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PREDUAL OF $M^{p,\,\alpha}(\mathbb{R}^d)$ SPACES

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Abstract. The space $M^{p,\,\alpha}(\mathbb{R}^d)$ introduced by I. Fofana is a subspace of the Wiener amalgam space of measures. In this note, we give a characterization of a predual space of this one.

1. Introduction

Let d be a positive integer. We denote by dx the Lebesgue measure on $\mathbb{R}^d$. For any Lebesgue measurable subset E of $\mathbb{R}^d$, $|E|$ stands for its Lebesgue measure and χ_E denotes its characteristic function. For $1 \leq q \leq \infty$, $\|\cdot\|_q$ denotes the usual norm on the classical Lebesgue space $L^q(\mathbb{R}^d)$ and q' is the conjugate exponent of q: $\frac{1}{q'} + \frac{1}{q} = 1$, with the convention $\frac{1}{\infty} = 0$.
For any $k = (k_1,\ ...,\ k_d) \in \mathbb{Z}^d$ and $r > 0$, set

$$I_k^r = \prod_{i=1}^{d} [k_i r, (k_i + 1)\, r)\,.$$

Let L^0 stands for the space of (equivalence classes modulo the equality Lebesgue almost everywhere of) all complex-valued functions defined on $\mathbb{R}^d$. By $L^1_{loc}(\mathbb{R}^d)$, we denote the set of all elements f of L^0 for which $\|f\chi_K\|_1 < \infty$ for any compact subset K of $\mathbb{R}^d$.
Let $1 \leq q, p \leq \infty$. For $f \in L^0$ and $r > 0$, we set

$${}_r\left\|f\right\|_{q,\,p} = \begin{cases} \left(\sum\limits_{k\in\mathbb{Z}^d} \left(\left\|f\chi_{I_k^r}\right\|_q\right)^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{k\in\mathbb{Z}^d} \left\|f\chi_{I_k^r}\right\|_q & \text{if } p = \infty. \end{cases}$$

The amalgam spaces $(L^q,\ l^p)(\mathbb{R}^d)$ are defined by

$$(L^q,\ l^p)(\mathbb{R}^d) = \left\{f \in L^0 \ \mid\ {}_1\left\|f\right\|_{q,\,p} < \infty\right\}.$$

They have been introduced by Wiener in 1926 (see [22]). But the first systematic study of these spaces is due to Holland [16]. Since then the amalgam spaces $(L^q,\ l^p)(\mathbb{R}^d)$ have been extensively studied (see [21], [14] and the references therein) and generalized in various directions (see [1], [6], [15] and the references therein).

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It is well-known that ${}_1\| \cdot \|_{q,p}$ is a norm which makes $(L^q,\ l^p)(\mathbb{R}^d)$ into Banach spaces. Furthermore, $(L^1,\ l^p)(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ is embedded in the Wiener amalgam space of measures $M^p(\mathbb{R}^d)$. For $1 \leq p \leq \infty$, $M^p(\mathbb{R}^d)$ is the space of Radon measures μ such that ${}_1\|\mu\|_p < \infty$, with

$$ {}_r\|\mu\|_p = \begin{cases} \left(\displaystyle\sum_{k \in \mathbb{Z}^d} |\mu|(I_k^r)^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \displaystyle\sup_{k \in \mathbb{Z}^d} |\mu|(I_k^r) & \text{if } p = \infty, \end{cases} $$

for all $r > 0$, where $|\mu|$ denotes the total variation of μ.

These spaces have been studied by several authors (see [14] and the references therein). They also occur as dual spaces. Actually, If $(\mathcal{C},\ l^p)$ denotes the space of continuous functions in $(L^\infty,\ l^p)(\mathbb{R}^d)$, where $1 \leq p < \infty$, then its dual space is $M^{p'}(\mathbb{R}^d)$ (see [2], [16] and [20]).

In [9], Fofana has introduced the spaces $(L^q,\ l^p)^\alpha(\mathbb{R}^d)$ defined as follows:

$$ (L^q,\ l^p)^\alpha(\mathbb{R}^d) = \left\{ f \in L^0 \ \mid \ \|f\|_{q,\,p,\,\alpha} < \infty \right\}, $$

where

$$ \|f\|_{q,\,p,\,\alpha} = \sup_{r>0}\ r^{d\left(\frac{1}{\alpha} - \frac{1}{q}\right)}\ {}_r\|f\|_{q,\,p}\,. $$

It is proved in [9] and [13] that, for $1 \leq p,\, q,\, \alpha \leq \infty$, the space $(L^q,\ l^p)^\alpha(\mathbb{R}^d)$ is non-trivial if and only if $q \leq \alpha \leq p$ and $\big((L^q,\ l^p)^\alpha(\mathbb{R}^d),\ \|\cdot\|_{q,\,p,\,\alpha}\big)$ is a Banach space. It is clearly a subspace of the amalgam space $(L^q,\ l^p)(\mathbb{R}^d)$. In addition, it is closely related to the Lebesgue spaces as follows :

$$ (L^q,\ l^p)^\alpha(\mathbb{R}^d) = L^\alpha(\mathbb{R}^d) \ \text{ if } \alpha \in \{p,\, q\}, $$

with equivalent norm and

$$ L^\alpha(\mathbb{R}^d) \subsetneq (L^q,\ l^p)^\alpha(\mathbb{R}^d) \text{ if } q < \alpha < p. $$

Several useful results in Fourier analysis, well-known in the Lebesgue spaces, have been extended to the framework of the spaces $(L^q,\ l^p)^\alpha(\mathbb{R}^d)$ (see for instance [3], [11], [12], [8], [17] and [19]). Let us recall that the space $(L^1,\ l^p)^\alpha(\mathbb{R}^d)$ is embedded in a space of measures denoted by $M^{p,\,\alpha}(\mathbb{R}^d)$ which has also been introduced by I. Fofana (see [13] and [11]). $M^{p,\,\alpha}(\mathbb{R}^d)$ is the space of Radon measures μ satisfying $\|\mu\|_{p,\,\alpha} < \infty$, where

$$ \|\mu\|_{p,\,\alpha} = \sup_{r>0} r^{d\left(\frac{1}{\alpha} - 1\right)}\ {}_r\|\mu\|_p. $$

Clearly, $M^{p,\,\alpha}(\mathbb{R}^d)$ is a subspace of $M^p(\mathbb{R}^d)$. It becomes a Banach space when equipped with the norm $\|\cdot\|_{p,\,\alpha}$ (see [13]). Furthermore, it is proved in [18] that if, for $1 \leq q \leq p < \infty$, there exists a constant C such that if a non-negative Radon measure μ satisfies

$$ \|\mu * f\|_p \leq C\|f\|_q, \qquad f \in L^q(\mathbb{R}^d), $$

then μ belongs to $M^{p,\,\alpha}(\mathbb{R}^d)$, with $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$.

Other interesting results involving the spaces $M^{p,\,\alpha}(\mathbb{R}^d)$ can be found in [4], [5] and [19].

Finally, we note that the dual spaces of $(L^q,\ l^p)^\alpha(\mathbb{R}^d)$ and $M^{p,\,\alpha}(\mathbb{R}^d)$ are still unknown. But recently, by using the idea of minimal invariant Banach spaces of functions with respect to a group of dilation operators, Feichtinger and Feuto have

characterized a predual space of $(L^q, l^p)^\alpha(\mathbb{R}^d)$, when $1 < q \leq \alpha \leq p \leq \infty$ (see [7]). They have denoted by $\mathcal{H}(q, p, \alpha)$ this space (see Section 2 for a precise definition of this one).
In this note, we shall describe a predual space of $M^{p,\alpha}(\mathbb{R}^d)$, for $1 \leq \alpha \leq p \leq \infty$ and $p > 1$. This one is closely related to $\mathcal{H}(1, p, \alpha)$.

The paper is organized as follows. In Section 2, we recall the definition of the spaces $\mathcal{H}(q, p, \alpha)$ and some of their basic properties including the fact that they are Banach spaces. Then, in section 3, we introduce a particular linear subspace of $\mathcal{H}(1, p, \alpha)$. Finally, we shall prove in Section 4 that the closure of this linear subspace in $\mathcal{H}(1, p, \alpha)$ is a predual space of $M^{p,\alpha}(\mathbb{R}^d)$.

2. A review of some basic properties of the spaces $\mathcal{H}(q, p, \alpha)$

For $1 \leq \alpha \leq \infty$, we set

$$St_\rho^\alpha f = \rho^{-\frac{d}{\alpha}} f(\rho^{-1}\cdot), \qquad \rho \in (0, \infty),\ f \in L^1_{loc}(\mathbb{R}^d).$$

The following remark summarizes some properties of the operator St_ρ^α.

Remark 2.1. *(See [7].) Assume that $1 \leq \alpha \leq \infty$.*
1) Then

a) *for any real number $\rho > 0$, St_ρ^α applies linearly $L^1_{loc}(\mathbb{R}^d)$ into itself;*
b) *for any $f \in L^1_{loc}(\mathbb{R}^d)$, $St_1^\alpha f = f$;*
c) *for $(\rho_1, \rho_2) \in (0, \infty)^2$ and $f \in L^1_{loc}(\mathbb{R}^d)$, we have*

$$St_{\rho_1}^\alpha \circ St_{\rho_2}^\alpha f = St_{\rho_1\rho_2}^\alpha f,$$

that is, $(St_\rho^\alpha)_{\rho>0}$ is a group of operators on $L^1_{loc}(\mathbb{R}^d)$ isomorphic to the multiplicative group $(0, \infty)$.

2) A direct calculation shows that for $1 \leq q, p \leq \infty$,

$$_1\|St_\rho^\alpha f\|_{q,p} = \rho^{-d(\frac{1}{\alpha}-\frac{1}{q})}\ {}_{\rho^{-1}}\|f\|_{q,p}, \qquad \rho > 0.$$

Since for $\rho > 0$, the mapping $f \mapsto {}_{\rho^{-1}}\|f\|_{q,p}$ is a norm on (L^q, l^p) equivalent to $_1\|\cdot\|_{q,p}$ with the equivalence constants depending only on ρ, then St_ρ^α applies (L^q, l^p) into itself.

We may now define the spaces $\mathcal{H}(q, p, \alpha)$.

Definition 2.2. *Let $1 \leq q \leq \alpha \leq p \leq \infty$. The space $\mathcal{H}(q, p, \alpha)$ is defined as the set of all elements f of $L^1_{loc}(\mathbb{R}^d)$ for which there exists a sequence $\{(c_n, \rho_n, f_n)\}_{n\geq 1}$ of elements of $\mathbb{C} \times (0, \infty) \times (L^{q'}, l^{p'})(\mathbb{R}^d)$ such that*

$$\begin{cases} \sum_{n\geq 1} |c_n| < \infty, \\ _1\|f_n\|_{q',p'} \leq 1, \qquad n \geq 1, \\ f = \sum_{n\geq 1} c_n St_{\rho_n}^{\alpha'} f_n \text{ in the sense of } L^1_{loc}(\mathbb{R}^d). \end{cases} \tag{2.1}$$

Any sequence $\{(c_n, \rho_n, f_n)\}_{n\geq 1}$ of elements of $\mathbb{C} \times (0, \infty) \times (L^{q'}, l^{p'})(\mathbb{R}^d)$ satisfying (2.1) is called an h-decomposition of f.

For $1 \leq q \leq \alpha \leq p \leq \infty$ and for any element f of $\mathcal{H}(q,\, p,\, \alpha)$, we set

$$\|f\|_{\mathcal{H}(q,\, p,\, \alpha)} = \inf \left\{ \sum_{n\geq 1} |c_n| \right\},$$

where the infimum is taken over all h-decompositions of f.
The result below states some basic properties of $\mathcal{H}(q,\, p,\, \alpha)$ and points out its connections with the amalgam spaces.

Proposition 2.3. *(See [7].) Let $1 \leq q \leq \alpha \leq p \leq \infty$.*

(i) *The space $\mathcal{H}(q,\, p,\, \alpha)$ endowed with $\|\cdot\|_{\mathcal{H}(q,\, p,\, \alpha)}$ is a Banach space.*
(ii) *For all $\rho \in (0,\, \infty)$, the operator $St_\rho^{\alpha'}$ is an isometric automorphism of $\mathcal{H}(q,\, p,\, \alpha)$.*
(iii) *The space $(L^{q'},\, l^{p'})(\mathbb{R}^d)$ is continuously embedded in $\mathcal{H}(q,\, p,\, \alpha)$:*

$$(L^{q'},\, l^{p'})(\mathbb{R}^d) \hookrightarrow \mathcal{H}(q,\, p,\, \alpha) \hookrightarrow L^{\alpha'}(\mathbb{R}^d).$$

3. A linear subspace of $\mathcal{H}(1, p, \alpha)$

Throughout the remainder of this paper, we assume that $1 \leq \alpha \leq p \leq \infty$ and $1 < p$. We shall denote by $\mathcal{C}$ the space of continuous functions and by $\mathcal{C}_c$ the one of continuous functions with compact support.

Definition 3.1. *The space X_0 is defined as the set of all elements f of $L^1_{loc}(\mathbb{R}^d)$ for which there exists a sequence $\{(c_n,\, \rho_n,\, f_n)\}_{n\geq 1}$ of elements of $\mathbb{C} \times (0,\, \infty) \times (\mathcal{C},\, l^{p'})$ such that*

(i) $\sum_{n\geq 1} |c_n| < \infty$,
(ii) *for all* $n \geq 1$, ${}_1\|f_n\|_{\infty,\, p'} \leq 1$,
(iii) $f = \sum_{n\geq 1} c_n St_{\rho_n}^{\alpha'} f_n$ *in the sense of* $L^1_{loc}(\mathbb{R}^d)$.

Remark 3.2. *From the above definition, it is easy to see that X_0 is a linear subspace of $\mathcal{H}(1,\, p,\, ,\alpha)$.*

In the sequel, we shall assume that X_0 is equipped with the norm $\|\cdot\|_{\mathcal{H}(1,\, p,\, \alpha)}$.

Proposition 3.3. *The space $(\mathcal{C},\, l^{p'})$ is continuously embedded in X_0.*

Proof. Let $g \in (\mathcal{C},\, l^{p'})$.
It is obvious that if $g = 0$ then $g \in X_0$ and

$$\|g\|_{\mathcal{H}(1,\, p,\, \alpha)} = 0 = {}_1\|g\|_{\infty,\, p'}.$$

Suppose that $g \neq 0$ and write

$$g = {}_1\|g\|_{\infty,\, p'} \frac{g}{{}_1\|g\|_{\infty,\, p'}}.$$

We have

$$\frac{g}{{}_1\|g\|_{\infty,\, p'}} \in (\mathcal{C},\, l^{p'}), \quad {}_1\|\frac{g}{{}_1\|g\|_{\infty,\, p'}}\|_{\infty,\, p'} = 1 \quad \text{and} \quad St_1^{\alpha'}\left(\frac{g}{{}_1\|g\|_{\infty,\, p'}}\right) = \frac{g}{{}_1\|g\|_{\infty,\, p'}}.$$

So $g \in X_0$ and

$$\|g\|_{\mathcal{H}(1,\, p,\, \alpha)} \leq {}_1\|g\|_{\infty,\, p'}.$$

This ends the proof. $\square$

Proposition 3.4. *The spaces* $\mathcal{C}_c$, $(\mathcal{C},\, l^{p'})$ *and* X_0 *have the same closure in* $\mathcal{H}(1,\, p, ,\alpha)$.

Proof. It is clear that $\mathcal{C}_c \subset (\mathcal{C},\, l^{p'})$. This fact together with Proposition 3.3 and Remark 3.2 implies that

$$\mathcal{C}_c \subset (\mathcal{C},\, l^{p'}) \subset X_0 \subset \mathcal{H}(1,\, p,\, \alpha).$$

Let $f \in X_0$. Let us consider an h-decomposition $\{(c_n,\, \rho_n,\, f_n)\}_{n\geq 1}$ of f with $f_n \in \mathcal{C}$ for all $n \geq 1$.
Let us set

$$g_m = \sum_{n=1}^{m} c_n St_{\rho_n}^{\alpha'} f_n, \qquad m \geq 1$$

and

$$g_{m,\,k} = g_m \max\left(1 - \frac{|\cdot|}{k},\, 0\right), \qquad m \geq 1,\ k \geq 1.$$

We notice that $g_m \in (\mathcal{C},\, l^{p'})$ for all $m \geq 1$ and

$$\lim_{m\to\infty} \|f - g_m\|_{\mathcal{H}(1,\, p,\, \alpha)} = 0.$$

Also, $g_{m,\,k} \in \mathcal{C}_c$ for all $m \geq 1$ and for all $k \geq 1$, and

$$\lim_{k\to\infty} {}_1\|g_m - g_{m,\,k}\|_{\infty,\, p'} = 0, \qquad m \geq 1.$$

Hence, by Proposition 3.3

$$\lim_{k\to\infty} \|g_m - g_{m,\,k}\|_{\mathcal{H}(1,\, p,\, \alpha)} = 0, \qquad m \geq 1.$$

It follows that for all $\varepsilon > 0$, there exists $m_\varepsilon \geq 1$ and $k_\varepsilon \geq 1$ such that

$$\|f - g_{m,\,k}\|_{\mathcal{H}(1,\, p,\, \alpha)} < \varepsilon, \qquad m \geq m_\varepsilon,\ \ k \geq k_\varepsilon.$$

So, X_0 is included in the closure of $\mathcal{C}_c$ in $\mathcal{H}(1,\, p,\, \alpha)$.
We deduce that, the spaces $\mathcal{C}_c$, $(\mathcal{C},\, l^{p'})$ and X_0 have the same closure in $\mathcal{H}(1,\, p, ,\alpha)$. □

In the sequel we shall denote by X the closure of $(\mathcal{C},\, l^{p'})$ in $\mathcal{H}(1,\, p,\, \alpha)$. It is clear that $(X,\, \|\cdot\|_{\mathcal{H}(1,\, p,\, \alpha)})$ is a Banach space. We shall denote by X^* its dual space.

4. A predual of $M^{p,\,\alpha}(\mathbb{R}^d)$ spaces

Proposition 4.1. *Let* μ *be an element of* $M^{p,\,\alpha}(\mathbb{R}^d)$. *There is a unique element* T_μ *of* X^* *satisfying*

$$\langle T_\mu,\, f\rangle = \int_{\mathbb{R}^d} f(x)\, d\mu(x), \qquad f \in X_0 \tag{4.1}$$

and

$$|\langle T_\mu,\, f\rangle| \leq \|\mu\|_{p,\,\alpha}\, \|f\|_{\mathcal{H}(1,\, p,\, \alpha)}, \qquad f \in X. \tag{4.2}$$

Proof. a) Assume that f is in X_0 and consider an h-decomposition $\{(c_n,\, \rho_n,\, f_n)\}_{n\geq 1}$ of f with $f_n \in \mathcal{C}$ for all $n \geq 1$. We have $f = \sum_{n\geq 1} c_n St^{\alpha'}_{\rho_n} f_n$.
For any $n \geq 1$,

$$\begin{aligned}
\int_{\mathbb{R}^d} |St^{\alpha'}_{\rho_n} f_n(x)|\, d|\mu|(x) &= \rho_n^{-\frac{d}{\alpha'}} \int_{\mathbb{R}^d} |f_n(\rho_n^{-1}x)|\, d|\mu|(x) \\
&= \rho_n^{-\frac{d}{\alpha'}} \sum_{k\in\mathbb{Z}^d} \int_{I_k^{\rho_n}} |f_n(\rho_n^{-1}x)|\, d|\mu|(x) \\
&\leq \rho_n^{-\frac{d}{\alpha'}} \sum_{k\in\mathbb{Z}^d} \|f_n \chi_{I_k^1}\|_\infty |\mu|(I_k^{\rho_n}) \\
&\leq \rho_n^{-\frac{d}{\alpha'}}\, {}_{\rho_n}\|\mu\|_{p\ 1} \|f_n\|_{\infty,\, p'} \\
&\leq \|\mu\|_{p,\,\alpha\ 1} \|f_n\|_{\infty,\, p'} \\
&\leq \|\mu\|_{p,\,\alpha}.
\end{aligned}$$

It follows that

$$\sum_{n\geq 1} \int_{\mathbb{R}^d} |c_n St^{\alpha'}_{\rho_n} f_n(x)|\, d|\mu|(x) \leq \left(\sum_{n\geq 1} |c_n|\right) \|\mu\|_{p,\,\alpha}$$

and consequently

$$\int_{\mathbb{R}^d} |f(x)|\, d|\mu|(x) \leq \left(\sum_{n\geq 1} |c_n|\right) \|\mu\|_{p,\,\alpha}.$$

Hence $f = \sum_{n\geq 1} c_n St^{\alpha'}_{\rho_n} f_n$ is μ-integrable and

$$\left|\int_{\mathbb{R}^d} f(x)\, d\mu(x)\right| \leq \left(\sum_{n\geq 1} |c_n|\right) \|\mu\|_{p,\,\alpha}.$$

As the above inequality holds for any h-decomposition of f, we have

$$\left|\int_{\mathbb{R}^d} f(x)\, d\mu(x)\right| \leq \|\mu\|_{p,\,\alpha}\, \|f\|_{\mathcal{H}(1,\,p,\,\alpha)}.$$

From the foregoing and the linearity of the integral,

$$J_\mu \,:\, f \mapsto \langle J_\mu,\, f\rangle = \int_{\mathbb{R}^d} f(x)\, d\mu(x)$$

is a bounded linear functional on X_0 such that

$$|\langle J_\mu,\, f\rangle| \leq \|\mu\|_{p,\,\alpha}\, \|f\|_{\mathcal{H}(1,\,p,\,\alpha)}, \qquad f \in X_0.$$

b) Since X_0 is a dense linear subspace of X, there exists a unique element T_μ of X^* satisfying (4.1) and (4.2). □

As a consequence of Proposition 4.1, we have the following result.

Corollary 4.2. *The operator*

$$T \,:\, \mu \mapsto T_\mu,$$

where T_μ is defined by (4.1) and (4.2), is linear and bounded from $M^{p,\,\alpha}(\mathbb{R}^d)$ to X^* and satisfies $\|T\| \leq 1$.

Proof. It follows from Proposition 4.1 that T is an operator from $M^{p,\alpha}(\mathbb{R}^d)$ to X^* satisfying

$$\|T_\mu\| \leq \|\mu\|_{p,\alpha}, \qquad \mu \in M^{p,\alpha}(\mathbb{R}^d).$$

In addition, T is clearly linear.
So, T is a bounded linear operator from $M^{p,\alpha}(\mathbb{R}^d)$ to X^* with $\|T\| \leq 1$. □

Proposition 4.3. *For any element Φ of X^*, there exists a unique measure μ belonging to $M^{p,\alpha}(\mathbb{R}^d)$ such that $\Phi = T_\mu$, where T_μ is defined by (4.1) and (4.2).*

Proof. Let $\Phi \in X^*$.
a) Let us set

$$\Phi_0(g) = \Phi(g), \qquad g \in (\mathcal{C},\, l^{p'}).$$

It follows from Proposition 3.3 that Φ_0 is a linear functional on $(\mathcal{C},\, l^{p'})$ such that, for all $g \in (\mathcal{C},\, l^{p'})$,

$$|\Phi_0(g)| = |\Phi(g)| \leq \|\Phi\|\, \|g\|_{\mathcal{H}(1,\,p,\,\alpha)} \leq \|\Phi\|\, {}_1\|g\|_{\infty,\,p'}.$$

Thus, Φ belongs to the dual space of $(\mathcal{C},\, l^{p'})$ (with respect to the norm ${}_1\|\cdot\|_{\infty,\,p'}$). Since $1 \leq p' < +\infty$, there exists an element μ of $M^p(\mathbb{R}^d)$ such that

$$\Phi(g) = \int_{\mathbb{R}^d} g(x)\, d\mu(x), \qquad g \in (\mathcal{C},\, l^{p'}),$$

(see [20] or [21]).
b) Let us consider a real number $\rho > 0$.
Let $\{(\psi_k,\, c_k)\}_{k\in\mathbb{Z}^d}$ be a subset of $\mathcal{C}_c \times \mathbb{C}$ such that

$$\left(\sum_{k\in\mathbb{Z}^d} |c_k|^{p'}\right)^{\frac{1}{p'}} < \infty, \tag{4.3}$$

$$supp(\psi_k) \subset \overset{\circ}{I^\rho_k} \ \text{ and } \ \|\psi_k\|_\infty \leq 1, \qquad k \in \mathbb{Z}^d, \tag{4.4}$$

where $supp(\psi_k)$ stands for the support of ψ_k and $\overset{\circ}{I^\rho_k}$ denotes the interior of I^ρ_k. Let us notice that

$$\phi_k := \psi_k(\rho\,\cdot) \in \mathcal{C}_c \text{ with } supp(\phi_k) \subset I^1_k \text{ and } \|\phi_k\|_\infty \leq 1.$$

Then

$$\sum_{k\in\mathbb{Z}^d} c_k\phi_k \in \mathcal{C} \quad \text{and} \quad {}_1\|\sum_{k\in\mathbb{Z}^d} c_k\phi_k\|_{\infty,\,p'} \leq \left(\sum_{k\in\mathbb{Z}^d} |c_k|^{p'}\right)^{\frac{1}{p'}} < \infty.$$

So

$$\sum_{k\in\mathbb{Z}^d} c_k\phi_k \in (\mathcal{C},\, l^{p'}), \quad St^{\alpha'}_\rho\left(\sum_{k\in\mathbb{Z}^d} c_k\phi_k\right) \in (\mathcal{C},\, l^{p'}) \subset X$$

and by Proposition 2.3 and Proposition 3.3,

$$\begin{aligned}\|St_\rho^{\alpha'}\left(\sum_{k\in\mathbb{Z}^d} c_k\phi_k\right)\|_{\mathcal{H}(1,\,p,\,\alpha)} &= \|\sum_{k\in\mathbb{Z}^d} c_k\phi_k\|_{\mathcal{H}(1,\,p,\,\alpha)} \\ &\leq {}_1\|\sum_{k\in\mathbb{Z}^d} c_k\phi_k\|_{\infty,\,p'} \leq \left(\sum_{k\in\mathbb{Z}^d}|c_k|^{p'}\right)^{\frac{1}{p'}}.\end{aligned}$$

We have

$$\begin{aligned}\left|\int_{\mathbb{R}^d} St_\rho^{\alpha'}\left(\sum_{k\in\mathbb{Z}^d} c_k\phi_k\right)(x)\,d\mu(x)\right| &\leq \left|\Phi\left(St_\rho^{\alpha'}\left(\sum_{k\in\mathbb{Z}^d} c_k\phi_k\right)\right)\right| \\ &\leq \|\Phi\|\left(\sum_{k\in\mathbb{Z}^d}|c_k|^{p'}\right)^{\frac{1}{p'}}.\end{aligned}$$

We also have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} c_k\int_{\mathbb{R}^d}\psi_k(x)\,d\mu(x) &= \sum_{k\in\mathbb{Z}^d} c_k\int_{\mathbb{R}^d}\phi_k(\rho^{-1}x)\,d\mu(x) \\ &= \rho^{\frac{d}{\alpha'}}\sum_{k\in\mathbb{Z}^d} c_k\int_{\mathbb{R}^d}\rho^{-\frac{d}{\alpha'}}\phi_k(\rho^{-1}x)\,d\mu(x) \\ &= \rho^{\frac{d}{\alpha'}}\sum_{k\in\mathbb{Z}^d} c_k\int_{\mathbb{R}^d} St_\rho^{\alpha'}(\phi_k)(x)\,d\mu(x) \\ &= \rho^{\frac{d}{\alpha'}}\int_{\mathbb{R}^d} St_\rho^{\alpha'}\left(\sum_{k\in\mathbb{Z}^d} c_k\phi_k\right)(x)\,d\mu(x).\end{aligned}$$

So

$$\left|\sum_{k\in\mathbb{Z}^d} c_k\int_{\mathbb{R}^d}\psi_k(x)\,d\mu(x)\right| \leq \rho^{\frac{d}{\alpha'}}\|\Phi\|\left(\sum_{k\in\mathbb{Z}^d}|c_k|^{p'}\right)^{\frac{1}{p'}}.$$

Since the above inequality holds for all $\{c_k\}_{k\in\mathbb{Z}^d}\subset\mathbb{C}$ satisfying (4.3), we have

$$\begin{cases}\left[\sum_{k\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}\psi_k(x)\,d\mu(x)\right|^p\right]^{\frac{1}{p}} \leq \|\Phi\|\rho^{\frac{d}{\alpha'}} & \text{if } 1<p<\infty, \\ \sup_{k\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}\psi_k(x)\,d\mu(x)\right| \leq \|\Phi\|\rho^{\frac{d}{\alpha'}} & \text{if } p=\infty.\end{cases}$$

Since the above inequalities hold for all $\{\psi_k\}_{k\in\mathbb{Z}^d}\subset\mathcal{C}_c$ satisfying (4.4), we have

$$\begin{cases}\left[\sum_{k\in\mathbb{Z}^d}|\mu|(\overset{\circ}{I_k^\rho})^p\right]^{\frac{1}{p}} \leq \|\Phi\|\rho^{\frac{d}{\alpha'}} & \text{if } 1<p<\infty, \\ \sup_{k\in\mathbb{Z}^d}|\mu|(\overset{\circ}{I_k^\rho}) \leq \|\Phi\|\rho^{\frac{d}{\alpha'}} & \text{if } p=\infty.\end{cases}$$

Then we deduce that μ belongs to $M^{p,\alpha}(\mathbb{R}^d)$ and

$$\|\mu\|_{p,\alpha} \leq C\|\Phi\|, \tag{4.5}$$

(see [5]), where C is a positive real number depending on d and p.
c) Let f be an element of X_0.
There exists a sequence $\{g_m\}_{m\geq 1}$ of elements of $(\mathcal{C}, l^{p'})$ that converges to f in X_0 and we have

$$\Phi(g_m) = \int_{\mathbb{R}^d} g_m(x)\, d\mu(x), \qquad m \geq 1.$$

It follows that

$$\Phi(f) = \lim_{m\to\infty} \int_{\mathbb{R}^d} g_m(x)\, d\mu(x).$$

In addition, by Proposition 4.1, f is μ-integrable and

$$\int_{\mathbb{R}^d} f(x)\, d\mu(x) = \lim_{m\to\infty} \int_{\mathbb{R}^d} g_m(x) d\mu(x).$$

Thus,

$$\Phi(f) = \int_{\mathbb{R}^d} f(x)\, d\mu(x).$$

Since the above equality holds for any element f of X_0, we have $\phi = T_\mu$.
d) The uniqueness of the measure μ belonging to $M^{p,\alpha}(\mathbb{R}^d)$ such that $\Phi = T_\mu$ follows easily from (4.5).
This ends the proof. □

Corollary 4.2 and Proposition 4.3 yield the following characterization of a predual space of $M^{p,\alpha}(\mathbb{R}^d)$.

Proposition 4.4. *The mapping $T : M^{p,\alpha}(\mathbb{R}^d) \to X^*$ given by $T(\mu) = T_\mu$ is an isomorphism and there exists a positive real number C such that*

$$\|T(\mu)\| \leq \|\mu\|_{p,\alpha} \leq C\|T(\mu)\|, \qquad \mu \in M^{p,\alpha}(\mathbb{R}^d).$$

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