)}$,$N^{(3)}$ and $N^{(4)}$, respectively by

$$N^{(1)}(X^H,Y^H) = N^H_\phi(X^H,Y^H), \tag{3.21}$$

$$N^{(2)}(X^H,Y^H) = (L^H_{\phi X}\eta^H)(Y^H) - (L^H_{\phi Y}\eta^H)(X^H) \tag{3.22}$$

$$N^{(3)}(X^H) = (L^H_\xi \phi)(X^H), N^{(4)}(X^H) = (L_{\xi^H}\eta^H)(X^H) \tag{3.23}$$

and

$$N^{(1)}(X^V,Y^V) = N^V_\phi(X^V,Y^V),$$

$$N^{(2)}(X^V,Y^V) = (L^V_{\phi X}\eta^V)(Y^V) - (L^V_{\phi Y}\eta^V)(X^V),$$

$$N^{(3)}(X^V) = (L^V_\xi \phi)(X^V), N^{(4)}(X^V) = (L_{\xi^V}\eta^V)(X^V).$$

Proposition 3.8. *The almost Kenmotsu pseudo-metric Finsler structures $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ on $(M^0)^h$ and $(M^0)^v$ are normal if and only if $N^H_\phi = 0$ and $N^V_\phi = 0$.*

Lemma 3.9. *The almost Kenmotsu pseudo-metric Finsler structures $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ are normal if and only if $N^{(1)} = N^{(2)} = N^{(3)} = N^{(4)} = 0$.*

Lemma 3.10. *For the almost Kenmotsu pseudo-metric Finsler structures* $(\phi^H, \xi^H, \eta^H, G^H)$ *and* $(\phi^V, \xi^V, \eta^V, G^V)$ *on* $(M^0)^h$ *and* $(M^0)^v$*, we get following equations.*

$$2G^H((\nabla^H_X\phi)Y^H, Z^H) = d\Omega(X^H, \phi Y^H, \phi Z^H) + \varepsilon N^{(2)}(Y^H, Z^H)\eta^H(X^H) \quad (3.24)$$
$$- d\Omega(X^H, Y^H, Z^H) + G(N^{(1)}(Y^H, Z^H), \phi X^H),$$

$$2G^V((\nabla^V_X\phi)Y^V, Z^V) = d\Omega(X^V, \phi Y^V, \phi Z^V) + \varepsilon N^{(2)}(Y^V, Z^V)\eta^V(X^V) \quad (3.25)$$
$$- d\Omega(X^V, Y^V, Z^V) + G(N^{(1)}(Y^V, Z^V), \phi X^V).$$

Proof. The Finsler connection ∇ with G is presented with following equations,

$$2G^H(\nabla^H_X Y^H, Z^H) = X^H G^H(Y^H, Z^H)$$
$$+ Y^H G^H(X^H, Z^H) - Z^H G^H(X^H, Y^H) + G^H([X^H, Y^H], Z^H)$$
$$+ G^H([Z^H, X^H], Y^H) - G^H([Y^H, Z^H], X^H), \quad (3.26)$$

$$2G^V(\nabla^V_X Y^V, Z^V) = X^V G^V(Y^V, Z^V)$$
$$+ Y^V G^V(X^V, Z^V) - Z^V G^V(X^V, Y^V) + G^V([X^V, Y^V], Z^V)$$
$$+ G^V([Z^V, X^V], Y^V) - G^V([Y^V, Z^V], X^V). \quad (3.27)$$

Also, we get

$$d\Omega(X^H, Y^H, Z^H) = X^H\Omega(Y^H, Z^H) + Y^H\Omega(Z^H, X^H)$$
$$+ Z^H\Omega(X^H, Y^H) - \Omega([X^H, Y^H], Z^H)$$
$$- \Omega([Z^H, X^H], Y^H) - \Omega([Y^H, Z^H], X^H). \quad (3.28)$$

By using (3.16), from(3.26), we obtain

$$2G((\nabla^H_X\phi)Y^H, Z^H) = \phi Y^H G^H(X^H, Z^H) - Z^H\Omega(X^H, Y^H)$$
$$+ G^H([X^H, \phi Y^H], Z^H) + \Omega([Z^H, X^H], Y^H) - G^H([\phi Y^H, Z^H], X^H)$$
$$+ Y^H\Omega(X^H, Z^H) - \phi Z^H G^H(X^H, Y^H) + \Omega([X^H, Y^H], Z^H)$$
$$+ G^H([\phi Z^H, X^H], Y^H) - G^H([Y^H, \phi Z^H], X^H). \quad (3.29)$$

Also from (3.21) by using (3.16), we get

$$G^H(N^{(1)}(Y^H, Z^H)\phi X^H) = -\Omega([Y^H, Z^H], X^H) + \Omega([\phi Y^H, \phi Z^H], X^H)$$
$$- G([\phi Y^H, Z^H], X^H) + \eta^H[\phi Y^H, Z^H]\eta^H(X^H)$$
$$- G^H([Y^H, \phi Z^H], X^H) + \eta^H[Y^H, \phi Z^H]\eta^H(X^H). \quad (3.30)$$

From (3.22), we have

$$N^{(2)}(Y^H, Z^H)\eta^H(X^H) = \phi Y^H(\eta^H(Y^H))\eta^H(X^H)$$
$$- \phi Z^H(\eta^H(Y^H))\eta^H(X^H) - \eta^H[\phi Y^H, Z^H]\eta^H(X^H)$$
$$- \eta^H[Y^H, \phi Z^H]\eta^H(X^H). \quad (3.31)$$

Then, from (3.28), we have

$$
\begin{aligned}
d\Omega(X^H, \phi Y^H, \phi Z^H) &= X^H \Omega(Y^H, Z^H) + \phi Y^H G(Z^H, X^H) \\
&- \varepsilon \phi Y^H (\eta^H(Z^H)\eta^H(X^H)) - \phi Z^H G(X^H, Y^H) \\
&+ \varepsilon \phi Z^H (\eta^H(X^H)\eta^H(Y^H)) + G([X^H, \phi Y^H], Z^H) \\
&- \varepsilon \eta^H [X^H, \phi Y^H]\eta^H(Z^H) + G^H([\phi Z^H, X^H], Y^H) \\
&- \varepsilon \eta^H [\phi Z^H, X^H]\eta^H(Y^H) - \Omega([\phi Y^H, \phi Z^H], X^H). \qquad (3.32)
\end{aligned}
$$

By using (3.30), (3.31) and (3.32), we have the equation (3.24). Similarly by using (3.16), (3.27) and (3.30) we get equation (3.25). □

Lemma 3.11. *For the almost Kenmotsu pseudo-metric Finsler structures* $(\phi^H, \xi^H, \eta^H, G^H)$ *and* $(\phi^V, \xi^V, \eta^V, G^V)$ *with* $d\Omega = \eta \wedge \Omega$ *and* $N^{(1)} = N^{(2)} = 0$*, we get*

$$(\nabla^V_X \phi)Y^V = \frac{1}{2}\{\varepsilon G^V(\phi X^V, Y^V)\xi^V - \eta^V(Y^V)\phi X^V\} \qquad (3.33)$$

and

$$(\nabla^H_X \phi)Y^H = \frac{1}{2}\{\varepsilon G^H(\phi X^H, Y^H)\xi^H - \eta^H(Y^H)\phi X^H\}. \qquad (3.34)$$

Proof. From (3.25), we can write

$$
\begin{aligned}
&2G^V((\nabla^V_X \phi)Y^V, Z^V) = -\eta^V(Y^V)G^V(\phi X^V, Z^V) - \varepsilon G^V(\xi^V, Z^V)G^V(X^V, \phi Y^V) \\
&= G^V(-\varepsilon G(X^V, \phi Y^V)\xi^V - \eta^V(Y^V)\phi X^V, Z^V),
\end{aligned}
$$

$$(\nabla^V_X \phi)Y^V = \frac{1}{2}\{\varepsilon G^V(\phi X^V, Y^V)\xi^V - \eta^V(Y^V)\phi X^V\}.$$

Similarly, from (3.24) we obtain (3.34). □

Theorem 3.12. *The almost Kenmotsu pseudo-metric Finsler structures* $(\phi^H, \xi^H, \eta^H, G^H)$ *and* $(\phi^V, \xi^V, \eta^V, G^V)$ *on* $(M^0)^h$ *and* $(M^0)^v$ *are the Kenmotsu pseudo-metric Finsler structures if and only if*

$$(\nabla^H_X \phi)Y^H = \frac{1}{2}\{\varepsilon G^H(\phi X^H, Y^H)\xi^H - \eta^H(Y^H)\phi X^H\},$$

$$(\nabla^V_X \phi)Y^V = \frac{1}{2}\{\varepsilon G^V(\phi X^V, Y^V)\xi^V - \eta^V(Y^V)\phi X^V\}.$$

Conversely, we suppose that the structures satisfy (3.33) *and* (3.34)*. Putting* $Y^V = \xi^V$ *in* (3.33)*, we have*

$$(\nabla^V_X \phi)\xi^V = \frac{1}{2}\{\varepsilon G(\phi X^V, \xi^V)\xi^V - \eta^V(\xi^V)\phi X^V\},$$

$$-\phi(\nabla^V_X \xi^V) = -\frac{1}{2}\phi X^V,$$

$$\nabla^V_X \xi^V = -\frac{1}{2}\phi^2 X^V = \frac{1}{2}(X^V - \eta^V(X^V)\xi^V). \qquad (3.35)$$

Similarly we obtain from (3.34)

$$\nabla^H_X \xi^H = -\frac{1}{2}\phi^2 X^H = \frac{1}{2}(X^H - \eta^H(X^H)\xi^H). \qquad (3.36)$$

Moreover from (3.35) *and* (3.36) *we get,*

$$(\nabla^H_X \eta^H)Y^H + (\nabla^H_Y \eta^H)X^H = \Omega(\phi X^H, Y^H) = G(\phi X^H, \phi Y^H), \tag{3.37}$$

$$(\nabla^V_X \eta^V)Y^V + (\nabla^V_Y \eta^V)X^V = \Omega(\phi X^V, Y^V) = G(\phi X^V, \phi Y^V), \tag{3.38}$$

$$2(\nabla^H_X \eta^H)Y^H = \Omega(\phi X^H, Y^H) = G(\phi X^H, \phi Y^H), \tag{3.39}$$

$$2(\nabla^V_X \eta^V)Y^V = \Omega(\phi X^V, Y^V) = G(\phi X^V, \phi Y^V). \tag{3.40}$$

Thus, these structures are the Kenmotsu pseudo-metric Finsler structures.

4. The Curvatures of Indefinite Kenmotsu Finsler Manifolds

In this section, we calculate curvatures of indefinite Kenmotsu Finsler manifolds. Firstly, we give the following theorem.

Theorem 4.1. *If* $(\phi^H, \xi^H, \eta^H, G^H)$ *and* $(\phi^V, \xi^V, \eta^V, G^V)$ *are the Kenmotsu pseudo-metric Finsler structures on the Finsler vector bundles* $(M^0)^h$ *and* $(M^0)^v$*, then from* (2.19)*,* (3.3)*,* (3.4)*,* (3.35) *and* (3.36)*, we have*

$$R(X^H, Y^H)\xi^H = \frac{1}{4}\{\eta^H(X^H)Y^H - \eta^H(Y^H)X^H\} \tag{4.1}$$

and

$$R(X^V, Y^V)\xi^V = \frac{1}{4}\{\eta^V(X^V)Y^V - \eta^V(Y^V)X^V\}. \tag{4.2}$$

That is, we have

$$\begin{aligned} R(X,Y)\xi &= R(X^H, Y^H)\xi^H + R(X^V, Y^V)\xi^V \\ &= \frac{1}{4}\{(\eta^H(X^H)Y^H + \eta^V(X^V)Y^V) - (\eta^H(Y^H)X^H + \eta^V(Y^V)X^V)\}. \end{aligned} \tag{4.3}$$

Theorem 4.2. *Let* $(\phi^H, \xi^H, \eta^H, G^H)$ *and* $(\phi^V, \xi^V, \eta^V, G^V)$ *be the Kenmotsu pseudo-metric Finsler structures on* $(M^0)^h$ *and* $(M^0)^v$*. From* (3.33)*,* (3.34)*,* (3.37)*,* (3.38)*,* (3.39) *and* (3.40)*, we find that,*

$$\begin{aligned} R(X^H, Y^H)\phi Z^H &= \phi R(X^H, Y^H)Z^H - \frac{\varepsilon}{4}\{G(\phi X^H, Z^H)Y^H \\ &- G(\phi Y^H, Z^H)X^H + G(X^H, Z^H)\phi Y^H - G(Y^H, Z^H)\phi X^H\} \end{aligned} \tag{4.4}$$

$$\begin{aligned} R(X^V, Y^V)\phi Z^V &= \phi R(X^V, Y^V)Z^V - \frac{\varepsilon}{4}\{G(\phi X^V, Z^V)Y^V \\ &- G(\phi Y^V, Z^V)X^V + G(X^V, Z^V)\phi Y^V - G(Y^V, Z^V)\phi X^V\} \end{aligned} \tag{4.5}$$

Result 4.3. *From* (4.4)*,* (4.5)*, we obtain the equations below.*

$$\begin{aligned} R(X^H, Y^H)Z^H &= -\frac{\varepsilon}{4}\{G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H - G(\phi Y^H, Z^H)\phi X^H \\ &+ G(\phi X^H, Z^H)\phi Y^H\} - \phi R(X^H, Y^H)\phi Z^H \end{aligned} \tag{4.6}$$

$$R(X^V,Y^V)Z^V = -\frac{\varepsilon}{4}\{G(Y^V,Z^V)X^V - G(X^V,Z^V)Y^V - G(\phi Y^V,Z^V)\phi X^V$$
$$+ G(\phi X^V,Z^V)\phi Y^V\} - \phi R(X^V,Y^V)\phi Z^V \tag{4.7}$$

Definition 4.4. *A plane section is called a horizontal (vertical) $\phi-$ section if there exists a unit vector X^H (X^V)in $(TM^0)^H$ $((TM^0)^V)$ orthogonal to ξ^H (ξ^V) such that $\{X^H,\phi X^H\}$ $(\{X^V,\phi X^V\})$. Thus, horizontal and vertical flag curvature can be given with following equations.*

$$K(X^H,\phi X^H) = \frac{G^H(R(X^H,\phi X^H)\phi X^H,X^H)}{G(X^H,X^H)G(\phi X^H,\phi X^H)} \tag{4.8}$$

is named a horizontal $\phi-$ sectional curvature, denoted by $K^H(X^H)$. Vertical flag curvature

$$K(X^V,\phi X^V) = \frac{G^V(R(X^V,\phi X^V)\phi X^V,X^V)}{G(X^V,X^V)G(\phi X^V,\phi X^V)} \tag{4.9}$$

is named a vertical $\phi-$ sectional curvature,denoted by $K^V(X^V)$. The $\phi-$ sectional curvature on a Kenmotsu pseudo-metric Finsler manifold is

$$K(X) = K^H(X^H) + K^V(X^V).$$

Proposition 4.5. *Let $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ be the Kenmotsu pseudo metric Finsler structures on $(M^0)^h$ and $(M^0)^v$. If $(M^0)^h$ and $(M^0)^v$ are locally symmetric, then $(M^0)^h$ and $(M^0)^v$ are indefinite Kenmotsu Finsler manifolds with constant curvature $-\frac{\varepsilon}{4}$.*

Proof. For $X^H,Y^H,Z^H,\xi^H \in (TM^0)^H$ from (3.33), (3.34), (4.1) and (4.2), we get

$$(\nabla^H_Z R)(X^H,Y^H,\xi^H) = -\frac{\varepsilon}{4}\{G(Y^H,Z^H)X^H - G(X^H,Z^H)Y^H\} - R(X^H,Y^H)Z^H \tag{4.10}$$

Since $(M^0)^h$ locally symmetric, that is, $\nabla^H_Z R = 0$, from (4.10) we get

$$R(X^H,Y^H)Z^H = -\frac{\varepsilon}{4}\{G^H(Y^H,Z^H)X^H - G^H(X^H,Z^H)Y^H\} \tag{4.11}$$

Y^H must be space-like vector when X^H is a time-like vector for any orthonormal pair $\{X^H,Y^H\}$. Two vectors both time-like(or space-like) can not be perpendicular to each other. Thus, we get

$$G(R(X^H,Y^H)Y^H,X^H) = -\frac{\varepsilon}{4}\{G(Y^H,Y^H)G(X^H,X^H)\} = \frac{\varepsilon}{4},$$
$$K(X^H,Y^H) = \frac{G(R(X^H,Y^H)Y^H,X^H)}{G(Y^H,Y^H)G(X^H,X^H)} = -\frac{\varepsilon}{4}$$

where if ξ^H is a time-like vector, then we have $K(X^H,Y^H) = \frac{1}{4}$, if ξ^H is a space-like vector, then we obtain $K(X^H,Y^H) = -\frac{1}{4}$.
In a similiar way, we get

$$R(X^V,Y^V)Z^V = -\frac{\varepsilon}{4}\{G^V(Y^V,Z^V)X^V - G^V(X^V,Z^V)Y^V\}.$$

for $X^V,Y^V,Z^V,\xi^V \in (TM^0)^V$.
We have for any orthonormal pair $\{X^V,Y^V\}$

$$G(R(X^V,Y^V)Y^V,X^V) = -\frac{\varepsilon}{4}\{G(Y^V,Y^V)G(X^V,X^V)\} = \frac{\varepsilon}{4},$$

$$K(X^V,Y^V)=\frac{G(R(X^V,Y^V)Y^V,X^V)}{G(Y^V,Y^V)G(X^V,X^V)}=-\frac{\varepsilon}{4}.$$

For any orthonormal pair $\{X,Y\}$ on TM^0, we obtain

$$K(X,Y)=\frac{G^H(R(X^H,Y^H)Y^H,X^H)+G^V(R(X^V,Y^V)Y^V,X^V)}{G^H(X^H,X^H)G^H(Y^H,Y^H)+G(X^V,X^V)G(Y^V,Y^V)}=-\frac{\varepsilon}{4}. \quad (4.12)$$

□

The horizontal Ricci tensor S^H of an indefinite Kenmotsu Finsler manifold $(M^0)^h$ is given with $\{E_1^H,...,E_{2n}^H,\xi^H\}$ is a local orthonormal frame of $(TM^0)^H$ as follows.

$$\begin{aligned} S^H(X^H,Y^H) &= \sum_{i=1}^{2n} G(R(X^H,E_i^H)E_i^H,Y^H)+G(R(X^H,\xi^H)\xi^H,Y^H) \\ &= \sum_{i=1}^{2n} G(R(E_i^H,X^H)Y^H,E_i^H)+G(R(\xi^H,X^H)Y^H,\xi^H). \end{aligned} \quad (4.13)$$

The vertical Ricci tensor S^V of an indefinite Kenmotsu Finsler manifold $(M^0)^v$ is given with $\{E_1^V,...,E_{2n}^V,\xi^V\}$ is a local orthonormal frame of $(TM^0)^V$ as follows.

$$\begin{aligned} S^V(X^V,Y^V) &= \sum_{i=1}^{2n} G(R(X^V,E_i^V)E_i^V,Y^V)+G(R(X^V,\xi^V)\xi^V,Y^V) \\ &= \sum_{i=1}^{2n} G(R(E_i^V,X^V)Y^V,E_i^V)+G(R(\xi^V,X^V)Y^V,\xi^V). \end{aligned} \quad (4.14)$$

Proposition 4.6. *A contact pseudo-metric structure $(\phi^H,\xi^H,\eta^H,G^H)$ on an indefinite Finsler vector bundle $(M^0)^h$ with index q is the Kenmotsu pseudo-metric structure if and only if*

$$S(\xi^H,\xi^H)= \begin{cases} \frac{q-2n}{4}, & \xi^H \text{ is a space-like vector} \\ \frac{q-2n-1}{4}, & \xi^H \text{ is a time-like vector} \end{cases} \quad (4.15)$$

Proof. From (4.1) and (4.13), we have

$$\begin{aligned} S(\xi^H,\xi^H) &= \sum_{i=1}^{2n} G(R(E_i^H,\xi^H)\xi^H,E_i^H) \\ &= \frac{1}{4}\sum_{i=1}^{2n} G(\eta^H(E_i^H)\xi^H-\eta^H(\xi^H)E_i^H,E_i^H) \\ &= \frac{1}{4}\sum_{i=1}^{2n} -G(E_i^H,E_i^H) \\ &= -\frac{\varepsilon_1+...+\varepsilon_{2n}}{4}. \end{aligned}$$

Since $F^{2n+1} = (M, M^0, F^*)$ is an indefinite Finsler manifold with index q, if $G(\xi^H, \xi^H) = \varepsilon = 1$, then ξ^H is a space-like vector and we obtain

$$S(\xi^H, \xi^H) = -\frac{1}{4}\sum_{i=1}^{q} G(E_i^H, E_i^H) - \frac{1}{4}\sum_{i=q+1}^{2n} G(E_i^H, E_i^H) = \frac{q-2n}{4}$$

If $G(\xi^H, \xi^H) = \varepsilon = -1$, then ξ^H is a time-like vector and we obtain

$$S(\xi^H, \xi^H) = -\frac{1}{4}\sum_{i=1}^{q-1} G(E_i^H, E_i^H) - \frac{1}{4}\sum_{i=q}^{2n} G(E_i^H, E_i^H) = \frac{q-2n-1}{4}$$

□

Proposition 4.7. *A contact pseudo-metric structure $(\phi^V, \xi^V, \eta^V, G^V)$ on an indefinite Finsler vector bundle $(M^0)^v$ of index q is the Kenmotsu pseudo-metric structure if and only if*

$$S(\xi^V, \xi^V) = \begin{matrix} \frac{q-2n}{4}, & \xi^V \text{ is a space-like vector} \\ \frac{q-2n-1}{4}, & \xi^V \text{ is a time-like vector} \end{matrix} \tag{4.16}$$

Proof. From (4.2) and (4.14), we have

$$S(\xi^V, \xi^V) = \sum_{i=1}^{2n} G(R(E_i^V, \xi^V)\xi^V, E_i^V) = -\frac{1}{4}\sum_{i=1}^{2n} G(E_i^V, E_i^V).$$

Since $F^{2n+1} = (M, M^0, F^*)$ is an indefinite Finsler manifold of index q, if $G(\xi^V, \xi^V) = \varepsilon = 1$, that is ξ^V is a space-like vector, we obtain

$$S(\xi^V, \xi^V) = \frac{q-2n}{4}.$$

If $G(\xi^V, \xi^V) = \varepsilon = -1$, that is ξ^V is a time-like vector we obtain

$$S(\xi^V, \xi^V) = \frac{q-2n-1}{4}.$$

□

Lemma 4.8. *The horizontal Ricci tensor S^H of an indefinite Kenmotsu Finsler vector bundle $(M^0)^h$ and the vertical Ricci tensor S^V of an indefinite Kenmotsu Finsler vector bundle $(M^0)^v$ satisfy the following equations:*

$$S(X^H, \xi^H) = \begin{cases} \left(\frac{-2n+q}{4}\right)\eta^H(X^H), & \xi^H \text{ is a space-like vector} \\ \left(\frac{-2n+q-1}{4}\right)\eta^H(X^H), & \xi^H \text{ is a time-like vector} \end{cases}, \tag{4.17}$$

$$S(X^V, \xi^V) = \begin{cases} \left(\frac{-2n+q}{4}\right)\eta^V(X^V), & \xi^V \text{ is a space-like vector} \\ \left(\frac{-2n+q-1}{4}\right)\eta^V(X^V), & \xi^V \text{ is a time-like vector} \end{cases} \tag{4.18}$$

$$S(X, \xi) = \begin{cases} \left(\frac{-2n+q}{4}\right)\eta(X), & \xi \text{ is a space-like vector} \\ \left(\frac{-2n+q-1}{4}\right)\eta(X), & \xi \text{ is a time-like vector} \end{cases} \tag{4.19}$$

Proof. Let ξ^H is a space like vector. By using (4.4) and (4.13) we obtain

$$S(X^H,\xi^H)=\sum_{i=1}^{2n}G(R(E_i^H,X^H)\xi^H,E_i^H)+G(R(\xi^H,X^H)\xi^H,\xi^H)$$
$$=-\frac{1}{4}\sum_{i=1}^{2n}G(\eta^H(X^H)E_i^H-\eta^H(E_i^H)X^H,E_i^H)-\frac{1}{4}G(\eta^H(X^H)\xi^H-\eta^H(\xi^H)X^H,\xi^H)$$
$$=-\frac{1}{4}\{\sum_{i=1}^{2n}\eta^H(X^H)G(E_i^H,E_i^H)\}-\frac{1}{4}\{\eta^H(X^H)G(\xi^H,\xi^H)-\varepsilon\eta^H(X^H)\}$$
$$=-\frac{2n-q}{4}\eta^H(X^H)=\frac{q-2n}{4}\eta^H(X^H)$$

If ξ^H is a time-like vector, we obtain

$$S(X^H,\xi^H)=(\frac{q-2n-1}{4})\eta^H(X^H)$$

Similarly, we obtain, if ξ^V is a space-like vector,

$$S(X^V,\xi^V)=(\frac{q-2n}{4})\eta^V(X^V).$$

If ξ^V is a time-like vector.

$$S(X^V,\xi^V)=(\frac{q-2n-1}{4})\eta^V(X^V).$$

□

5. Conclusion

Let $(M,\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ be an $(2n+1)$- dimensional Kenmotsu pseudo-metric manifold, where $\bar{\phi}$ is a $(1,1)$ tensor field, $\bar{\eta}$ is a 1-form, $\bar{g}$ is the pseudo-Riemannian metric on M and $F^{2n+1}=(M,M^0,F^*)$ is an indefinite Finsler manifold. It is well known that $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ satisfy

$$\bar{\eta}(\bar{\xi})=1$$
$$\bar{\phi}^2\bar{X}=-\bar{X}+\bar{\eta}(\bar{X})\bar{\xi}$$
$$\bar{\phi}\bar{\xi}=0$$
$$\bar{\eta}(\bar{\phi}\bar{X})=0$$
$$rank\bar{\phi}=2n$$
$$\bar{g}(\bar{\phi}\bar{X},\bar{\phi}\bar{Y})=\bar{g}(\bar{X},\bar{Y})-\varepsilon\bar{\eta}(\bar{X})\bar{\eta}(\bar{Y})$$
$$\bar{\eta}(\bar{X})=\varepsilon\bar{g}(\bar{X},\bar{\xi}),\bar{g}(\bar{\xi},\bar{\xi})=\varepsilon$$
$$\nabla_{\bar{X}}\bar{\xi}=-\bar{\phi}^2\bar{X}$$
$$(\nabla_{\bar{X}}\bar{\phi})\bar{Y}=\varepsilon\bar{g}(\bar{\phi}\bar{X},\bar{Y})\bar{\xi}-\bar{\eta}(\bar{Y})\bar{\phi}\bar{X}$$
$$(\nabla_{\bar{X}}\bar{\eta})\bar{Y}=\bar{g}(\bar{X},\bar{Y})-\varepsilon\bar{\eta}(\bar{X})\bar{\eta}(\bar{Y})$$
$$R(\bar{X},\bar{Y})\bar{\xi}=\bar{\eta}(\bar{X})\bar{Y}-\bar{\eta}(\bar{Y})\bar{X}$$
$$R(\bar{X},\bar{Y})\bar{Z}=-\varepsilon\{\bar{g}(\bar{Y},\bar{Z})\bar{X}-\bar{g}(\bar{X},\bar{Z})\bar{Y}\}$$

$$S(\bar{X},\bar{\xi}) = \begin{cases} (q-2n)\,\eta(X), & \bar{\xi} \text{ is a space-like vector} \\ (q-2n-1)\,\eta(X), & \bar{\xi} \text{ is a time-like vector} \end{cases},$$
$$S(\bar{\xi},\bar{\xi}) = \begin{cases} (q-2n)\,, & \bar{\xi} \text{ is a space-like vector} \\ (q-2n-1)\,, & \bar{\xi} \text{ is a time-like vector} \end{cases}$$

for any vector fields $\bar{X},\bar{Y},\bar{Z} \in T_xM$, where ∇ is the Levi-Civita connection of pseudo-Riemannian metric $\bar{g}$, R is the Riemannian curvature tensor, S is the Ricci tensor.
For the Kenmotsu pseudo-metric Finsler structures $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ on subbundles $(M^0)^h$ and $(M^0)^v$, the following relations hold;

$$\eta^H(\xi^H)=1, \eta^V(\xi^V)=1$$
$$\phi^H(\xi^H)=0, \phi^V(\xi^V)=0$$
$$\eta^H(\phi^H X^H)=0, \eta^V(\phi^V X^V)=0$$
$$(\phi^H)^2(X^H)=-X^H+\eta^H(X^H)\xi^H, (\phi^V)^2(X^V)=-X^V+\eta^V(X^V)\xi^V$$
$$\eta^H(X^H)=\varepsilon G^H(X^H,\xi^H), \eta^V(X^V)=\varepsilon G^V(X^V,\xi^V)$$
$$G^H(\phi X^H,\phi Y^H)=G^H(X^H,Y^H)-\varepsilon\eta^H(X^H)\eta^H(Y^H)$$
$$G^V(\phi X^V,\phi Y^V)=G^V(X^V,Y^V)-\varepsilon\eta^V(X^V)\eta^V(Y^V)$$
$$(\nabla_{X^H}\phi^H)Y^H=\frac{1}{2}\{\varepsilon G^H(\phi X^H,Y^H)\xi^H-\eta^H(Y^H)\phi X^H\}$$
$$(\nabla_{X^V}\phi^V)Y^V=\frac{1}{2}\{\varepsilon G^V(\phi X^V,Y^V)\xi^V-\eta^V(Y^V)\phi X^V\}$$
$$(\nabla_{X^H}\eta^H)Y^H=\frac{1}{2}G(\phi X^H,\phi Y^H)$$
$$(\nabla_{X^V}\eta^V)Y^V=\frac{1}{2}G(\phi X^V,\phi Y^V)$$
$$\nabla_{X^H}\xi^H=-\frac{1}{2}\phi^2 X^H$$
$$\nabla_{X^V}\xi^V=-\frac{1}{2}\phi^2 X^V$$
$$R(X^H,Y^H)\xi^H=\frac{1}{4}\{\eta^H(X^H)Y^H-\eta^H(Y^H)X^H\}$$
$$R(X^V,Y^V)\xi^V=\frac{1}{4}\{\eta^V(X^V)Y^V-\eta^V(Y^V)X^V\}$$

$$S(X^H,\xi^H) = \begin{cases} \left(\frac{-2n+q}{4}\right)\eta^H(X^H), & \xi^H \text{ is a space-like vector} \\ \left(\frac{-2n+q-1}{4}\right)\eta^H(X^H), & \xi^H \text{ is a time-like vector} \end{cases},$$
$$S(X^V,\xi^V) = \begin{cases} \left(\frac{-2n+q}{4}\right)\eta^V(X^V), & \xi^V \text{ is a space-like vector} \\ \left(\frac{-2n+q-1}{4}\right)\eta^V(X^V), & \xi^V \text{ is a time-like vector} \end{cases}$$

$$S(\xi^H,\xi^H) = \begin{cases} \left(\frac{-2n+q}{4}\right), & \xi^H \text{ is a space-like vector} \\ \left(\frac{-2n+q-1}{4}\right), & \xi^H \text{ is a time-like vector} \end{cases},$$
$$S(\xi^V,\xi^V) = \begin{cases} \left(\frac{-2n+q}{4}\right), & \xi^V \text{ is a space-like vector} \\ \left(\frac{-2n+q-1}{4}\right), & \xi^V \text{ is a time-like vector} \end{cases}$$

$$R(X^H, Y^H)Z^H = -\frac{\varepsilon}{4}\{G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H\}$$
$$R(X^V, Y^V)Z^V = -\frac{\varepsilon}{4}\{G(Y^V, Z^V)X^V - G(X^V, Z^V)Y^V\}$$

where $X^H, Y^H, Z^H \in (TM^0)^H$ and $X^V, Y^V, Z^V \in (TM^0)^V$.

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