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EXISTENCE OF SOLUTIONS FOR SINGULAR FRACTIONAL NAVIER BOUNDARY VALUE PROBLEMS

SARA ALTALHI, ABDELWAHEB DHIFLI, ABDELJABBAR GHANMI

Abstract. In this paper, we are concerned with the following fractional Navier boundary value problem:

$$D^{\beta}(D^{\alpha}u)(x) = -g(u), \quad x \in (0,1),$$
$$\lim_{x\to 0^+} x^{1-\beta}D^{\alpha}u(x) = -a, \quad u(1) = b,$$

where $\alpha, \beta \in (0,1]$ such that $\alpha+\beta > 1$, D^{α} and D^{β} stand for the standard Riemann-Liouville fractional derivatives, the function g is continuous and non-increasing on $(0,\infty)$ and the reals $a, b \in (0,\infty)$. Using Schäuder's fixed point theorem, we prove the existence of positive continuous solutions.

1. Introduction

Fractional differential equations have extensive applications in various fields of science and engineering. Many phenomena in electrochemistry, control theory, porous media, electromagnetism and other fields, can be modeled by fractional differential equations. Concerning the development of theory methods and applications of fractional calculus, we refer to [4, 7, 8, 9, 10, 11, 13, 21, 22, 24] and the references therein for discussions of various applications.

The theory of fractional differential equations with various boundary conditions has been developed very quickly and the investigation for the existence, uniqueness and asymptotic behavior of positive continuous solutions attracted a considerable attention of researchers; see, for instance [1, 2, 3, 5, 6, 12, 14, 15, 16, 17, 18, 19, 20, 23, 25, 26] and the references therein..

Recently, in [18] the authors studied the following fractional Navier boundary value problem

$$\begin{cases} D^{\beta}(D^{\alpha}u)(x) = -p(x)u^{\sigma}, \quad x \in (0,1), \\ \lim_{x\to 0^+} x^{1-\beta}D^{\alpha}u(x) = 0, \quad u(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha, \beta \in (0,1]$ such that $\alpha+\beta > 1$, $\sigma \in (-1,1)$ and p is a nonnegative continuous function on $(0,1)$. Under some appropriate condition on the function p

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and using the Schäuder fixed point theorem, the authors proved the existence of a unique positive solution to problem (1.1).

Inspired by the above-mentioned works, in this paper, we consider the following nonlinear Navier boundary value problem

$$\begin{cases} D^{\beta}(D^{\alpha}u)(x) = -g(u(x)), \quad x \in (0,1), \\ \lim\limits_{x \longrightarrow 0^+} x^{1-\beta} D^{\alpha}u(x) = -a, \ u(1) = b, \end{cases} \tag{1.2}$$

where $\alpha, \beta \in (0,1]$ such that $\alpha+\beta > 1$, D^{α} and D^{β} stand for the standard Riemann-Liouville fractional derivatives and a, b are positive real numbers. The nonlinear term g is a nonnegative function defined on $(0,\infty)$ and satisfying the following hypotheses

(H1) $g : (0,\infty) \longrightarrow (0,\infty)$ is continuous and nonincreasing.
(H2) $\int_0^1 (1-t)^{\alpha+\beta-1} g(bt^{\alpha-1})dt < \infty$.
(H3) There exists $c > 0$, such that $|g(bt^{\alpha-1})| \leq ct^{-\delta}$ for t near 0 with $\delta < 1$.

To illustrate, let us present the following example

Example 1.1. *Let $\sigma > 0$ and let $g(t) = t^{-\sigma}$, $t > 0$. Then g satisfies (H1)-(H3).*

To state our main results in this paper, we need to introduce some convenient notations. For $\lambda \in \mathbb{R}$, we put $\lambda^+ = \max(\lambda, 0)$ and for $\alpha, \beta \in (0,1]$, such that $\alpha+\beta > 1$, we denote by $G(x,t)$ the Green function of the operator $u \to -D^{\beta}(D^{\alpha}u)$, with boundary conditions $\lim\limits_{x \longrightarrow 0} x^{1-\beta} D^{\alpha}u(x) = u(1) = 0$. From [18], $G(x,t)$ is explicitly given by

$$G(x,t) = \frac{1}{\Gamma(\alpha+\beta)} \left(x^{\alpha-1}(1-t)^{\alpha+\beta-1} - ((x-t)^+)^{\alpha+\beta-1} \right) \tag{1.3}$$

where Γ is the Euler gamma function.

We denote by $\mathcal{B}((0,1))$ the set of Borel measurable functions in $(0,1)$, by $\mathcal{B}^+((0,1))$ the set of nonnegative ones and by $C((0,1))$ the space of all continuous real functions on $(0,1)$. For a positive real number r, we use $C_r([0,1])$ to denote the set of continuous functions f on $(0,1]$ such that $x \to x^r f(x)$ is continuous on $[0,1]$.

Moreover, for $f \in B^+((0,1))$ and $x \in (0,1)$, we put

$$Vf(x) := \int_0^1 G(x,t) f(t) dt.$$

The authors in ([18, 19]) proved the following results,

Lemma 1.2. (i) For $(x,t) \in (0,1) \times (0,1)$, the Green's function $G(x,t)$ satisfies

$$\begin{aligned} \frac{(\alpha+\beta-1)x^{\alpha-1}(1-x)(1-t)^{\alpha+\beta-1}}{\beta\Gamma(\alpha+\beta)} & \\ &\leq G(x,t) \\ &\leq \frac{x^{\alpha-1}(1-t)^{\alpha+\beta-2}\min(1-t, 1-x)}{\Gamma(\alpha+\beta)}. \end{aligned}$$

(ii) Let $f \in \mathcal{B}^+((0,1))$, then the function $x \to Vf(x)$ is in $C_{1-\alpha}([0,1])$ if and only if $\int_0^1 (1-t)^{\alpha+\beta-1} f(t)dt < \infty$.

(iii) Let $\alpha, \beta \in (0,1]$ such that $\alpha + \beta > 1$. Let $f \in C((0,1))$ such that the map $t \to (1-t)^{\alpha+\beta-1} f(t)$ is integrable and $|f(t)| \leq t^{-\delta}$ near 0, with $\delta < 1$. Then Vf is the unique solution in $C_{1-\alpha}([0,1])$ of the following boundary value problem

$$\begin{cases} D^\beta(D^\alpha u)(x) = -f(x), & x \in (0,1), \\ \lim_{x \to 0^+} x^{1-\beta} D^\alpha u(x) = 0, & u(1) = 0. \end{cases} \tag{1.4}$$

In the sequel, we denote by ω the unique solution of the homogeneous problem corresponding to (1.2). We can easily verify that, for $x \in (0,1)$

$$\omega(x) = a \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\alpha-1}(1 - x^\beta) + b x^{\alpha-1}. \tag{1.5}$$

Our main results are the following.

Theorem 1.3. *Assume (H1)-(H3), then problem* (1.2) *has a positive solution* u *in* $C_{1-\alpha}([0,1])$ *satisfying for* $x \in (0,1]$,

$$\omega(x) \leq u(x) \leq \gamma \omega(x), \tag{1.6}$$

where $\gamma > 1$.

This paper is organized as follows. In Section 2, we give some basic preliminary results of fractional calculus. In Section 3 we prove our main results.

2. Fractional calculus

For the convenience of the reader, we recall in the following some basic definitions and some elementary properties of fractional calculus (see [8, 22, 23]).

Definition 2.1. The Riemann-Liouville fractional integral of order $\gamma > 0$ for a measurable function $f : (0,\infty) \to \mathbb{R}$ is defined as

$$I^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t) dt, \quad x > 0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here Γ is the Euler Gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\gamma > 0$ of a measurable function $f : (0,\infty) \to \mathbb{R}$ is defined as

$$D^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \Big(\frac{d}{dx}\Big)^n \int_0^x (x-t)^{n-\gamma-1} f(t) dt = \Big(\frac{d}{dx}\Big)^n I^{n-\gamma} f(x),$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here $n = [\gamma] + 1$, where $[\gamma]$ denotes the integer part of the number γ.

Example 2.3. Let $\alpha > 0$ and $\lambda > -1$ and let $f(t) = t^\lambda$. Then by simple calculus, we have for $x \in (0,1]$

$$I^\alpha f(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} x^{\lambda+\alpha}$$

and

$$D^\alpha f(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} x^{\lambda-\alpha}.$$

In particular $D^\alpha x^{\alpha-m} = 0$, $m = 1, 2, ..., N$, where N is the smallest integer greater than or equal to α.

Lemma 2.4. (i) Let $\alpha > 0$ and let $v \in C((0,1)) \cap L^1(0,1)$, then we have

$$D^\alpha I^\alpha v = v.$$

(ii) Let $\alpha > 0$ and $v \in C((0,1)) \cap L^1(0,1)$, then

$$D^\alpha v(x) = 0 \text{ if and only if } v(x) = \sum_{j=1}^{N} c_j x^{\alpha - j},$$

where N is the smallest integer greater than or equal to α and $(c_1, ..., c_N) \in \mathbb{R}^N$.

(iii) Let $\alpha > 0$ and $v \in C((0,1)) \cap L^1(0,1)$ such that $D^\alpha v \in C((0,1)) \cap L^1(0,1)$, then

$$I^\alpha D^\alpha v(x) = v(x) + \sum_{j=1}^{N} c_j x^{\alpha - j},$$

where N is the smallest integer greater than or equal to α and $(c_1, ..., c_N) \in \mathbb{R}^N$.

The following lemma due to [19]

Lemma 2.5. Let $\beta \in (0, 1]$ and $\alpha \in (0, \infty)$. Let $f \in C((0,1))$ such that the map $t \to (1-t)^{\alpha+\beta-1} f(t)$ is integrable and $|f(t)| \leq t^{-\delta}$ for t near 0, with $\delta < 1$. Then the function $x \to I^\beta f(x) \in C((0,1)) \cap L^1((0,1))$ and $\lim\limits_{x \to 0} x^{1-\beta} I^\beta f(x) = 0$.

3. Proofs of main results

In this section, we aim at proving Theorem 1.3. First we need the following

Proposition 3.1. Let φ be a nonnegative function that satisfies $\displaystyle\int_0^1 (1-t)^{\alpha+\beta-1} \varphi(t) dt < \infty$. Then the family of functions defined in $(0,1)$ by

$$\mathcal{F} = \left\{ x \longrightarrow S(f)(x) := \frac{1}{\omega(x)} \int_0^1 G(x,t) f(t) dt; \ |f| \leq \varphi \right\}$$

is uniformly bounded and equicontinuous in $[0,1]$. Consequently, $\mathcal{F}$ is relatively compact in $C([0,1])$.

Proof. Let φ be a nonnegative function that satisfies $\displaystyle\int_0^1 (1-t)^{\alpha+\beta-1} \varphi(t) dt < \infty$ and let f be a measurable function such that $|f| \leq \varphi$. By Lemma (1.2) (i) and (1.5), we have

$$\big|S(f)(x)\big| \leq \frac{1}{b\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-1} \varphi(t) dt.$$

Hence, $\mathcal{F}$ is uniformly bounded.

Now, let's show that the family $\mathcal{F}$ is equicontinuous in $[0,1]$. For $x, x' \in (0,1)$, we have

$$\begin{aligned}
\big|S(f)(x) - S(f)(x')\big| &= \Big|\int_0^1 \Big(\frac{G(x,t)}{\omega(x)} - \frac{G(x',t)}{\omega(x')}\Big) f(t) dt\Big| \\
&\leq \int_0^1 \Big|\frac{G(x,t)}{\omega(x)} - \frac{G(x',t)}{\omega(x')}\Big| \varphi(t) dt.
\end{aligned}$$

Using the fact that ω is continuous on $(0,1]$ and the Green function $G(x,t)$ is continuous with respect to the first variable, then

$$\Big|\frac{G(x,t)}{\omega(x)} - \frac{G(x',t)}{\omega(x')}\Big| \longrightarrow 0 \quad \text{as } \big|x - x'\big| \longrightarrow 0.$$

On the other hand, by using (1.3) and (1.5) for $t \in (0,1)$, we have

$$\Big|\frac{G(x,t)}{\omega(x)} - \frac{G(x',t)}{\omega(x')}\Big| \leq \frac{2}{b\Gamma(\alpha+\beta)}(1-t)^{\alpha+\beta-1}.$$

So, by the dominated convergence theorem, we deduce that $\big|S(f)(x) - S(f)(x')\big|$ tends to zero as $\big|x - x'\big| \longrightarrow 0$, uniformly with respect to $f \in \mathcal{F}$. This implies that, the family $\mathcal{F}$ is equicontinuous in $(0,1)$. Finally, we need to verify that the function $S(f)$ has a limits at $x = 0$ and $x = 1$. From Lemma (1.2) (i) and (1.5) for $x \in (0,1)$, we have

$$\big|S(f)(x)\big| \leq \frac{1}{b\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-2} \min(1-t, 1-x)\varphi(t)dt.$$

Hence, we deduce that $\lim_{x\to 1^-} S(f)(x) = 0$, uniformly with respect to $f \in \mathcal{F}$. Moreover, by (1.3) and (1.5), we have

$$\begin{aligned}
&\Big|S(f)(x) - \frac{1}{a\Gamma(\beta) + b\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-1} f(t)dt\Big| \\
&\quad = \Big|\int_0^1 \Big(\frac{G(x,t)}{\omega(x)} - \frac{(1-t)^{\alpha+\beta-1}}{a\Gamma(\beta) + b\Gamma(\alpha+\beta)}\Big)\Big|\big|f(t)\big|dt
\end{aligned}$$

and

$$\begin{aligned}
\Big|\frac{G(x,t)}{\omega(x)} - \frac{(1-t)^{\alpha+\beta-1}}{a\Gamma(\beta) + b\Gamma(\alpha+\beta)}\Big| &= \Big|\frac{(1-t)^{\alpha+\beta-1} - x^{1-\alpha}((x-t)^+)^{\alpha+\beta-1}}{a\Gamma(\beta)(1-x^\beta) + b\Gamma(\alpha+\beta)} \\
&\quad - \frac{(1-t)^{\alpha+\beta-1}}{a\Gamma(\beta) + b\Gamma(\alpha+\beta)}\Big| \\
&\leq \frac{3}{b\Gamma(\alpha+\beta)}(1-t)^{\alpha+\beta-1}.
\end{aligned}$$

It follows from the dominated convergence theorem that $\big|S(f)(x) - \frac{1}{a\Gamma(\beta)+b\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-1} f(t)dt\big|$ tend to zero as $x \longrightarrow 0^+$, uniformly with respect to $f \in \mathcal{F}$.

Hence, we conclude the family $\mathcal{F}$ is equicontinuous in $[0,1]$ and by Ascoli's theorem $\mathcal{F}$ is relatively compact in $C([0,1])$. This ends the proof. □

Proof of Theorem 1.3. Let $\alpha, \beta \in (0,1]$ such that $\alpha + \beta > 1$ and let $a, b \in (0,\infty)$. We shall use a fixed point argument to construct a solution to problem (1.2). For this end, put $\gamma = 1 + \frac{1}{b\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-1} g(bt^{\alpha-1})dt$ and consider the closed convex set

$$\Lambda = \{v \in C([0,1]) \,:\, 1 \leq v(x) \leq \gamma\}.$$

We define the operator T on Λ by

$$Tv(x) = 1 + \frac{1}{\omega(x)} \int_0^1 G(x,t) g(\omega(t)v(t))dt, \quad x \in (0,1). \tag{3.1}$$

We shall prove that T has a fixed point in Λ. First, we have clearly that for $v \in \Lambda$ $1 \leq Tv \leq \gamma$. By same arguments as in the Proof of Proposition (3.1), we show that $T\Lambda$ is relatively compact in $C([0,1])$. So, we deduce that $T\Lambda \subset \Lambda$.

Next, let us prove the continuity of the operator T in Λ. Consider a sequence (v_k) in Λ which converges uniformly to a function $v \in \Lambda$ and let $x \in [0,1]$. We have

$$\big|Tv_k(x) - Tv(x)\big| \leq \frac{1}{\omega(x)} \int_0^1 G(x,t)\big|g(\omega(t)v_k(t)) - g(\omega(t)v(t))\big|dt$$

and using the hypothesis (H1), we get

$$\big|g(\omega(t)v_k(t)) - g(\omega(t)v(t))\big| \leq 2g(\omega(t)) \leq 2g(bt^{\alpha-1}).$$

Now, since g is continuous, we deduce by the dominated convergence theorem that for $x \in [0,1]$, $Tv_k(x) \longrightarrow Tv(x)$ as $k \longrightarrow \infty$. Since $T\Lambda$ is relatively compact family in $C([0,1])$, we have the uniform convergence, namely

$$\|Tv_k - Tv\|_\infty \longrightarrow 0 \; as \, k \longrightarrow \infty.$$

Thus we have proved that T is a compact mapping from Λ to itself. So, by the Schäuder fixed point theorem, T has a fixed point $v \in \Lambda$. Put $u(x) = \omega(x)v(x)$, for $x \in (0,1]$. Then $u \in C_{1-\alpha}([0,1])$ and satisfies for $x \in (0,1]$

$$u(x) = V(g(u))(x) + \omega(x)$$

and

$$\omega(x) \leq u(x) \leq \gamma\omega(x). \tag{3.2}$$

It remains to prove that u is a positive solution of problem (1.2). Indeed, by (3.2) $u(x) \geq \omega(x) \geq bx^{\alpha-1}$ and since the function g is nonnegative and nonincreasing, we have obviously $g(u(x)) \leq g(bx^{\alpha-1})$. We deduce from (H2),(H3) and Lemma (1.2) (ii) that $V(g(u))$ is a positive continuous solution of the following boundary value problem

$$\begin{cases} D^\beta(D^\alpha u)(x) = -g(u(x)), \quad x \in (0,1), \\ \lim\limits_{x\to 0^+} x^{1-\beta}D^\alpha u(x) = 0, \quad u(1) = 0. \end{cases}$$

In addition, since ω is the unique solution of the homogeneous problem associated to (1.2), then u is a positive solution of problem (1.2). This completes the proof. □

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Sara Altalhi
Department of Mathematics, University of Jeddah, P.O.Box 80327, Jeddah 21589, Saudi Arabia,
E-mail address: sara3455757@gmail.com

Abdelwaheb Dhifli
Department of Mathematics, Faculty of Science, University of Jeddah, Saudi Arabia.
E-mail address: amdhifli@uj.edu.sa

Abdeljabbar Ghanmi
Department of Mathematics, Faculty of Science, University of Jeddah, Saudi Arabia.
E-mail address: AAALGANMY1@uj.edu.sa