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FIXED POINT THEOREMS FOR GENERALIZED $(\alpha\eta)_{EB}$-CONTRACTIONS IN EXTENDED B-METRIC SPACES WITH APPLICATIONS

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Abstract. The aim of this paper is to define generalized $(\alpha\eta)_{EB}$-contraction in extended b-metric space to obtain some generalized fixed point theorems. As application, we apply our fixed point theorem to prove the existence theorem for Fredholm integral equation

$$\vartheta(t) = \int_a^b K(t,q,\vartheta(q))dq + g(t),$$

for all $t,q \in [a,b]$, where $f : [a,b] \to \mathbb{R}$ and $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Our results generalize and extend several known results of literature.

1. Introduction and Preliminaries

In 1906, M. Frechet introduced the notion of metric space which is one of pillar of not only mathematics but also physical sciences. Because to its importance and simplicity, this notion has been extended, improved and generalized in many different ways [1,2]. The famous extensions of the concept of metric spaces have been done by Bakhtin [3] which was formally defined by Czerwik [4] in 1993 with a view of generalizing Banach contraction principle.

Definition 1.1. *(see.[4])Let $\mathcal{M}$ be a nonempty set and $s \geq 1$ be a constant. A function $d_b : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ is called a b-metric if the following assertions hold:*
$(b1)$ $d_b(\vartheta,\theta) = 0 \Leftrightarrow \vartheta = \theta$;
$(b2)$ $d_b(\vartheta,\theta) = d_b(\theta,\vartheta)$ *for all* $\vartheta,\theta \in \mathcal{M}$;
$(b3)$ $d_b(\vartheta,\varphi) \leq s[d_b(\vartheta,\theta) + d_b(\theta,\varphi)]$, *for all* $\vartheta,\theta,\varphi \in \mathcal{M}$.
The pair $(\mathcal{M}, d_b)$ is then said to be a b- metric space.

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In 2017, Kamran et al. [5] introduced the notion of extended b-metric spaces.

Definition 1.2. *Let $\mathcal{M}$ be a non-empty set and $\mathfrak{s} : \mathcal{M} \times \mathcal{M} \to [1, \infty)$. A function $d_{eb} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is called an extended b-metric if the following conditions hold:*

(i) $d_{eb}(\vartheta, \theta) = 0 \iff \vartheta = \theta$;
(ii) $d_{eb}(\vartheta, \theta) = d_{eb}(\theta, \vartheta)$;
(iii) $d_{eb}(\vartheta, \theta) \leq \mathfrak{s}(\vartheta, \theta)[d_{eb}(\vartheta, \varphi) + d_{eb}(\varphi, \theta)]$,
for all $\vartheta, \theta, \varphi \in \mathcal{M}$.
The pair $(\mathcal{M}, d_{eb})$ is called an extended b-metric space.

Note that if $\mathfrak{s}(\vartheta, \theta) = \mathfrak{s}$ for $\mathfrak{s} \geq 1$, then we get b-metric space from extended b-metric space.

Example 1.3. [5] *Let $\mathcal{M} = \mathfrak{C}([a, b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a, b]$. Note that $\mathcal{M}$ is complete extended b-metric space by considering*

$$d_{eb}(\vartheta, \theta) = \sup_{t \in [a,b]} |\vartheta(\mathfrak{t}) - \theta(\mathfrak{t})|^2$$

with $\mathfrak{s}(\vartheta, \theta) = |\vartheta(\mathfrak{t})| + |\theta(\mathfrak{t})| + 2$, *where* $\mathfrak{s} : \mathcal{M} \times \mathcal{M} \to [1, \infty)$.

For more details in this direction, we refer the readers to [6, 7, 8].

In 2012, Samet et al. [9] introduced the concept of α-admissible mapping on complete metric space in this way.

Definition 1.4. [9]*Let $\mathcal{H}$ be a self-mapping on $\mathcal{M}$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be a function. We say that $\mathcal{H}$ is an α-admissible mapping if*

$$\alpha(\vartheta, \theta) \geq 1 \Longrightarrow \alpha(\mathcal{H}\vartheta, \mathcal{H}\theta) \geq 1$$

$\forall \vartheta, \theta \in \mathcal{M}$.

Hussain et al. [10] extended the above notion of α-admissible mapping as follows.

Definition 1.5. [10]*Let $\mathcal{H}$ be a self-mapping on $\mathcal{M}$ and $\alpha, \eta : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be two functions. We say that $\mathcal{H}$ is an α-admissible mapping with respect to η if*

$$\alpha(\vartheta, \theta) \geq \eta(\vartheta, \theta) \Longrightarrow \alpha(\mathcal{H}\vartheta, \mathcal{H}\theta) \geq \eta(\mathcal{H}\vartheta, \mathcal{H}\theta)$$

$\forall \vartheta, \theta \in \mathcal{M}$.

If $\eta(\vartheta, \theta) = 1$, then Definition 1.5 reduces to Definition 1.4.

Forfurther details in the direction of F-contractions, we refer the following [10,11] to the readers.

In this paper, we define the notion of generalized $(\alpha\eta)_{EB}$-contraction and establish some new fixed point theorems in the context of extended b-metric spaces. We also furnish a notable example to describe the significance of established results.

2. Results and Discussions

Definition 2.1. *Let $(\mathcal{M}, d_{eb})$ be an extended b-metric space and $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Then $\mathcal{H}$ is said to be generalized $(\alpha\eta)_{EB}$-contraction if there exists two functions $\alpha, \eta : \mathcal{M}\times\mathcal{M}\to [0,\infty)$ and $k \in [0,1)$ such that*

$$\begin{aligned} \alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) &\geq \eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \\ \Longrightarrow d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta) &\leq k \max\{d_{eb}(\vartheta, \theta), \min\{d_{eb}(\vartheta, \mathcal{H}\vartheta), d_{eb}(\theta, \mathcal{H}\theta)\}\} \quad (2.1) \end{aligned}$$

$\forall \vartheta, \theta \in \mathcal{M}$.

Theorem 2.2. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Suppose that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping with respect to η,
(ii) $\mathcal{H}$ is generalized $(\alpha\eta)_{EB}$-contraction,
(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \eta(\vartheta_0, \mathcal{H}\vartheta_0)$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq \eta(\vartheta_n, \vartheta_{n+1})$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq \eta(\vartheta, \mathcal{H}\vartheta)$
Then $\mathcal{H}$ has a fixed point.

Proof. Let $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \eta(\vartheta_0, \mathcal{H}\vartheta_0)$ and construct $\{\vartheta_n\}$ in $\mathcal{M}$ by $\vartheta_{n+1} = \mathcal{H}^n\vartheta_0 = \mathcal{H}\vartheta_n$, $\forall\ n \in \mathbb{N}$. By (i), we have

$$\alpha(\vartheta_0, \vartheta_1) = \alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \eta(\vartheta_0, \mathcal{H}\vartheta_0) = \eta(\vartheta_0, \vartheta_1).$$

Continuing in this way, we get

$$\alpha(\vartheta_{n-1}, \vartheta_n) = \alpha(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}) \geq \eta(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}) = \eta(\vartheta_{n-1}, \vartheta_n) \quad (2.2)$$

$\forall\ n \in \mathbb{N}$. Then

$$\alpha(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1})\alpha(\vartheta_n, \mathcal{H}\vartheta_n) \geq \eta(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1})\eta(\vartheta_n, \mathcal{H}\vartheta_n)$$

$\forall\ n \in \mathbb{N}$. Clearly, if $\exists\ n_0 \in \mathbb{N}$ for which $\vartheta_{n_0+1} = \vartheta_{n_0}$, then $\mathcal{H}\vartheta_{n_0} = \vartheta_{n_0}$ and the proof is completed. Hence, we assume that $\vartheta_{n+1} \neq \vartheta_n$ or $d_{eb}(\mathcal{H}\vartheta_{n-1}, \mathcal{H}\vartheta_n) > 0$ for every $n \in \mathbb{N}$. Now as $\mathcal{H}$ is generalized $(\alpha\eta)_{EB}$-contraction, so we have

$$\begin{aligned} d_{eb}(\vartheta_n, \vartheta_{n+1}) &= d_{eb}(\mathcal{H}\vartheta_{n-1}, \mathcal{H}\vartheta_n) \\ &\leq k \max\{d_{eb}(\vartheta_{n-1}, \vartheta_n), \min\{d_{eb}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}), d_{eb}(\vartheta_n, \mathcal{H}\vartheta_n)\}\} \end{aligned}$$

$\forall\ n \in \mathbb{N}$. Now if $d_{eb}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}) < d_{eb}(\vartheta_n, \mathcal{H}\vartheta_n)$, then

$$\max\{d_{eb}(\vartheta_{n-1}, \vartheta_n), \min\{d_{eb}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}), d_{eb}(\vartheta_n, \mathcal{H}\vartheta_n)\}\} = d_{eb}(\vartheta_{n-1}, \vartheta_n)$$

$\forall\ n \in \mathbb{N}$. If $d_{eb}(\vartheta_n, \mathcal{H}\vartheta_n) < d_{eb}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1})$, then

$$\max\{d_{eb}(\vartheta_{n-1}, \vartheta_n), \min\{d_{eb}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}), d_{eb}(\vartheta_n, \mathcal{H}\vartheta_n)\}\} = d_{eb}(\vartheta_{n-1}, \vartheta_n)$$

$\forall\ n \in \mathbb{N}$. Thus in all case, we have

$$d_{eb}(\vartheta_n, \vartheta_{n+1}) \leq k d_{eb}(\vartheta_{n-1}, \vartheta_n)$$

$\forall\ n \in \mathbb{N}$. Continuing in this way, we get

$$d_{eb}(\vartheta_n, \vartheta_{n+1}) \leq k^n d_{eb}(\vartheta_0, \vartheta_1) \tag{2.3}$$

$\forall\ n \in \mathbb{N}$. For all $n, m \in \mathbb{N}(n < m)$, we get

$$\begin{aligned} d_{eb}(\vartheta_n, \vartheta_m) &\leq s(\vartheta_n, \vartheta_m)\left[d_{eb}(\vartheta_n, \vartheta_{n+1}) + d_{eb}(\vartheta_{n+1}, \vartheta_m)\right] \\ &\leq s(\vartheta_n, \vartheta_m)d_{eb}(\vartheta_n, \vartheta_{n+1}) + s(\vartheta_n, \vartheta_m)s(\vartheta_{n+1}, \vartheta_m)\left[d_{eb}(\vartheta_{n+1}, \vartheta_{n+2}) + d_{eb}(\vartheta_{n+2}, \vartheta_m)\right] \\ &\leq s(\vartheta_n, \vartheta_m)d_{eb}(\vartheta_n, \vartheta_{n+1}) + s(\vartheta_n, \vartheta_m)s(\vartheta_{n+1}, \vartheta_m)d_{eb}(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots \\ &+ s(\vartheta_n, \vartheta_m)s(\vartheta_{n+1}, \vartheta_m)s(\vartheta_{n+2}, \vartheta_m)\cdots s(\vartheta_{m-2}, \vartheta_m)s(\vartheta_{m-1}, \vartheta_m)d_{eb}(\vartheta_{m-1}, \vartheta_m) \\ &\leq s(\vartheta_1, \vartheta_m)s(\vartheta_2, \vartheta_m)\cdots s(\vartheta_n, \vartheta_m)d_{eb}(\vartheta_n, \vartheta_{n+1}) \\ &+ s(\vartheta_1, \vartheta_m)s(\vartheta_2, \vartheta_m)\cdots s(\vartheta_{n+1}, \vartheta_m)d_{eb}(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots \\ &+ s(\vartheta_1, \vartheta_m)s(\vartheta_2, \vartheta_m)\cdots s(\vartheta_{m-1}, \vartheta_m)d_{eb}(\vartheta_{m-1}, \vartheta_m) \\ &< \sum_{n=1}^{\infty} d_{eb}(\vartheta_n, \vartheta_{n+1})\left(\prod_{j=0}^{n} s(\vartheta_j, \vartheta_m)\right) \\ &\leq \sum_{n=1}^{\infty} k^n d_{eb}(\vartheta_0, \vartheta_1)\left(\prod_{j=0}^{n} s(\vartheta_j, \vartheta_m)\right). \end{aligned}$$

Thus

$$d_{eb}(\vartheta_n, \vartheta_m) \leq \sum_{n=1}^{\infty} k^n d_{eb}(\vartheta_0, \vartheta_1)\left(\prod_{j=0}^{n} s(\vartheta_j, \vartheta_m)\right). \tag{2.4}$$

Since $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, so the series $\sum_{n=1}^{\infty} k^n \left(\prod_{j=0}^{n} s(\vartheta_j, \vartheta_m)\right)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} k^n \left(\prod_{j=0}^{n} s(\vartheta_j, \vartheta_m)\right)$$

$$S_n = \sum_{i=1}^{n} k^i \left(\prod_{j=1}^{i} s(\vartheta_j, \vartheta_m)\right).$$

Thus, for $m > n$, (2.4) implies

$$d_{eb}(\vartheta_n, \vartheta_m) \leq d_{eb}(\vartheta_0, \vartheta_1)[S_{m-1} - S_{n-1}].$$

Taking $m, n \to \infty$, we get that $\{\vartheta_n\}$ is a Cauchy. As $\mathcal{M}$ is complete, so $\vartheta_n \to \vartheta^* \in \mathcal{M}$. Now if $\mathcal{H}$ is continuous, then, $\vartheta_{n+1} = \mathcal{H}\vartheta_n \to \mathcal{H}\vartheta^*$ as $n \to \infty$.

Thus, $\mathcal{H}\vartheta^* = \vartheta^*$. Thus ϑ^* is a fixed point of $\mathcal{H}$. Secondly as $\vartheta_n \to \vartheta^*$ and $\alpha(\vartheta_n, \vartheta_{n+1}) \geq \eta(\vartheta_n, \vartheta_{n+1})$, then $\alpha(\vartheta^*, \mathcal{H}\vartheta^*) \geq \eta(\vartheta^*, \mathcal{H}\vartheta^*)$.
Thus

$$\alpha(\vartheta^*, \mathcal{H}\vartheta^*)\alpha(\vartheta_n, \mathcal{H}\vartheta_n) \geq \eta(\vartheta^*, \mathcal{H}\vartheta^*)\eta(\vartheta_n, \mathcal{H}\vartheta_n)$$

By (1), we have

$$\begin{aligned} d_{eb}(\mathcal{H}\vartheta^*, \vartheta_{n+1}) &= d_{eb}(\mathcal{H}\vartheta^*, \mathcal{H}\vartheta_n) \\ &\leq k \max\{d_{eb}(\vartheta^*, \vartheta_n), \min\{d_{eb}(\vartheta^*, \mathcal{H}\vartheta^*), d_{eb}(\vartheta_n, \mathcal{H}\vartheta_n)\}\} \\ &= k \max\{d_{eb}(\vartheta^*, \vartheta_n), \min\{d_{eb}(\vartheta^*, \mathcal{H}\vartheta^*), d_{eb}(\vartheta_n, \vartheta_{n+1})\}\}. \end{aligned}$$

Letting n$\to \infty$ and using the supposition that d_{eb} is continuous functional, we have $d_{eb}(\mathcal{H}\vartheta^*, \vartheta^*) = 0$. Thus, $\mathcal{H}\vartheta^* = \vartheta^*$ and ϑ^* is a fixed point of $\mathcal{H}$.

□

If $\eta(\vartheta, \theta) = 1$, then we have the following corollaries.

Corollary 2.3. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d_{eb}(\vartheta, \theta), \min\{d_{eb}(\vartheta, \mathcal{H}\vartheta), d_{eb}(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then $\mathcal{H}$ has a fixed point.

Corollary 2.4. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$. and $l > 0$

$$(d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta) + l)^{\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)} \leq k \max \{d_{eb}(\vartheta, \theta), \min \{d_{eb}(\vartheta, \mathcal{H}\vartheta), d_{eb}(\theta, \mathcal{H}\theta)\}\} + l.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.5. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{k \max\{d_{eb}(\vartheta,\theta), \min\{d_{eb}(\vartheta,\mathcal{H}\vartheta), d_{eb}(\theta,\mathcal{H}\theta)\}\}}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.6. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max \{d_{eb}(\vartheta, \theta), \min \{d_{eb}(\vartheta, \mathcal{H}\vartheta), d_{eb}(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

If $\alpha(x,y)=1$, then we have the following corollaries.

Corollary 2.7. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H}:\mathcal{M}\to\mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta,\theta\in\mathcal{M}$ such that $\forall\vartheta,\theta\in\mathcal{M}$

$$\eta(\vartheta,\mathcal{H}\vartheta)\eta(\theta,\mathcal{H}\theta)\leq 1 \Longrightarrow d_{eb}(\mathcal{H}\vartheta,\mathcal{H}\theta)\leq k\max\left\{d_{eb}(\vartheta,\theta),\min\left\{d_{eb}(\vartheta,\mathcal{H}\vartheta),d_{eb}(\theta,\mathcal{H}\theta)\right\}\right\}.$$

(iii) $\exists\ \vartheta_0\in\mathcal{M}$ such that $\eta(\vartheta_0,\mathcal{H}\vartheta_0)\leq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n,\vartheta_m)k<1$, for each $\vartheta_0\in\mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n\to\vartheta$, $\eta(\vartheta_n,\vartheta_{n+1})\leq 1$, then $\eta(\vartheta,\mathcal{H}\vartheta)\leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.8. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H}:\mathcal{M}\to\mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta,\theta\in\mathcal{M}$ such that $\forall\vartheta,\theta\in\mathcal{M}$. and $l>0$

$$d_{eb}(\mathcal{H}\vartheta,\mathcal{H}\theta)+l\leq\left[k\max\left\{d_{eb}(\vartheta,\theta),\min\left\{d_{eb}(\vartheta,\mathcal{H}\vartheta),d_{eb}(\theta,\mathcal{H}\theta)\right\}\right\}+l\right]^{\eta(\vartheta,\mathcal{H}\vartheta)\eta(\theta,\mathcal{H}\theta)}.$$

(iii) $\exists\ \vartheta_0\in\mathcal{M}$ such that $\eta(\vartheta_0,\mathcal{H}\vartheta_0)\leq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n,\vartheta_m)k<1$, for each $\vartheta_0\in\mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n\to\vartheta$, $\eta(\vartheta_n,\vartheta_{n+1})\leq 1$, then $\eta(\vartheta,\mathcal{H}\vartheta)\leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.9. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H}:\mathcal{M}\to\mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta,\theta\in\mathcal{M}$ such that $\forall\vartheta,\theta\in\mathcal{M}$

$$2^{d_{eb}(\mathcal{H}\vartheta,\mathcal{H}\theta)}\leq\left(\eta(\vartheta,\mathcal{H}\vartheta)\eta(\theta,\mathcal{H}\theta)+1\right)^{k\max\left\{d_{eb}(\vartheta,\theta),\min\left\{d_{eb}(\vartheta,\mathcal{H}\vartheta),d_{eb}(\theta,\mathcal{H}\theta)\right\}\right\}}.$$

(iii) $\exists\ \vartheta_0\in\mathcal{M}$ such that $\eta(\vartheta_0,\mathcal{H}\vartheta_0)\leq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n,\vartheta_m)k<1$, for each $\vartheta_0\in\mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n\to\vartheta$, $\eta(\vartheta_n,\vartheta_{n+1})\leq 1$, then $\eta(\vartheta,\mathcal{H}\vartheta)\leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.10. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \max \{d_{eb}(\vartheta, \theta), \min \{d_{eb}(\vartheta, \mathcal{H}\vartheta), d_{eb}(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.11. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq kd_{eb}(\vartheta, \theta).$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.12. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{kd_{eb}(\vartheta, \theta)}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 2.13. *Let $(\mathcal{M}, d_{eb})$ be a complete extended b-metric space such that d_{eb} is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d_{eb}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq kd_{eb}(\vartheta, \theta).$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) $\lim_{m,n\to\infty} s(\vartheta_n, \vartheta_m)k < 1$, for each $\vartheta_0 \in \mathcal{M}$,,
(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

3. Consequences

3.1. **Fixed Point Results in b-Metric Spaces.**

Theorem 3.1. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Suppose that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping with respect to η,
(ii) there exist two functions $\alpha, \eta : \mathcal{M}\times\mathcal{M}\to [0, \infty)$ and $k \in [0, 1)$ such that for all $\vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq \eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \Longrightarrow d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\}$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \eta(\vartheta_0, \mathcal{H}\vartheta_0)$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq \eta(\vartheta_n, \vartheta_{n+1})$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq \eta(\vartheta, \mathcal{H}\vartheta)$
Then $\mathcal{H}$ has a fixed point.

Corollary 3.2. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.3. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$. and $l > 0$

$$(d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) + l)^{\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)} \leq k \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\} + l.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.4. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d_b(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{k \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\}}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.5. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

If $\alpha(x, y) = 1$, then we have the following corollaries.

Corollary 3.6. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \leq 1 \Longrightarrow d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.7. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$. and $l > 0$

$$d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) + l \leq [k \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\} + l]^{\eta(\vartheta,\mathcal{H}\vartheta)\eta(\theta,\mathcal{H}\theta)}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.8. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$2^{d_b(\mathcal{H}\vartheta,\mathcal{H}\theta)} \leq (\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) + 1)^{k \max\{d_b(\vartheta,\theta),\min\{d_b(\vartheta,\mathcal{H}\vartheta),d_b(\theta,\mathcal{H}\theta)\}\}}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.9. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \max\{d_b(\vartheta, \theta), \min\{d_b(\vartheta, \mathcal{H}\vartheta), d_b(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.10. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d_b(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k d_b(\vartheta, \theta).$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.11. *Let $(\mathcal{M}, d_b)$ be a complete b-metric space such that d_b is a continuous functional and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d_b(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{k d_b(\vartheta, \theta)}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

3.2. **Fixed Point Results in Metric Spaces.**

Theorem 3.12. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Suppose that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping with respect to η,
(ii) there exist two functions $\alpha, \eta : \mathcal{M}\times\mathcal{M} \to [0, \infty)$ and $k \in [0, 1)$ such that for all $\vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq \eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \Longrightarrow d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d(\vartheta, \theta), \min\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\}\}$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \eta(\vartheta_0, \mathcal{H}\vartheta_0)$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq \eta(\vartheta_n, \vartheta_{n+1})$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq \eta(\vartheta, \mathcal{H}\vartheta)$
Then $\mathcal{H}$ has a fixed point.

Corollary 3.13. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d(\vartheta, \theta), \min\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\}\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.14. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$ and $l > 0$

$$(d(\mathcal{H}\vartheta, \mathcal{H}\theta) + l)^{\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)} \leq k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\} + l.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.15. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\}}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.16. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.
Then $\mathcal{H}$ has a fixed point.

If $\alpha(x, y) = 1$, then we have the following corollaries.

Corollary 3.17. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \leq 1 \Longrightarrow d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.18. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$. and $l > 0$

$$d(\mathcal{H}\vartheta, \mathcal{H}\theta) + l \leq \left[k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\} + l\right]^{\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta)}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.19. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$2^{d(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq \left(\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) + 1\right)^{k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\}}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,
(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.
Then $\mathcal{H}$ has a fixed point.

Corollary 3.20. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an η-subadmissible mapping,
(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k\eta(\vartheta, \mathcal{H}\vartheta)\eta(\theta, \mathcal{H}\theta) \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\}.$$

(iii) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\eta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\eta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\eta(\vartheta, \mathcal{H}\vartheta) \leq 1$.

Then $\mathcal{H}$ has a fixed point.

Corollary 3.21. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq kd(\vartheta, \theta).$$

(iii) $\exists\, \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then $\mathcal{H}$ has a fixed point.

Corollary 3.22. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\forall \vartheta, \theta \in \mathcal{M}$

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{kd(\vartheta, \theta)}.$$

(iii) $\exists\, \vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then $\mathcal{H}$ has a fixed point.

4. Applications

In this section, we present an application of Theorem 2.2 in establishing the existence of solutions for a Fredholm integral equation:

$$\vartheta(t) = \int_a^b K(t, q, \vartheta(q))dq + f(t), \tag{4.1}$$

for all $t, q \in [a, b]$, where $f : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $\mathcal{M}$ be the set of all continuous real valued functions defined on $[a, b]$. i.e., $\mathcal{M} = C([a, b], \mathbb{R})$. Define $d_{eb} : \mathcal{M}\times\mathcal{M} \to \mathbb{R}$ by

$$d_{eb}(\vartheta, \theta) = \sup_{t\in[a,b]} |\vartheta(t) - \theta(t)|^2, \text{ with } \mathfrak{s}(\vartheta, \theta) = |\vartheta(t) + \theta(t)| + 1$$

where $\mathfrak{s} : \mathcal{M} \times \mathcal{M} \to [1, \infty)$. Then $(\mathcal{M},\, d_{eb})$ is a complete extended b-metric space.

Theorem 4.1. *Suppose that ($\forall \vartheta, \theta \in \mathcal{M}$), we have*

$$|K(t,q,\vartheta(q)) - K(t,q,\theta(q))| \leq \sqrt{k}|\vartheta(q) - \theta(q)|$$

$\forall\ t, q \in [a,b]$ *and* $\tau > 0$, *then, the integral equation (21) has a solution.*

Proof. Define $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{H}(\vartheta(t)) = \int_a^b K(t,q,\vartheta(q))dq + f(t),$$

for all $t, q \in [a,b]$ and $\alpha, \eta : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ by $\alpha(\vartheta,\theta) = \eta(\vartheta,\theta) = 1$, $\forall \vartheta, \theta \in \mathcal{M}$ and $Q : \mathbb{R}^{+^4} \to \mathbb{R}^+$ by $Q(l_1,l_2,l_3,l_4) = \tau$, where $\tau > 0$. Now we show that $\mathcal{H}$ satisfies all the assertions of Theorem 2.2. For any $\vartheta(t), \theta(t) \in \mathcal{M}$. Consider

$$\begin{aligned} |\mathcal{H}(\vartheta(t)) - \mathcal{H}(\theta(t))|^2 &= \left(\int_a^b |K(t,q,\vartheta(q)) - K(t,q,\theta(q))|\right)^2 dq \\ &\leq \left(\int_a^b \sqrt{k}|\vartheta(q) - \theta(q)|\right)^2 dq \\ &= k\left(\int_a^b |\vartheta(q) - \theta(q)|\right)^2 dq \\ &= kd_{eb}\left(\vartheta(t), \theta(t)\right) \end{aligned}$$

which implies

$$d_{eb}(\mathcal{H}(\vartheta(t)), \mathcal{H}(\theta(t))) \leq kd_{eb}(\vartheta, \theta).$$

Thus all the assertions of the Theorem 2.2 are satisfied. Hence $\mathcal{H}$ has a unique fixed point and the Fredholm integral equation has a solution. □

5. Conclusion

In this article, we have defined generazlied $(\alpha\eta)_{EB}$-contraction to obtain new fixed point theorems in the setting of complete extended b-metric spaces. As application of our main theorems, the existence of solution for a Fredholm integral inclusion is also explored. We hope that the theorems proved in this paper will form new connections for those who are working in extended b-metric space .

Conflict of Interests
The authors declare that they have no competing interests.

Author's Contribution
All authors contributed equally and significantly in writing this paper. Both authors read and approved the final paper.

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