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MULTI-VALUED VARIATIONAL INCLUSION PROBLEM IN HADAMARD MANIFOLDS

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Abstract. We consider a multi-valued variational inclusion problem in Hadamard manifold and study the Korpelevich-type algorithm to estimate the approximate solution of a multi-valued variational inclusion problem. We used the properties of the multi-valued monotone vector field to prove that the sequence generated by the proposed algorithm converges to the solution of multi-valued variational inclusion problem. An example is also presented in support of our problem. The results presented in this paper improve and generalize some known results given in the literature.

1. Introduction

Variational inequalities in Euclidean spaces are very powerful tool for studying optimization problems, equilibrium problems, problems of finding zero of operators as well as complementary problems and have been studied extensively, see for example [1, 2, 3, 4, 5, 6, 7]. Modern interests are concentrated on extending some classical and important results from linear spaces to nonlinear spaces [8, 9]. Therefore, the span of idea and techniques of the theory of variational inequalities and related topics from Euclidean spaces to Riemannian manifold are logical and fascinating.

Variational inequalities on Hadamard manifold were first introduced and studied in [10]: Find $x \in D$ such that

$$\langle F(x), \exp_x^{-1} y\rangle \geq 0, \ \forall \ y \in D,$$

where D is nonempty closed, convex subset of Hadamard manifold $\mathcal{M}$. $F : D \to T\mathcal{M}$ is vector field, that is $F(x) \in T_x\mathcal{M}$ for each $x \in D$ and $\exp^{-1}$ is the inverse of exponential mapping. Li et al. [11] extend the variational inequality problem from Hadamard manifold to Riemannian manifolds.

Fang et. al. [12] extend the work of Németh to study a multi-valued pseudomonotone variational inequality problem in Hadamard manifold: Find $x \in D$

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and $u \in F(x)$ such that

$$\langle u,\ \exp_x^{-1} y\rangle \geq 0,\ \forall\ y \in D,$$

where $F : D \rightrightarrows T\mathcal{M}$ is a multi-valued vector field, that is $F(x) \subseteq T_x\mathcal{M}$ for each $x \in D$. Later Jana and Nahak [13] also studied multi-valued variational inequality problem in Hadamard manifold with similar assumption on F.

An natural generalization of variational inequalities are variational inclusions. For a given multi-valued maximal monotone operator $G : \mathcal{H} \rightrightarrows \mathcal{H}$, the variational inclusion problem on Hilbert space $\mathcal{H}$ is to find $x \in D$ such that

$$0 \in f(x) + G(x),$$

where $f : \mathcal{H} \to \mathcal{H}$ be any single-valued operator. Due to the fact that the zeros of maximal monotone operator are the fixed point sets of resolvent operator, the resolvent associated to a multi-valued maximal monotone operator plays an important role to find the zeros of monotone operators. Many authors have discussed how to find the zeros of monotone operator, see for example [14, 15, 16, 17, 18, 19, 20].

Recently, many authors have extended the results related to the zero of monotone operators from linear spaces to Riemannian manifold. Li. et al. [21] proved the convergence of proximal point algorithm on Hadamard manifolds. The idea of firmly nonexpansive and resolvent of multi-valued monotone vector field are introduced in [22]. Furthermore, Tang and Huang [23] studied a variant of Korpelevich's method for pseudomonotone variational inequalities. Recently, Ansari et. al [24] introduced Korpelevich's method for variational inclusion problems on Hadamard manifolds.

Motivated by the work of Fang, Tang and Ansari et al., our motive in this work is to study the solution of following multi-valued variational inclusion problem in Hadamard manifolds: Find $x \in D$ such that $u \in F(x)$ and

$$0 \in u + G(x), \tag{1.1}$$

where $F, G : \mathcal{M} \rightrightarrows \mathcal{M}$ are two multi-valued monotone vector fields.

2. Preliminaries

Let $\mathcal{M}$ be a finite dimensional differentiable manifold. For a given $x \in \mathcal{M}$, the tangent space of $\mathcal{M}$ at x is denoted by $T_x\mathcal{M}$ and the tangent bundle is denoted by $T\mathcal{M} = \cup_{x\in\mathcal{M}} T_x\mathcal{M}$, which is naturally a manifold. An inner product $\Re_x(.,.)$ on $T_x\mathcal{M}$ is called the Riemannian metric on $T_x\mathcal{M}$. A tensor field $\Re(.,.)$ is said to be Riemannian metric on $\mathcal{M}$ if for every $x \in \mathcal{M}$, the tensor $\Re_x(.,.)$ is a Riemannian metric on $T_x\mathcal{M}$. The norm corresponding to the inner product on $T_x\mathcal{M}$ is denoted by $\|.\|_x$. A differentiable manifold $\mathcal{M}$ endowed with the Riemannian metric $\Re(.,.)$ is called a Riemannian manifold. Given a piecewise smooth curve $\gamma : [a, b] \to \mathcal{M}$ joining x to y ($i.e., \gamma(a) = x$ and $\gamma(b) = y$), we can define the length of γ by $L(\gamma) = \int_a^b \|\gamma^{'}(t)\|dt$. The Riemannian distance $d(x, y)$, which included the original topology on $\mathcal{M}$, is the minimal length over the set of all such curves joining x to y.

Let Δ be the Levi-Civita connection associated with Riemannian manifold $\mathcal{M}$. Let γ be a smooth curve on $\mathcal{M}$. A vector field X is said to be parallel along γ if $\Delta_{\gamma'} X = 0$. If $\gamma^{'}$ is parallel along γ, i.e., $\Delta_{\gamma'}\gamma^{'} = 0$, then $\gamma^{'}$ is said to be geodesic and in this case $\|\gamma^{'}\|$ is a constant. When $\|\gamma^{'}\| = 1$, γ is said to be normalized. A geodesic joining x and y in $\mathcal{M}$ is said to be minimal geodesic if its length equal to $d(x, y)$.

A Riemannian manifold is complete if for any $x \in \mathcal{M}$, all geodesic emanating from x are defined for all $t \in (-\infty, \infty)$. By the Hopf-Rinow Theorem [25], we know that if $\mathcal{M}$ is complete, then any pair of point in $\mathcal{M}$ can be joined by a minimal geodesic. Furthermore, $(\mathcal{M}, d)$ is a complete metric space, and hence, every bounded closed subset is compact.

Assuming $\mathcal{M}$ is complete, the exponential mapping $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ at x is defined by $\exp_x(v) = \gamma_v(1, x)$ for each $v \in T_x\mathcal{M}$, where $\gamma(.) = \gamma_v(., x)$ is the geodesic starting at x with velocity v ($i.e.\gamma(0) = 0$ and $\gamma^{'}(0) = v$). It is known that $\exp_x(tv) = \gamma_v(t, x)$ for each real number t.

The parallel transport on the tangent bundle $T\mathcal{M}$ along with γ with respect to Δ, is denoted by $\mathcal{P}_{\gamma,.,.}$ and is defined as

$$\mathcal{P}_{\gamma,\gamma(a),\gamma(b)}(v) = V(\gamma(b)), \ \forall \ a, b \in \mathbb{R} \text{ and } T_{\gamma(a)}\mathcal{M},$$

where V is a unique vector field satisfying $\Delta_{\gamma^{'}(t)}V = 0$ for all t and $V(\gamma(a)) = v$. Then for any $a, b \in \mathbb{R}, \mathcal{P}_{\gamma,\gamma(a),\gamma(b)}$ is an isometry from $T_{\gamma(a)}\mathcal{M}$ to $T_{\gamma(b)}\mathcal{M}$. When γ is a minimal geodesic joining x to y, we write $\mathcal{P}_{y,x}$ instead of $\mathcal{P}_{\gamma,y,x}$.

A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard manifold. Throughout the remainder of the paper, we will assume that $\mathcal{M}$ is a finite dimensional Hadamard manifold with constant curvature.

Proposition 2.1. *([25]) Let $\mathcal{M}$ be a Hadamard manifold and $x \in \mathcal{M}$. Then $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ is a diffeomorphism and for any two points x and $y \in \mathcal{M}$, there exists a unique normalized geodesic joining x to y, which is in fact a minimal geodesic.*

If $\mathcal{M}$ be a finite dimensional manifold with dimension n, then above proposition shows that $\mathcal{M}$ is diffeomorphism to the Euclidean space $\mathcal{R}^n$. Thus we see that $\mathcal{M}$ has the same topology and differential structure as $\mathcal{R}^n$. Moreover, Hadamard manifold and Euclidean space have some similar geometrical properties. We describe some of them in the following results.

Recall that a geodesic triangle $\Delta(x_1, x_2, x_3)$ of Riemannian manifold is a set consisting of three points x_1, x_2 and x_3 and the three minimal geodesic γ_i joining x_i to x_{i+1}, where $\mathrm{i} = 1, 2, 3 \bmod (3)$.

Proposition 2.2. *(Comparison Theorem for Triangle) ([25]) Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $\mathrm{i} = 1, 2, 3 \bmod (3)$, by $\gamma_i : [0, l_i] \to \mathcal{M}$ geodesic joining x_i to x_{i+1} and set $l_i = L(\gamma_i), \alpha_1 = \angle(\gamma_i^{'}(0), -\gamma_{i-1}^{'}(l_{i-1}))$. Then*

$$l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2.$$

In terms of distance and exponential mapping, above inequality can be rewritten as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \rangle \leq d^2(x_{i-1}, x_i), \tag{2.1}$$

since

$$\langle \exp^{-1}_{x_{i+1}} x_i, \exp^{-1}_{x_{i+1}} x_{i+2} \rangle = d(x_i, x_{i+1}) d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}.$$

A subset $D \subset \mathcal{M}$ is said to be convex if for any two points $x, y \in D$, the geodesic joining x and y is contained in D, that is, if $\gamma : [a, b] \to \mathcal{M}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma(1-t)a + tb \in D$ for all $t \in [0, 1]$. From now on, $D \subset \mathcal{M}$ will denote a nonempty, closed and convex subset of a Riemannian manifold. The projection of v onto D is defined by

$$P_D(v) = \{u \in D : d(v, u) \leq d(v, w),\ \forall\ w \in D\},\ \forall\ v \in \mathcal{M}. \tag{2.2}$$

Lemma 2.3. *([21]) Let $x_0 \in \mathcal{M}$ and $\{x_n\} \subset \mathcal{M}$ with $x_n \to x_0$. Then the following assertion holds:*

(i) *For any $y \in \mathcal{M}$, we have*

$$\exp^{-1}_{x_n} y \to \exp^{-1}_{x_0} y \text{ and } \exp^{-1}_{y} x_n \to \exp^{-1}_{y} x_0$$

(ii) *If $v_n \in T_{x_n}\mathcal{M}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathcal{M}$.*

(iii) *Given $u_n, v_n \in T_{x_n}\mathcal{M}$ and $u_0, v_0 \in T_{x_0}\mathcal{M}$, if $u_n \to u_0$ and $v_n \to v_0$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.*

(iv) *For any $u \in T_{x_0}\mathcal{M}$, the function $F : \mathcal{M} \to T\mathcal{M}$ defined by $F(x) = \mathcal{P}_{x,x_0} u$ for each $x \in \mathcal{M}$ is continuous on $\mathcal{M}$.*

Lemma 2.4. [17] *Let $\mathcal{M}$ be a Riemannian manifold with constant curvature. For given $x \in \mathcal{M}$ and $u \in T_x\mathcal{M}$, the set*

$$L_{x,u} = \{y \in \mathcal{M} : \mathcal{R}(\exp^{-1}_x y, u) \leq 0\},$$

is convex.

Proposition 2.5. *([26]) If $x \in \mathcal{M}$ and P_D is singleton, then*

$$\Re\Big(\exp^{-1}_{P_D(x)} x,\ \exp^{-1}_{P_D(x)} y\Big) \leq 0,\ \forall\ y \in \mathcal{M}.$$

Lemma 2.6. *([23]) Let D be nonempty closed convex subset of $\mathcal{M}$. Then*

$$d^2(P_D(x), x^*) \leq d^2(x, x^*) - d^2(x, P_D(x)),\ \forall\ x \in \mathcal{M}, x^* \in D.$$

The set of all single-valued vector fields on $\mathcal{M}$ is denoted by $\Omega(\mathcal{M})$. We denote the set of all multi-valued vector fields on $\mathcal{M}$ by $\chi(\mathcal{M})$. Let $G \in \mathcal{M}$, then $G \rightrightarrows T\mathcal{M}$ such that $G(x) \subseteq T_x(\mathcal{M})$ for all $x \in \mathcal{D}(G)$, where $\mathcal{D}(G)$ is the domain of G defined as $\mathcal{D}(G) = \{x \in \mathcal{M} : G(x) \neq \phi\}$.

Definition 2.7. *A vector field $F \in \Omega(\mathcal{M})$ is said to be*

(i) *monotone if for all $x, y \in \mathcal{M}$,*

$$\Re\big(F(x), \exp^{-1}_x y\big) \leq \Re\big(F(y), -\exp^{-1}_y x\big);$$

(ii) *pseudomonotone if for all $x, y \in \mathcal{M}$,*

$$\Re\big(F(x), \exp^{-1}_x y\big) \geq 0 \Rightarrow \Re\big(F(y), \exp^{-1}_y x\big) \leq 0;$$

Definition 2.8. *A vector field $G \in \chi(\mathcal{M})$ is said to be*

(i) *monotone if for all $x, y \in \mathcal{D}(\mathcal{M})$,*

$$\Re\big(u, \exp^{-1}_x y\big) \leq \Re\big(v, -\exp^{-1}_y x\big),\ \forall\ u \in G(x), v \in G(y);$$

(ii) *pseudomonotone if for all* $x, y \in \mathcal{D}(\mathcal{M})$ *and* $\forall\ u \in G(x)$ *and* $\forall\ v \in G(y)$

$$\Re\big(u, \exp_x^{-1} y\big) \geq 0 \Rightarrow \Re\big(v, \exp_y^{-1} x\big) \leq 0;$$

(iii) *maximal monotone if it is a monotone and for all* $x \in \mathcal{M}$ *and all* $u \in T_x\mathcal{M}$, *the condition*

$$\Re\big(u, \exp_x^{-1} y\big) \leq \Re\big(v, -\exp_y^{-1} x\big),\ \forall\ y \in \mathcal{D}(G), v \in G(y),$$

implies that $u \in G(x)$.

Definition 2.9. *([21]) Given* $\lambda > 0$ *and* $G \in \chi(\mathcal{M})$, *the resolvent of* G *of order* λ *is a multi-valued mapping* $J_\lambda^G : \mathcal{M} \to \mathcal{D}(\mathcal{M})$ *defined by*

$$J_\lambda^G(x) = \{z \in \mathcal{M} : x \in \exp_z \lambda G(z)\},\ \forall\ x \in \mathcal{M}.$$

Definition 2.10. *([11]) Let* $F \in \chi(\mathcal{M})$, *then* F *is said to be*

(i) *lower semicontinuous at* x_0 *if given any sequence* $\{x_k\} \subseteq \mathcal{D}(F)$ *converging to* x *and* $y \in F(x)$, *there exists a sequence* $\{y_k\} \subseteq T\mathcal{M}$ *satisfying* $\{y_k\} \in F(x_k)$ *that converges to* y.

(ii) *upper Kuratowski semicontinuous at* x_0 *if given any sequence* $\{x_k\} \subseteq \mathcal{D}(F)$ *and* $\{u_k\} \subseteq T\mathcal{M}$ *with each* $\{u_k\} \in F(x_k)$, *the relation* $\lim_{k\to\infty} x_k = x_0$ *and* $\lim_{k\to\infty} u_k = u_0$ *imply that* $u_0 \in F(x_0)$.

(iii) *lower semicontinuous (upper Kuratowski semicontinuous) on* D *if it is lower semicontinuous (upper Kuratowski semicontinuous) at each point* $x \in \mathcal{D}(F)$.

Theorem 2.11. *([21]) Let* $\lambda > 0$ *and* $G \in \chi(\mathcal{M})$. *The the following assertion holds:*

(i) *The vector field* G *is monotone if and only if* J_λ^G *is single-valued and firmly nonexpansive.*

(ii) *If* $\mathcal{D}(G) = \mathcal{M}$, *the vector field* G *is maximal if and only if* J_λ^G *is single-valued and firmly nonexpansive and domain* $D(J_\lambda^G) = \mathcal{M}$.

Németh [10] present the following version of Brouwer's fixed point theorem in the setting of Hadamard manifolds.

Lemma 2.12. *If* D *be a compact subset of* $\mathcal{M}$, *then every continuous function* $f : D \to D$ *has a fixed point.*

Definition 2.13. *([27]) Let* X *be a complete matric space and* $D \subset X$ *be a nonempty set. A sequence* $\{x_n\} \subset X$ *is called Fejér convergent to* D *if for all* $y \in D$

$$d(x_{n+1}, y) \leq d(x_n, y)\ \forall\ n \in \mathbb{N}.$$

Lemma 2.14. *([27]) Let* X *be a complete matric space. If and* $\{x_n\} \subset X$ *is a Fejér convergent to a nonempty set* $D \subset X$, *then* $\{x_n\}$ *is bounded. Moreover, if a cluster point* x *of* $\{x_n\}$ *belong to* D, *then* $\{x_n\}$ *converges to* x.

3. Main Results

We denote the solution set of problem (1.1) by $S = \{x \in \mathbb{M} : u \in F(x)$ and $0 \in u + G(x)\}$.

Lemma 3.1. *If $G \in \chi(\mathcal{M})$ is a monotone vector field on D and F is any vector filed on D, then for any $x \in \mathcal{M}$*

$$d^2\Big(x, J_\lambda^G(\exp_x(-\lambda u_x))\Big) \leq -\lambda \Re\Big(u_x + v_x, \exp_x^{-1}\big[J_\lambda^G(\exp_x(-\lambda u_x))\big]\Big), \tag{3.1}$$

where $u_x \in F(x)$ and $v_x \in G(x)$.

Proof. Let $x \in \mathcal{M}$ and $u_x \in F(x)$. Consider the geodesic triangle $\triangle(x, y, z)$, where $z = \exp_x(-\lambda u_x)$ and $y = J_\lambda^G(z)$. From inequality (1.2), we have

$$d^2(x, y) + d^2(z, y) - 2\Re\big(\exp_y^{-1} x, \exp_y^{-1} z\big) \leq d^2(x, z), \tag{3.2}$$

and

$$d^2(x, y) + d^2(x, z) - 2\Re\big(\exp_x^{-1} z, \exp_x^{-1} y\big) \leq d^2(z, y). \tag{3.3}$$

Since $y = J_\lambda^G(z)$, this implies that $\frac{1}{\lambda}\exp_y^{-1} z \in G(y)$. By monotonicity of G, for all $v_x \in G(x)$, we have

$$\Re\Big(\frac{1}{\lambda}\exp_y^{-1} z, \exp_y^{-1} x\Big) \leq \Re\Big(v_x, -\exp_x^{-1} y\Big). \tag{3.4}$$

Combining (3.2) and (3.3), we have

$$d^2(x, y) \leq -\lambda \Re\Big(u_x, \exp_x^{-1} y\Big) + \Re\Big(\exp_y^{-1} z, \exp_y^{-1} x\Big). \tag{3.5}$$

From (3.4) and (3.5), we have

$$d^2(x, y) \leq -\lambda \Re\Big(u_x, \exp_x^{-1} y\Big) + \lambda \Re\Big(v_x, -\exp_x^{-1} y\Big), \tag{3.6}$$

that is

$$d^2\Big(x, J_\lambda^G(\exp_x(-\lambda u_x))\Big) \leq -\lambda \Re\Big(u_x + v_x, \exp_x^{-1} y\Big). \tag{3.7}$$

This completes the proof. $\square$

Following proposition can be seen as an extended version of Proposition 2.6 of [13] and Proposition 2.4 of [12].

Proposition 3.2. *Let $G \in \chi(\mathbb{M})$ such that G is monotone and $x \in D$ and $u \in F(x)$. The following statement are equivalent:*

(i) *x is a solution of problem (1.1).*
(ii) *$x = J_\lambda^G(\exp_x(-\lambda u))$, for all $\lambda > 0$.*
(iii) *$r_\lambda(x, u) = 0$, where $r_\lambda(x, u)$ is defined by*

$$r_\lambda(x, u) = \exp_x^{-1}\big[J_\lambda^G(\exp_x(-\lambda u))\big].$$

Proof. $(i) \Leftrightarrow (ii)$

$$\begin{aligned} x &= J_\lambda^G(\exp_x(-\lambda u)) \\ \Leftrightarrow\ & \exp_x(-\lambda u) \in \exp_x(\lambda G(x)) \\ \Leftrightarrow\ & -\lambda u \in \lambda G(x) \\ \Leftrightarrow\ & 0 \in u + G(x) \\ \Leftrightarrow\ & x \textit{ is a solution of problem } (1) \end{aligned}$$

$(ii) \Leftrightarrow (iii)$ It follows directly by the definition of exponential mapping. $\square$

Proposition 3.3. *Let D be a nonempty bounded closed and convex subset of Hadamard manifold $\mathcal{M}$ with constant curvature. If $F \in \chi(\mathcal{M})$ such that $F(x)$ compact and convex for each $x \in \mathcal{M}$. $G \in \chi(\mathcal{M})$ is a maximal monotone vector field on D, then problem (1.1) has a solution.*

Proof. K is compact convex subset of Hadamard manifold by Hopf-Rinow Theorem on $\mathcal{M}$ and $F(x)$ is compact convex valued for each $x \in K$. Since G is maximal monotone, hence J_λ^G is single valued. Since $J_\lambda^G(-\lambda(\cdot))$ is continuous with compact domain. Therefore by Lemma 2.4, $J_\lambda^G(-\lambda(\cdot))$ has a fixed point. In view of Proposition 3.2, the proof is complete. □

Now, we describe Korpelevich-type algorithm to compute the approximate solution of multi-valued variational inclusion problem (1.1).

Algorithm 3.4. *Let D be a nonempty bounded, closed and convex subset of Hadamard Manifold $\mathbb{M}$, $F \in \chi(\mathbb{M})$ be a monotone vector field and $G \in \chi(\mathbb{M})$ be a maximal monotone vector field on D.*
Step 0. *Choose any $\lambda > 0, \zeta > 1$, $t \in (0,1)$ and initial point $x_0 \in D$*
Set k=0
Step 1. *Compute $r_\lambda(x_k, u)$. If $r_\lambda(x_k, u) = 0$ for some $u \in F(x_k)$ then stop. Otherwise take arbitrary $u_k \in F(x_k)$, compute*

$$\gamma_k(t) = \exp_{x_k} t \, \exp_{x_k}^{-1}\Big[J_\lambda^G(\exp_{x_k}(-\lambda u_k))\Big] \tag{3.8}$$

and

$$y_k = \gamma_k(\mu_k)$$

where

$$\mu_k = \zeta^{-j(k)}$$

with

$$\begin{aligned} j(k) = \min\Bigg\{ j \in \mathbb{N}_+ : \Re\Big(u_{\gamma_k(\zeta^{-j})} + v_{\gamma_k(\zeta^{-j})},\ \gamma_k'(\zeta^{-j})\Big) \\ \leq -\frac{1}{\lambda} d^2\Big(x_k,\ J_\lambda^G(\exp_{x_k}(-\lambda u_k))\Big)\Bigg\} \end{aligned} \tag{3.9}$$

where $u_{\gamma_k(\zeta^{-j})} \in F(\gamma_k(\zeta^{-j}))$ and $v_{\gamma_k(\zeta^{-j})} \in G(\gamma_k(\zeta^{-j}))$. Let $u_{y_k} \in F(y_k)$ and $v_{y_k} \in G(y_k)$. Compute

$$\mathrm{Q_k} = \bigcap_{u_{y_k} \in F(y_k), v_{y_k} \in G(y_k)} \Big\{ x \in \mathbb{M} : \Re\big(u_{y_k} + v_{y_k},\ \exp_{y_k}^{-1} x\big) \leq 0 \Big\}, \tag{3.10}$$

define

$$x_{k+1} = \mathbf{P}_{\mathrm{Q_k}} x_k. \tag{3.11}$$

Update k=k+1 and return to **Step 1**.

From now on, we adopt the following assumptions:

(C1) F is upper Kuratowski semicontinuous and lower semicontinuous vector filed such that $F(x)$ is compact valued for all $x \in D$.
(C2) F is bounded on bounded sets.

Following proposition proved that the Algorithm 3.4 is well defined.

Proposition 3.5. *Let $\{x_k\}$ and $\{y_k\}$ are the sequences defined in Algorithm (3.4), then the following declaration hold:*

(i) *If $r_\lambda(x_k, u_k) = 0$, then current term x_k is a solution of problem (1.1).*
(ii) *If $r_\lambda(x_k, u_k) \neq 0$ then $j(k)$ is well defined and $y_k \in D$.*
(iii) $\mathrm{Q_k}$ *is nonempty, closed and convex and x_{k+1} is well defined.*

Proof. (i) This proof is trivial and can be obtained directly using Proposition 3.2. (ii) Suppose that for all k, $u_k \in F(x_k)$. We have

$$\gamma_k^{'}(t) \quad = \quad \mathcal{P}_{\gamma_k(t),x_k}\exp_{x_k}^{-1}\big[J_\lambda^G \exp_{x_k}(-\lambda u_k)\big].$$

Since the parallel transport is an isometry and using Lemma 2.3(iv), we have

$$\begin{aligned}\lim_{j\to\infty}\gamma_k^{'}(\zeta^{-j}) &= \lim_{j\to\infty}\mathcal{P}_{\gamma_k(\zeta^{-j}),x_k}\exp_{x_k}^{-1}\big[J_\lambda^G \exp_{x_k}(-\lambda u_k)\big]\\ &= \exp_{x_k}^{-1}\big[J_\lambda^G \exp_{x_k}(-\lambda u_k)\big]\\ &= r_\lambda(x_k,u_k).\end{aligned}$$

Since F is lower semicontinuous, $u_k \in F(x_k)$ and $\lim\limits_{j\to\infty}\gamma_k(\zeta^{-j}) = x_k$, there exists a sequence $u_{k_j} \in F(\gamma_k(\zeta^{-j})$ such that $\lim\limits_{j\to\infty} u_{k_j} = u_{x_k}$, therefore for $v_{k_j} \in G(\gamma_k(\zeta^{-j})$ and each j we have

$$\begin{aligned}\lim_{j\to\infty}\Re\big(u_{\gamma_k(\zeta^{-j})}+v_{\gamma_k(\zeta^{-j})},\gamma_k^{'}(\zeta^{-j})\big) &= \Re\bigg(u_k+v_k,\exp_{x_k}^{-1}\big[J_\lambda^G\exp_{x_k}(-\lambda u_{x_k})\big]\bigg)\\ &\le -\frac{1}{\lambda}d^2\big(x_k,J_\lambda^G(\exp_{x_k}(-\lambda u_k))\big).\end{aligned}\tag{3.12}$$

If $r_\lambda(x_k, u_k) \neq 0$, then $d\big(x_k, J_\lambda^G(\exp_{x_k}(-\lambda u_k))\big) > 0$. It follows from the inequality that whatever we choose large j, the inequality (3.9) holds good. Thus $j(k)$ is well defined. Moreover $y_k = \gamma_k(\mu_k)$ is geodesic joining x_k and $J_\lambda^G(\exp_{x_k}(-\lambda u_k))$ and $x_k \in D$. By the definition of y_k and convexity of D, it follows that $y_k \in D$.
(iii) To show that x_{k+1} is well defined it is sufficient to show that $\mathrm{Q_k}$ is nonempty, closed and convex subset of Hadamard manifold. $\mathrm{Q_k}$ is closed by Lemma 2.3 (i) and $u_{y_k} + v_{y_k} \in T_{y_k}D$. In view of Lemma 2.4, we conclude that $\mathrm{Q_k}$ is convex being the intersection of convex sets and $y_k \in \mathrm{Q_k}$. This completes the proof. □

Theorem 3.6. *Let D be a nonempty bounded, closed and convex subset of Hadamard Manifold $\mathcal{M}$ with constant curvature. $F \in \chi(\mathcal{M})$ be a monotone vector field on D satisfying the condition C1, C2 and $G \in \chi(\mathcal{M})$ be a maximal monotone vector field on D. Then the sequence $\{x_k\}$ achieved by Algorithm (3.4) converges to a solution of problem (1.1).*

Proof. Let x^* is a solution of problem (1.1) such that $u^* \in F(x^*)$ and $0 \in u^* + G(x^*)$, that is $-u^* \in G(x^*)$. For any $x \in \mathcal{M}$, $v_x \in G(x)$ and using monotonicity of G, we have

$$\Re(v_x, \exp_x^{-1}x^*) \le \Re(u^*, \exp_{x^*}^{-1}x). \tag{3.13}$$

For any $u_x \in F(x)$, monotonicity of F implies that

$$\Re(u^*, \exp_{x^*}^{-1}x) \le \Re(u_x, -\exp_x^{-1}x^*). \tag{3.14}$$

Taking together (3.13) and (3.14), we have

$$\Re(u_x + v_x, \exp_x^{-1}x^*) \le 0.$$

In particular, $y_k \in \mathcal{M}, u_{y_k} \in F(y_k)$ and $v_{y_k} \in G(y_k)$, we have

$$\Re(u_{y_k} + v_{y_k}, \exp_{y_k}^{-1} x^*) \leq 0.$$

From (3.10), we conclude that $x^* \in \mathrm{Q}_k$ and $x_{k+1} = \mathrm{Q}_k x_k$. By Lemma 2.6, we have

$$d^2(x_{k+1}, x^*) + d^2(x_k, x_{k+1}) \leq d^2(x_k, x^*). \tag{3.15}$$

This implies that

$$d^2(x_{k+1}, x^*) \leq d^2(x_k, x^*). \tag{3.16}$$

Thus the sequence generated by Algorithm (3.4) is Fe$\acute{j}$er's convergent with respect to S. This implies that $\{x_k\}$ is bounded. Also from (3.15), since $\{x_k\}$ is bounded

$$d^2(x_k, x_{k+1}) \leq d^2(x_k, x^*) - d^2(x_{k+1}, x^*), \tag{3.17}$$

implies that $\{d(x_k, x^*)\}$ is nonincreasing and bounded and hence convergent. Therefore by (3.17), we have

$$\lim_{k \to \infty} d(x_{k+1}, x_k) = 0. \tag{3.18}$$

Boundedness of $\{x_k\}$ implies that there exists a subsequence $\{x_{k_j}\}$ converging to $\bar{x}$. Since F is bounded on bounded sets implies that $\{F(x_k) : k \in \mathbb{N}\}$ is bounded hence $u_k \in F(x_k)$ is also bounded, hence there exists a subsequence $\{u_{k_j}\}$ converging to $\bar{u}$. Furthermore, since J_λ^G is nonexpansive, we have $\{J_\lambda^G(\exp(-\lambda u_k))\}$ is also bounded and so $\{y_k\}$ is bounded.

To completes the proof, it is sufficient to show that any cluster point $\bar{x}$ of $\{x_k\}$ belong to S. We have $\lim_{j\to\infty} x_{k_j} = \bar{x}$. By (3.18), we can also have $\lim_{j\to\infty} x_{k_j+1} = \bar{x}$.

Since $\{\Re(u_{y_k}+v_{y_k}, \exp_{y_k}^{-1} x_k)\}$ is bounded, we can easily obtain that $\lim_{j\to\infty} \Re(u_{y_{k_j}} + v_{y_{k_j}}, \exp_{y_{k_j}}^{-1} x_{k_j})$ exists. From (3.9), we have

$$\begin{aligned} \Re(u_{y_k} + v_{y_k}, \gamma_k^{'}(\mu_k)) &\leq -\frac{1}{\lambda} d^2\big(x_k, J_\lambda^G(\exp_{x_k}(-\lambda u_k))\big) \\ \Re(u_{y_k} + v_{y_k}, -\mu_k \gamma_k^{'}(\mu_k)) &\geq \frac{\mu_k}{\lambda} d^2\big(x_k, J_\lambda^G(\exp_{x_k}(-\lambda u_k))\big). \end{aligned} \tag{3.19}$$

Define $\varphi(s) = \gamma_k(1-s)\mu_k, \ \forall\ s \in [0,1]$. Then $\varphi(s)$ is a geodesic joining y_k and x_k and

$$\varphi^{'}(s) = -\mu_k \gamma_k^{'}(\mu_k), \tag{3.20}$$

and $\varphi(s) = \exp_{y_k} s \ \exp_{y_k}^{-1} x_k, \ \forall\ s \in [0,1]$ is also a geodesic joining y_k to x_k and

$$\varphi^{'}(0) = -\exp_{y_k}^{-1} x_k. \tag{3.21}$$

From (3.19), (3.20) and (3.21), we conclude

$$\Re(u_{y_k} + v_{y_k}, \exp_{y_k}^{-1} x_k) \geq \frac{\mu_k}{\lambda} d^2\big(x_k, J_\lambda^G(\exp_{x_k}(-\lambda u_k))\big). \tag{3.22}$$

On the other hand

$$x_{k_j+1} \in \mathrm{Q}_{k_j} = \bigcap_{\substack{u_{y_{k_j}} \in F(y_{k_j}), \\ v_{y_{k_j}} \in G(y_{k_j})}} \Big\{ x \in \mathbb{M} : \Re\big(u_{y_{k_j}} + v_{y_{k_j}}, \exp_{y_{k_j}}^{-1} x\big) \leq 0 \Big\}, \tag{3.23}$$

we have $\lim_{j\to\infty} x_{k_j} = x_{k_{j+1}} = \bar{x}$. From (3.21) and Lemma 2.3 (i), we have

$$\begin{aligned}\lim_{j\to\infty} \Re\big(u_{y_{k_j}} + v_{y_{k_j}},\ \exp^{-1}_{y_{k_j}} x\big) &\leq \lim_{j\to\infty} \Re\big(u_{y_{k_j}} + v_{y_{k_j}},\ \exp^{-1}_{y_{k_j}} x_{k_j+1}\big)\\ &\leq 0. \end{aligned} \tag{3.24}$$

From (3.22) and (3.24), we have

$$\lim_{j\to\infty} \mu_{k_j} d(x_{k_j}, J^G_\lambda(\exp_{x_{k_j}}(-\lambda u_{k_j})) = 0. \tag{3.25}$$

Now, we have two possible cases.
Suppose first that $\mu_{k_j} \nrightarrow 0$, then there exists $\mu > 0$ such that $\mu_{k_j} > \mu$ for all j. Thus following (3.25), we have

$$\lim_{j\to\infty} d(x_{k_j}, J^G_\lambda(\exp_{x_{k_j}}(-\lambda u_{k_j})) = 0,$$

and so

$$d(\bar{x}, J^G_\lambda(\exp_{\bar{x}}(-\lambda\bar{u})) = 0,$$

that is $\bar{x} \in S$.
Suppose now that $\lim_{j\to\infty} d(x_{k_j}, J^G_\lambda(\exp_{x_{k_j}}(-\lambda u_{k_j})) \neq 0$, then $\lim_{j\to\infty} \mu_{k_j} = 0$. Again from (3.9), we have

$$\Re\big(u_{\gamma_{k_j}(\zeta^{-j})} + v_{\gamma_{k_j}(\zeta^{-j})}, \gamma'_{k_j}(\mu_{k_j})\big) > -\frac{1}{\lambda} d^2\bigg(x_{k_j}, J^G_\lambda(\exp_{x_{k_j}}(-\lambda u_{k_j}))\bigg), \tag{3.26}$$

for all $u_{\gamma_{k_j}} \in F(\gamma_{k_j}(\zeta^{-j})), v_{\gamma_{k_j}} \in G(\gamma_{k_j}(\zeta^{-j}))$. Taking into account that

$$\gamma'_{k_j}(t) = \mathcal{P}_{\gamma_{k_j}(t)x_{k_j}}\bigg\{\exp^{-1}_{x_{k_j}}\big[J^G_\lambda \exp_{x_{k_j}}(-\lambda u_{k_j})\big]\bigg\}, \tag{3.27}$$

we have for all $u_{\gamma_{k_j}} \in F(\gamma_{k_j}(\zeta^{-j}))$ and $v_{\gamma_{k_j}} \in G(\zeta^{-j}))$.

$$\begin{aligned}&\Re\big(u_{\gamma_{k_j}(\zeta^{-j})} + v_{\gamma_{k_j}(\zeta^{-j})}, \mathcal{P}_{\gamma_{k_j}(\mu_{k_j})x_{k_j}}\big\{\exp^{-1}_{x_{k_j}}\big[J^G_\lambda \exp_{x_{k_j}}(-\lambda u_{k_j})\big]\big\}\big)\\ &\qquad > -\frac{1}{\lambda} d^2\bigg(x_{k_j}, J^G_\lambda(\exp_{k_j}(-\lambda u_{k_j})\bigg).\end{aligned} \tag{3.28}$$

If $\lim_{j\to\infty} \gamma_{k_j}(\zeta^{-j}) = \bar{x}$. Since $\bar{u} \in F(\bar{x})$, the lower continuity of F implies the existence of $u_{k_j} \in F(\gamma_{k_j}(\zeta^{-1}))$ such that $u_{k_j} \to \bar{u}$ as $j \to \infty$. Since the parallel transport is an isometry, letting $\lim_{j\to\infty}$ in (3.28), we have

$$-\lambda\Re\bigg(u_{\bar{x}} + v_{\bar{x}}, \exp^{-1}_{\bar{x}}\big[J^G_\lambda(\exp_{\bar{x}}(-\lambda\bar{u}))\big]\bigg) < d^2\bigg(\bar{x}, J^G_\lambda[\exp_{\bar{x}}(-\lambda\bar{u})]\bigg). \tag{3.29}$$

It follows from (3.29) and (3.1), that

$$\begin{aligned} d^2\bigg(\bar{x}, \big[J^G_\lambda(\exp_{\bar{x}}(-\lambda\bar{u})\big]\bigg) &\leq -\lambda\Re\bigg(u_{\bar{x}} + v_{\bar{x}}, \exp^{-1}_{\bar{x}}\big[J^G_\lambda(\exp_{\bar{x}}(-\lambda\bar{u}))\big]\bigg)\\ &< d^2\bigg(\bar{x}, J^G_\lambda[\exp_{\bar{x}}(-\lambda\bar{u})]\bigg),\end{aligned}$$

which is a contradiction to our assumption. Hence, we have

$$d\bigg(\bar{x}, J^G_\lambda[\exp_{\bar{x}}(-\lambda\bar{u})]\bigg) = 0.$$

Thus $\bar{x} \in S$. This completes the proof. $\square$

In support of problem (1.1), we have the following example in Hadamard manifold.

Example 3.7. *Let*

$$\mathbb{M} = \mathbb{L}^2 = \{a = (a_1, a_2, a_3) \in \mathbb{R}^3 : \langle a, a \rangle = -1, a_3 > 0\},$$

be the 2-dimensional hyperbolic space endowed with the Lorentz metric

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3, \ \forall \ a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{L}^2.$$

It is well known that $\mathbb{L}^2$ *is Hadamard manifold with sectional curvature -1. The normalized geodesic* $\gamma : \mathbb{R} \to \mathbb{L}^2$*, starting from* $a \in \mathbb{L}^2$ *is given by*

$$\gamma(t) = (cosht)a + (sinht)v, \ \forall \ t \in \mathbb{R},$$

where $v \in T_a\mathbb{L}^2$ *is a unit vector, with the distance on* $\mathbb{L}^2$ *is*

$$d(a, b) = arccosh(-\langle a, b \rangle), \ \forall \ a, b \in \mathbb{L}^2.$$

this implies that $\exp_a(tv) = (cosht)a + (sinht)v$*, and the inverse exponential mapping is given by*

$$\exp_a^{-1} b = arccosh(-\langle a, b \rangle) \frac{b + \langle a, b \rangle a}{\sqrt{\langle a, b \rangle^2 - 1}}, \ \forall \ a, b \in \mathbb{L}^2.$$

Let us consider a bounded, closed and convex sunset D *of* $\mathbb{L}^2$*,* $D = \{a = (a_1, a_2, a_3) \in \mathbb{L}^2 : 1 \leq a_3 \leq 2\}$ *and* $F, G : D \rightrightarrows T\mathbb{L}^2$ *are two multi-valued monotone vector field on* D *defined as* $F(a) = (a_1, a_2, ta_3)$ *and* $G(a) = (a_1, a_2, -t^2 a_3^2), \ \forall \ a = (a_1, a_2, a_3) \in D, t \in [-1, 0]$*, for more detail see* [24, 23]*. One can check that the* $x = (0, 0, 1)$ *and* $u = (0, 0, 0) \in F(0, 0, 1)$ *is a solution of multi-valued variational inclusion problem (1.1).*

4. Conclusion

This paper is devoted to the study of multi-valued variational inclusion problem in Hadamard manifold. We prove the convergence of Korpelevich's type algorithm to solve a multi-valued variational inclusion problem. The results presented in this paper are new and some existing results followed as special cases of our results.

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