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$\mathcal{I}_2$-STATISTICALLY AND $\mathcal{I}_2$-LACUNARY STATISTICALLY CONVERGENT DOUBLE SET SEQUENCES OF ORDER η

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Abstract. In this study, for double set sequences, as a new approach to the notion of statistical convergence of order η, the notions of Wijsman $\mathcal{I}_2$-statistically convergence of order η, Wijsman strong $\mathcal{I}_2$-Cesàro summability of order η, Wijsman $\mathcal{I}_2$-lacunary statistically convergence of order η and Wijsman strong $\mathcal{I}_2$-lacunary summability of order η are introduced, where $0 < \eta \leq 1$. Also, some properties of these notions are investigated, some investigations about these are made and the existence of some relationships between them are examined.

1. Introduction

The notion of statistical convergence, introduced in the 1950's, was extended to double sequences by Mursaleen and Edely [17], which generalizes the notion of convergence for double sequences introduced by Pringshiem [25]. Then, using double lacunary sequence notion, Patterson and Savaş [24] studied the notion of lacunary statistical convergence for double sequences. Moreover, Das et. al [7] presented the concept of $\mathcal{I}$-convergence for double sequences via ideals. Recently, for double sequences, Çolak and Altın [6] defined the concept of statistically convergence of order α and investigated some properties of this notion. From past to present, many authors have studied and have developed these notions in their papers. More information on the notions of convergence for real sequences can be found in [1, 5, 8, 9, 12, 13, 16, 26, 27, 29]. The readers should refer to the monographs [2] and [18] for the background on the sequence spaces and related topics.

For years, many authors have examined on the notions of various convergence for set sequences. One of these convergence notions, handled in this study, is the notion of Wijsman convergence (see, [3, 4, 35]). Using the notions of statistical convergence, lacunary sequence, ideal and invariant mean, many authors have extended the notion of Wijsman convergence to the new convergence notions in Wijsman sense for set sequences (for examples, see [10, 15, 19, 23, 31]).

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The basic notions of Wijsman convergence for double sequences of sets were introduced by Nuray et. al [21, 22] and Dündar et. al [11], some of which are Wijsman statistically, Wijsman lacunary statistically, Wijsman $\mathcal{I}_2$-statistically and Wijsman $\mathcal{I}_2$-lacunary statistically convergence. Recently, the new concepts about Wijsman convergence of order α for double sequences of sets were studied by Gülle and Ulusu [14, 34]. More information on the notions of convergence for set sequences can be found in [20, 28, 30, 33].

2. Preliminaries

First of all, we give the basic notations necessary for a better understanding of our study (see, [3, 4, 7, 11, 16, 22, 21, 25, 24, 33]).

A double sequence (a_{mn}) is called convergent to L (in Pringsheim sense) if every $\xi > 0$, there exists $N_\xi \in \mathbb{N}$, the set of natural numbers, such that $|a_{mn} - L| < \xi$, when ever $m, n > N_\xi$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$, the power set of $\mathbb{N}$, is said to be an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,
(ii) $E \cup F \in \mathcal{I}$ for each $E, F \in \mathcal{I}$,
(iii) $F \in \mathcal{I}$ for each $E \in \mathcal{I}$ and $F \subseteq E$.

An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be admissible if $\{m\} \in \mathcal{I}$ for each $m \in \mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be strong admissible if $\{m\} \times \mathbb{N}$ and $\mathbb{N} \times \{m\}$ belongs to $\mathcal{I}_2$ for each $m \in \mathbb{N}$. Obviously, a strong admissible ideal is admissible.

Throughout the study, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ will taken a strong admissible ideal.

Let Y be non-empty set. A function $g : \mathbb{N} \to 2^Y$ is defined $g(m) = V_m \in 2^Y$ for each $m \in \mathbb{N}$. The sequence $\{V_m\} = (V_1, V_2, \ldots)$, which the range elements of g, is called sequences of sets.

Let (Y, d) be a metric space. For any $y \in Y$ and any non-empty $V \subseteq Y$, the distance from y to V is defined

$$\rho(y, V) = \inf_{v \in V} d(y, v).$$

Throughout the study, (Y, d) will taken a metric space and V, V_{mn} will taken any non-empty closed subsets of Y.

A double sequence $\{V_{mn}\}$ is called Wijsman convergent to V if each $y \in Y$,

$$\lim_{m,n\to\infty} \rho(y, V_{mn}) = \rho(y, V).$$

A double sequence $\{V_{mn}\}$ is called Wijsman statistically convergent to V if every $\xi > 0$ and each $y \in Y$,

$$\lim_{i,j\to\infty} \frac{1}{ij}\left|\Big\{(m,n) :\ m \le i, n \le j,\ |\rho(y, V_{mn}) - \rho(y, V)| \ge \xi\Big\}\right| = 0.$$

A double sequence $\{V_{mn}\}$ is called Wijsman strong $\mathcal{I}_2$-Cesàro summable to V if every $\xi > 0$ and each $y \in Y$,

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{ij}\sum_{m,n=1,1}^{i,j} |\rho(y, V_{mn}) - \rho(y, V)| \ge \xi\right\} \in \mathcal{I}_2.$$

A double sequence $\{V_{mn}\}$ is called Wijsman $\mathcal{I}_2$-statistically convergent to V if every $\xi > 0$, $\delta > 0$ and each $y \in Y$,

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{ij}\left|\left\{(m,n) : \ m \le i, n \le j, \ |\rho(y, V_{mn}) - \rho(y,V)| \ge \xi\right\}\right| \ge \delta\right\} \in \mathcal{I}_2.$$

The class of Wijsman $\mathcal{I}_2$-statistically convergent double sequences is denoted by $S\left(\mathcal{I}_{W_2}\right)$.

A double sequence $\theta_2 = \{(j_s, k_t)\}$ is said to be a double lacunary sequence if there exists an increasing sequences (j_s) and (k_t) of the integers such that

$$j_0 = 0, \ h_s = j_s - j_{s-1} \to \infty \ \text{ and } \ k_0 = 0, \ \bar{h}_t = k_t - k_{t-1} \to \infty \ \text{ as } \ s,t \to \infty.$$

Throughout the study, regarding lacunary sequence $\theta_2 = \{(j_s, k_t)\}$, we will use the following notations:

$$h_{st} = h_s \bar{h}_t, \quad I_{st} = \{(j,k) : j_{s-1} < j \le j_s \ \text{ and } \ k_{t-1} < k \le k_t\}$$

$$q_s = \frac{j_s}{j_{s-1}} \ \text{ and } \ q_t = \frac{k_t}{k_{t-1}}.$$

Throughout the study, $\theta_2 = \{(j_s, k_t)\}$ will taken a double lacunary sequence.

A double sequence $\{V_{mn}\}$ is called Wijsman lacunary statistically convergent to V if every $\xi > 0$ and each $y \in Y$,

$$\lim_{s,t \to \infty} \frac{1}{h_{st}}\left|\left\{(m,n) \in I_{st} : |\rho(y, V_{mn}) - \rho(y,V)| \ge \xi\right\}\right| = 0.$$

A double sequence $\{V_{mn}\}$ is called Wijsman strong $\mathcal{I}_2$-lacunary summable to V if every $\xi > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{st}} \sum_{(m,n) \in I_{st}} |\rho(y, V_{mn}) - \rho(y,V)| \ge \xi\right\} \in \mathcal{I}_2.$$

The class of Wijsman strong $\mathcal{I}_2$-lacunary summable double sequences is denoted by $N_\theta\left[\mathcal{I}_{W_2}\right]$.

A double sequence $\{V_{mn}\}$ is called Wijsman $\mathcal{I}_2$-lacunary statistically convergent to V if every $\xi > 0$, $\delta > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{st}}\left|\left\{(m,n) \in I_{st} : |\rho(y, V_{mn}) - \rho(y,V)| \ge \xi\right\}\right| \ge \delta\right\} \in \mathcal{I}_2.$$

The class of Wijsman $\mathcal{I}_2$-lacunary statistically convergent double sequences is denoted by $S_\theta\left(\mathcal{I}_{W_2}\right)$.

From now on, for short, we use $\rho_y(V)$ and $\rho_y(V_{mn})$ instead of $\rho(y,V)$ and $\rho(y, V_{mn})$, respectively.

3. New Concepts

In this section, for double set sequences, as a new approach to the notion of statistical convergence of order η, the notions of Wijsman $\mathcal{I}_2$-statistically convergence of order η, Wijsman strong $\mathcal{I}_2$-Cesàro summability of order η, Wijsman $\mathcal{I}_2$-lacunary statistically convergence of order η and Wijsman strong $\mathcal{I}_2$-lacunary summability of order η are introduced, where $0 < \eta \le 1$.

Definition 3.1. *Let $0 < \eta \leq 1$. A double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent of order η to V or $S(I^{\eta}_{W_2})$-convergent to V if every $\xi > 0$, $\delta > 0$ and each $y \in Y$,*

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(ij)^{\eta}} \Big|\big\{(m,n): \ m \leq i, n \leq j, \ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \geq \delta\right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{S(I^{\eta}_{W_2})}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(S(I^{\eta}_{W_2})\big)$.

The class of Wijsman $\mathcal{I}_2$-statistical convergent double sequences is denoted by $S(I^{\eta}_{W_2})$.

Example 3.1. *Let $Y = \mathbb{R}^2$ and double sequence $\{V_{mn}\}$ be defined as follows:*

$$V_{mn} := \begin{cases} \big\{(a,b) \in \mathbb{R}^2 : \ (a+m)^2 + (b+n)^2 = 1\big\} & ; \ \text{if } m \text{ and } n \text{ are square integers} \\ \{(1,1)\} & ; \ \text{otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}^{\delta}_2$, ($\mathcal{I}^{\delta}_2$ is the class of $E \subset \mathbb{N} \times \mathbb{N}$ with density of E equals to 0), then double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent of order η to $V = \{(1,1)\}$.

Remark. *For $\eta = 1$, the concept of $S(I^{\eta}_{W_2})$-convergence coincides with the notion of Wijsman $\mathcal{I}_2$-statistical convergence for double sequences of sets in* [11].

Definition 3.2. *Let $0 < \eta \leq 1$. A double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-Cesàro summable of order η to V or $C_1(I^{\eta}_{W_2})$-summable to V if every $\xi > 0$ and each $y \in Y$,*

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \left|\frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \rho_y(V_{mn}) - \rho_y(V)\right| \geq \xi\right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{C_1(I^{\eta}_{W_2})}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(C_1(I^{\eta}_{W_2})\big)$.

Definition 3.3. *Let $0 < \eta \leq 1$. A double sequence $\{V_{mn}\}$ is Wijsman strong $\mathcal{I}_2$-Cesàro summable of order η to V or $C_1[I^{\eta}_{W_2}]$-summable to V if every $\xi > 0$ and each $y \in Y$,*

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \big|\rho_y(V_{mn}) - \rho_y(V)\big| \geq \xi\right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{C_1[I^{\eta}_{W_2}]}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(C_1[I^{\eta}_{W_2}]\big)$.

The class of Wijsman strong $\mathcal{I}_2$-Cesàro summable double sequences is denoted by $C_1[I^{\eta}_{W_2}]$.

Example 3.2. *Let $Y = \mathbb{R}^2$ and double sequence $\{V_{mn}\}$ be defined as follows:*

$$V_{mn} := \begin{cases} \big\{(a,b) \in \mathbb{R}^2 : \ (a+1)^2 + b^2 = \frac{1}{mn}\big\} & ; \ \text{if } m \text{ and } n \text{ are square integers} \\ \{(0,1)\} & ; \ \text{otherwise.} \end{cases}$$

If $\mathcal{I}_2 = \mathcal{I}^{f}_2$, ($\mathcal{I}^{f}_2$ is the class of finite subsets of $\mathbb{N} \times \mathbb{N}$), then double sequence $\{V_{mn}\}$ is Wijsman strong $\mathcal{I}_2$-Cesàro summable of order η to $V = \{(0,1)\}$.

Remark. *For $\eta = 1$, the concepts of $C_1(I_{W_2}^{\eta})$-summability and $C_1[I_{W_2}^{\eta}]$-summability coincide with the notions of Wijsman $\mathcal{I}_2$-Cesàro summability, Wijsman strong $\mathcal{I}_2$-Cesàro summability for double sequences of sets in* [33], *respectively.*

Definition 3.4. *Let $0 < \eta \leq 1$ and $0 < p < \infty$. A double sequence $\{V_{mn}\}$ is Wijsman strong $p-\mathcal{I}_2$-Cesàro summable of order η to V or $C_1[I_{W_2}^{\eta}]^p$-summable to V if every $\xi > 0$ and each $y \in Y$,*

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \left|\rho_y(V_{mn}) - \rho_y(V)\right|^p \geq \xi\right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{C_1[I_{W_2}^{\eta}]^p}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(C_1[I_{W_2}^{\eta}]^p\big)$.

The class of Wijsman strong $p-\mathcal{I}_2$-Cesàro summable double sequences is denoted by $C_1[I_{W_2}^{\eta}]^p$.

Definition 3.5. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. Double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-lacunary statistically convergent of order η to V or $S_{\theta}(I_{W_2}^{\eta})$-convergent to V if every $\xi > 0$, $\delta > 0$ and each $y \in Y$,*

$$\left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(h_{st})^{\eta}} \left|\left\{(m,n) \in I_{st} : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{S_{\theta}(I_{W_2}^{\eta})}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(S_{\theta}(I_{W_2}^{\eta})\big)$.

The class of Wijsman $\mathcal{I}_2$-lacunary statistical convergent double sequences is denoted by $S_{\theta}(I_{W_2}^{\eta})$.

Example 3.3. *Let $Y = \mathbb{R}^2$ and double sequence $\{V_{mn}\}$ be defined as follows:*

$$V_{mn} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : \ (a-m)^2 + (b+n)^2 = 1\right\} & ; \ \text{if } (m,n) \in I_{st}, \ m \text{ and } n \text{ are square integers} \\ \{(-1,1)\} & ; \ \text{otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^{\delta}$, then double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-lacunary statistical convergent of order η to $V = \{(-1,1)\}$.

Remark. *For $\eta = 1$, the concept of $S_{\theta}(I_{W_2}^{\eta})$-convergence coincides with the notion of Wijsman $\mathcal{I}_2$-lacunary statistical convergence for double sequences of sets in* [11].

Definition 3.6. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. Double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-lacunary summable of order η to V or $N_{\theta}(I_{W_2}^{\eta})$-summable to V if every $\xi > 0$ and each $y \in Y$,*

$$\left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \left|\frac{1}{(h_{st})^{\eta}} \sum_{(m,n)\in I_{st}} \rho_y(V_{mn}) - \rho_y(V)\right| \geq \xi\right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{N_{\theta}(I_{W_2}^{\eta})}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(N_{\theta}(I_{W_2}^{\eta})\big)$.

Definition 3.7. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. Double sequence $\{V_{mn}\}$ is Wijsman strong $\mathcal{I}_2$-lacunary summable of order η to V or $N_\theta[I^{\eta}_{W_2}]$-summable to V if every $\xi > 0$ and each $y \in Y$,*

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(h_{st})^{\eta}} \sum_{(m,n) \in I_{st}} \big|\rho_y(V_{mn}) - \rho_y(V)\big| \geq \xi \right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{N_\theta[I^{\eta}_{W_2}]}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(N_\theta[I^{\eta}_{W_2}]\big)$.

The class of Wijsman strong $\mathcal{I}_2$-lacunary summable double sequences is denoted by $N_\theta[I^{\eta}_{W_2}]$.

Example 3.4. *Let $Y = \mathbb{R}^2$ and double sequence $\{V_{mn}\}$ be defined as follows:*

$$V_{mn} := \begin{cases} \big\{(a,b) \in \mathbb{R}^2 : \ a^2 + (b-1)^2 = \frac{1}{mn}\big\} & ; \ \text{if } (m,n) \in I_{st}, \ m \text{ and } n \text{ are square integers} \\ \{(1,0)\} & ; \ \text{otherwise.} \end{cases}$$

If $\mathcal{I}_2 = \mathcal{I}_2^f$, then double sequence $\{V_{mn}\}$ is Wijsman strong $\mathcal{I}_2$-lacunary summable of order η to $V = \{(1,0)\}$.

Remark. *For $\eta = 1$, the concepts of $N_\theta(I^{\eta}_{W_2})$-summability and $N_\theta[I^{\eta}_{W_2}]$-summability coincide with the notions of Wijsman $\mathcal{I}_2$-lacunary convergence, Wijsman strong $\mathcal{I}_2$-lacunary convergence for double sequences of sets in* [11].

Definition 3.8. *Let $0 < \eta \leq 1$, $0 < p < \infty$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. Double sequence $\{V_{mn}\}$ is Wijsman strong $p - \mathcal{I}_2$-lacunary summable of order η to V or $N_\theta[I^{\eta}_{W_2}]^p$-summable to V if every $\xi > 0$ and each $y \in Y$,*

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(h_{st})^{\eta}} \sum_{(m,n) \in I_{st}} \big|\rho_y(V_{mn}) - \rho_y(V)\big|^p \geq \xi \right\} \in \mathcal{I}_2.$$

Also, we write $V_{mn} \overset{N_\theta[I^{\eta}_{W_2}]^p}{\longrightarrow} V$ or $V_{mn} \longrightarrow V\big(N_\theta[I^{\eta}_{W_2}]^p\big)$.

The class of Wijsman strong $p - \mathcal{I}_2$-lacunary summable double sequences is denoted by $N_\theta[I^{\eta}_{W_2}]^p$.

4. Inclusions Teorems

In this section, firstly, some properties of the new notions introduced in Section 3 are examined with some investigations and emphasized on the existence of some relationships between them.

Theorem 4.1. *If $0 < \eta \leq \mu \leq 1$, then $S(\mathcal{I}^{\eta}_{W_2}) \subseteq S(\mathcal{I}^{\mu}_{W_2})$.*

Proof. Let $0 < \eta \leq \mu \leq 1$. Also, we suppose that $V_{mn} \overset{S(\mathcal{I}^{\eta}_{W_2})}{\longrightarrow} V$. For every $\xi > 0$ and each $y \in Y$, we have

$$\frac{1}{(ij)^{\mu}} \big|\{(m,n) : \ m \leq i, n \leq j, \ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\}\big|$$
$$\leq \frac{1}{(ij)^{\eta}} \big|\{(m,n) : \ m \leq i, n \leq j, \ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\}\big|$$

and so for every $\delta > 0$,

$$\Big\{(i,j) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{(ij)^{\mu}}\big|\{(m,n):\ m \le i, n \le j,\ |\rho_y(V_{mn}) - \rho_y(V)| \ge \xi\}\big| \ge \delta\Big\}$$

$$\subseteq \Big\{(i,j) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{(ij)^{\eta}}\big|\{(m,n):\ m \le i, n \le j,\ |\rho_y(V_{mn}) - \rho_y(V)| \ge \xi\}\big| \ge \delta\Big\}.$$

Hence, by our assumption, the set on right side belongs to the ideal $\mathcal{I}_2$, obviously the set on left side also belongs to $\mathcal{I}_2$. Consequently, $S(\mathcal{I}^{\eta}_{W_2}) \subseteq S(\mathcal{I}^{\mu}_{W_2})$. □

If $\mu = 1$ is taken in Theorem 4.1, then the following corollary is obtained.

Corollary 4.2. *A double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to V if the double sequence is Wijsman $\mathcal{I}_2$-statistical convergent of order η to V for some $0 < \eta \le 1$, i.e., $S(\mathcal{I}^{\eta}_{W_2}) \subseteq S(\mathcal{I}_{W_2})$.*

Similarly, we can give the following theorem without proof.

Theorem 4.3. *Let $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. Then,*

i. *If $0 < \eta \le \mu \le 1$, then $S_\theta(\mathcal{I}^{\eta}_{W_2}) \subseteq S_\theta(\mathcal{I}^{\mu}_{W_2})$.*
ii. *Particularly, for $\mu = 1$, $S_\theta(\mathcal{I}^{\eta}_{W_2}) \subseteq S_\theta(\mathcal{I}_{W_2})$.*

Theorem 4.4. *If $0 < \eta \le \mu \le 1$ and $0 < p < \infty$, then $C_1[\mathcal{I}^{\eta}_{W_2}]^p \subseteq C_1[\mathcal{I}^{\mu}_{W_2}]^p$.*

Proof. Let $0 < \eta \le \mu \le 1$. Also, we suppose that $V_{mn} \overset{C_1[\mathcal{I}^{\eta}_{W_2}]^p}{\longrightarrow} V$. For each $y \in Y$, we have

$$\frac{1}{(ij)^{\mu}} \sum_{m,n=1,1}^{i,j} \big|\rho_y(V_{mn}) - \rho_y(V)\big|^p \le \frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \big|\rho_y(V_{mn}) - \rho_y(V)\big|^p$$

and so for every $\xi > 0$,

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(ij)^{\mu}} \sum_{m,n=1,1}^{i,j} \big|\rho_y(V_{mn}) - \rho_y(V)\big|^p \ge \xi\right\}$$

$$\subseteq \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \big|\rho_y(V_{mn}) - \rho_y(V)\big|^p \ge \xi\right\}.$$

Hence, by our assumption, the set on right side belongs to the ideal $\mathcal{I}_2$, obviously the set on left side also belongs to $\mathcal{I}_2$. Consequently, $C_1[\mathcal{I}^{\eta}_{W_2}]^p \subseteq C_1[\mathcal{I}^{\mu}_{W_2}]^p$. □

If $\mu = 1$ is taken in Theorem 4.4, then the following corollary is obtained.

Corollary 4.5. *A double sequence $\{V_{mn}\}$ is Wijsman strong $p - \mathcal{I}_2$-Cesàro summable to V if the double sequence is Wijsman strong $p - \mathcal{I}_2$-Cesàro summable of order η to V for some $0 < \eta \le 1$.*

Now, we can state the theorem giving the relation between $C_1[\mathcal{I}^{\eta}_{W_2}]^p$ and $C_1[\mathcal{I}^{\eta}_{W_2}]^q$, where $0 < \eta \le 1$ and $0 < p < q < \infty$.

Theorem 4.6. *Let $0 < \eta \le 1$ and $0 < p < q < \infty$. Then, $C_1[\mathcal{I}^{\eta}_{W_2}]^q \subset C_1[\mathcal{I}^{\eta}_{W_2}]^p$.*

Similarly, we can give the following theorem without proof.

Theorem 4.7. *Let $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. Then,*

i. *If $0 < \eta \leq \mu \leq 1$, then $N_\theta[\mathcal{I}^{\eta}_{W_2}] \subseteq N_\theta[\mathcal{I}^{\mu}_{W_2}]$.*
ii. *Particularly, for $\mu = 1$, $N_\theta[\mathcal{I}^{\eta}_{W_2}] \subseteq N_\theta[\mathcal{I}_{W_2}]$.*
iii. *If $0 < \eta \leq 1$ and $0 < p < q < \infty$, then $N_\theta[\mathcal{I}^{\eta}_{W_2}]^q \subset N_\theta[\mathcal{I}^{\eta}_{W_2}]^p$.*

Theorem 4.8. *Let $0 < \eta \leq \mu \leq 1$ and $0 < p < \infty$. If a double sequence $\{V_{mn}\}$ is Wijsman strong $p-\mathcal{I}_2$-Cesàro summable of order η to V, then the sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent of order μ to V.*

Proof. Let $0 < \eta \leq \mu \leq 1$ and $0 < p < \infty$. Also, we suppose that the sequence $\{V_{mn}\}$ is Wijsman strong $p-\mathcal{I}_2$-Cesàro summable of order η to V. For every $\xi > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\sum_{m,n=1,1}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)|^p &= \sum_{\substack{m,n=1,1 \\ |\rho_y(V_{mn})-\rho_y(V)|\geq\xi}}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)|^p \\
&\quad + \sum_{\substack{m,n=1,1 \\ |\rho_y(V_{mn})-\rho_y(V)|<\xi}}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)|^p \\
&\geq \sum_{\substack{m,n=1,1 \\ |\rho_y(V_{mn})-\rho_y(V)|\geq\xi}}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)|^p \\
&\geq \xi^p \Big| \big\{ (m,n) : \ m \leq i, n \leq j, \ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi \big\} \Big|
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{\xi^p \, (ij)^\eta} \sum_{m,n=1,1}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)|^p \\
&\qquad \geq \frac{1}{(ij)^\eta} \Big| \big\{ (m,n) : \ m \leq i, n \leq j, \ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi \big\} \Big| \\
&\qquad \geq \frac{1}{(ij)^\mu} \Big| \big\{ (m,n) : \ m \leq i, n \leq j, \ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi \big\} \Big|.
\end{aligned}$$

Thus, for every $\delta > 0$

$$\begin{aligned}
&\left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(ij)^\mu} \Big| \big\{ (m,n) : \ m \leq i, n \leq j, \ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi \big\} \Big| \geq \delta \right\} \\
&\qquad \subseteq \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(ij)^\eta} \sum_{m,n=1,1}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)|^p \geq \xi^p \, \delta \right\}.
\end{aligned}$$

Hence, by our assumption, the set on right side belongs to the ideal $\mathcal{I}_2$, obviously the set on left side also belongs to $\mathcal{I}_2$. Consequently, the sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent of order μ to V. □

If $\mu = \eta$ is taken in Theorem 4.8, then the following corollary is obtained.

Corollary 4.9. *Let $0 < \eta \leq 1$ and $0 < p < \infty$. A double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent of order η to V if the double sequence is Wijsman strong $p - \mathcal{I}_2$-Cesàro summable of order η to V.*

Similarly, we can give the following theorem without proof.

Theorem 4.10. *Let $0 < \eta \leq \mu \leq 1$ and $0 < p < \infty$. If the sequence $\{V_{mn}\}$ is Wijsman strong $p-\mathcal{I}_2$-lacunary summable of order η to V, then the double sequence is Wijsman $\mathcal{I}_2$-lacunary statistical convergent of order μ to V.*

If $\mu = \eta$ is taken in Theorem 4.10, then the following corollary is obtained.

Corollary 4.11. *Let $0 < \eta \leq 1$ and $0 < p < \infty$. A double sequence $\{V_{mn}\}$ is Wijsman $\mathcal{I}_2$-lacunary statistical convergent of order η to V if the double sequence is Wijsman strong $p - \mathcal{I}_2$-lacunary summable of order η to V.*

Theorem 4.12. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. If $\liminf_s q_s^\eta > 1$ and $\liminf_t q_t^\eta > 1$, then $V_{mn} \overset{S(\mathcal{I}_{W_2}^\eta)}{\longrightarrow} V$ implies $V_{mn} \overset{S_\theta(\mathcal{I}_{W_2}^\eta)}{\longrightarrow} V$.*

Proof. Let $0 < \eta \leq 1$ be given. Also, we assume that $\liminf_s q_s^\eta > 1$ and $\liminf_t q_t^\eta > 1$. Then, there exist $\alpha > 0$, $\beta > 0$ such that $q_s^\eta \geq 1+\alpha$ and $q_t^\eta \geq 1+\beta$ for all s and t, which implies that

$$\frac{h_{st}^\eta}{(j_s k_t)^\eta} \geq \frac{\alpha\beta}{(1+\alpha)(1+\beta)}.$$

Suppose that $V_{mn} \overset{S(\mathcal{I}_{W_2}^\eta)}{\longrightarrow} V$. For every $\xi > 0$ and each $y \in Y$, we have

$$\begin{aligned} \frac{1}{(j_s k_t)^\eta}&\Big|\big\{(m,n):\ m \leq j_s, n \leq k_t : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \\ &\geq \frac{1}{(j_s k_t)^\eta}\Big|\big\{(m,n) \in I_{st} : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \\ &= \frac{h_{st}^\eta}{(j_s k_t)^\eta}\frac{1}{h_{st}^\eta}\Big|\big\{(m,n) \in I_{st} : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \\ &\geq \frac{\alpha\beta}{(1+\alpha)(1+\beta)}\frac{1}{h_{st}^\eta}\Big|\big\{(m,n) \in I_{st} : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big|. \end{aligned}$$

Thus, for any $\delta > 0$

$$\begin{aligned} &\Big\{(s,t) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{(h_{st})^\eta}\Big|\big\{(m,n) \in I_{st} : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \geq \delta\Big\} \\ &\subseteq \Big\{(s,t) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{(j_s k_t)^\eta}\Big|\big\{(m,n):\ m \leq j_s, n \leq k_t : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \\ &\qquad \geq \tfrac{\alpha\,\beta\,\delta}{(1+\alpha)(1+\beta)}\Big\}. \end{aligned}$$

Hence, by our assumption, the set on right side belongs to the ideal $\mathcal{I}_2$, obviously the set on left side also belongs to $\mathcal{I}_2$. Consequently, $V_{mn} \overset{S_\theta(I_{W_2}^\eta)}{\longrightarrow} V$. □

Theorem 4.13. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. If $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$, then $V_{mn} \overset{S_\theta(\mathcal{I}_{W_2}^\eta)}{\longrightarrow} V$ implies $V_{mn} \overset{S(\mathcal{I}_{W_2}^\eta)}{\longrightarrow} V$.*

Proof. Let $0 < \eta \leq 1$ be given. Also, we assume that $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$. Then, there exist $M, N > 0$ such that $q_s < M$ and $q_t < N$ for all s and t. Suppose that $V_{mn} \overset{S_\theta(\mathcal{I}^\eta_{W_2})}{\longrightarrow} V$ and let

$$T_{st} := \Big|\big\{(m,n) \in I_{st} : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big|.$$

Since $V_{mn} \overset{S_\theta(\mathcal{I}^\eta_{W_2})}{\longrightarrow} V$, it holds for every $\xi > 0$, $\delta > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(h_{st})^\eta}\Big|\big\{(m,n) \in I_{st} : |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \geq \delta\right\}$$

$$= \left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{T_{st}}{(h_{st})^\eta} \geq \delta\right\} \in \mathcal{I}_2.$$

Hence, we can choose positive integers $s_0, t_0 \in \mathbb{N}$ such that

$$\frac{T_{st}}{(h_{st})^\eta} < \delta$$

for all $s \geq s_0$, $t \geq t_0$. Now let

$$H := \max\{T_{st} : 1 \leq s \leq s_0,\ 1 \leq t \leq t_0\}$$

and let i and j be integers satisfying $j_{s-1} < i \leq j_s$ and $k_{t-1} < j \leq k_t$. Then, for every $\xi > 0$ and each $y \in Y$ we have

$$\frac{1}{(ij)^\eta}\Big|\big\{(m,n) :\ m \leq i, n \leq j,\ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big|$$

$$\leq \frac{1}{(j_{s-1}k_{t-1})^\eta}\Big|\big\{(m,n) :\ m \leq j_s, n \leq k_t :\ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big|$$

$$= \frac{1}{(j_{s-1}k_{t-1})^\eta}\big\{T_{11} + T_{12} + T_{21} + T_{22} + \cdots + T_{s_0t_0} + \cdots + T_{st}\big\}$$

$$\leq \frac{s_0t_0}{(j_{s-1}k_{t-1})^\eta}\left(\max_{\substack{1\leq m\leq s_0\\1\leq n\leq t_0}}\{T_{mn}\}\right) + \frac{1}{(j_{s-1}k_{t-1})^\eta}\left\{h^\eta_{s_0(t_0+1)}\frac{T_{s_0(t_0+1)}}{h^\eta_{s_0(t_0+1)}}\right.$$

$$\left.+h^\eta_{(s_0+1)t_0}\frac{T_{(s_0+1)t_0}}{h^\eta_{(s_0+1)t_0}} + h^\eta_{(s_0+1)(t_0+1)}\frac{T_{(s_0+1)(t_0+1)}}{h^\eta_{(s_0+1)(t_0+1)}} + \cdots + h^\eta_{st}\frac{T_{st}}{h^\eta_{st}}\right\}$$

$$\leq \frac{s_0t_0H}{(j_{s-1}k_{t-1})^\eta} + \frac{1}{(j_{s-1}k_{t-1})^\eta}\left(\sup_{\substack{s>s_0\\t>t_0}}\frac{T_{st}}{h^\eta_{st}}\right)\left(\sum_{\substack{m\geq s_0\\n\geq t_0}}^{s,t} h^\eta_{mn}\right)$$

$$\leq \frac{s_0t_0H}{(j_{s-1}k_{t-1})^\eta} + \frac{1}{(j_{s-1}k_{t-1})}\left(\sup_{\substack{s>s_0\\t>t_0}}\frac{T_{st}}{h^\eta_{st}}\right)\left(\sum_{\substack{m\geq s_0\\n\geq t_0}}^{s,t} h_{mn}\right)$$

$$\leq \frac{s_0 t_0 H}{(j_{s-1}k_{t-1})^{\eta}} + \delta \frac{(j_s - j_{s_0})(k_t - k_{t_0})}{j_{s-1}k_{t-1}}$$

$$\leq \frac{s_0 t_0 H}{(j_{s-1}k_{t-1})^{\eta}} + \delta\, q_s\, q_t$$

$$\leq \frac{s_0 t_0 H}{(j_{s-1}k_{t-1})^{\eta}} + \delta\, M\, N.$$

Since $j_{s-1}, k_{t-1} \to \infty$ as $i, j \to \infty$, it follows that for each $y \in Y$

$$\frac{1}{(ij)^{\eta}}\Big|\big\{(m,n):\ m \leq i, n \leq j,\ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \to 0$$

and so for any $\delta_1 > 0$,

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(ij)^{\eta}}\Big|\big\{(m,n):\ m \leq i, n \leq j,\ |\rho_y(V_{mn}) - \rho_y(V)| \geq \xi\big\}\Big| \geq \delta_1\right\} \in \mathcal{I}_2.$$

Consequently, $V_{mn} \stackrel{S(\mathcal{I}^{\eta}_{W_2})}{\longrightarrow} V$. □

Theorem 4.14. *Let $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. If*

$$1 < \liminf_s q_s^{\eta} \leq \limsup_s q_s < \infty \quad \textit{and} \quad 1 < \liminf_t q_t^{\eta} \leq \limsup_t q_t < \infty,$$

then $V_{mn} \stackrel{S_\theta(\mathcal{I}^{\eta}_{W_2})}{\longrightarrow} V$ if and only if $V_{mn} \stackrel{S(\mathcal{I}^{\eta}_{W_2})}{\longrightarrow} V$.

Proof. This can be obtained from Theorem 4.12 and Theorem 4.13, immediately. □

Theorem 4.15. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. If $\liminf_s q_s^{\eta} > 1$ and $\liminf_t q_t^{\eta} > 1$, then $C_1[I^{\eta}_{W_2}] \subseteq N_\theta[I^{\eta}_{W_2}]$.*

Proof. Let $0 < \eta \leq 1$ be given. Also, we assume that $\liminf_s q_s^{\eta} > 1$ and $\liminf_t q_t^{\eta} > 1$. Then, there exist $\alpha > 0$ and $\beta > 0$ such that $q_s^{\eta} \geq 1 + \alpha$ and $q_t^{\eta} \geq 1 + \beta$ for all s and t, which implies that

$$\frac{(j_s k_t)^{\eta}}{h_{st}^{\eta}} \leq \frac{(1+\alpha)(1+\beta)}{\alpha\beta} \quad \text{and} \quad \frac{(j_{s-1}k_{t-1})^{\eta}}{h_{st}^{\eta}} \leq \frac{1}{\alpha\beta}.$$

Suppose that $V_{mn} \stackrel{C_1[I_{W_2}^{\eta}]}{\longrightarrow} V$. For each $y \in Y$, we have

$$\begin{aligned}\frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} |\rho_y(V_{mn}) - \rho_y(V)| &= \frac{1}{h_{st}^{\eta}} \sum_{r,u=1,1}^{j_s,k_t} |\rho_y(V_{ru}) - \rho_y(V)| \\ &\quad - \frac{1}{h_{st}^{\eta}} \sum_{r,u=1,1}^{j_{s-1},k_{t-1}} |\rho_y(V_{ru}) - \rho_y(V)| \\ &= \frac{(j_s k_t)^{\eta}}{h_{st}^{\eta}} \left(\frac{1}{(j_s k_t)^{\eta}} \sum_{r,u=1,1}^{j_s,k_t} |\rho_y(V_{ru}) - \rho_y(V)| \right) \\ &\quad - \frac{(j_{s-1} k_{t-1})^{\eta}}{h_{st}^{\eta}} \left(\frac{1}{(j_{s-1} k_{t-1})^{\eta}} \sum_{r,u=1,1}^{j_{s-1},k_{t-1}} |\rho_y(V_{ru}) - \rho_y(V)| \right).\end{aligned}$$

Since $V_{mn} \stackrel{C_1[I_{W_2}^{\eta}]}{\longrightarrow} V$, then for each $y \in Y$

$$\frac{1}{(j_s k_t)^{\eta}} \sum_{r,u=1,1}^{j_s,k_t} |\rho_y(V_{ru}) - \rho_y(V)| \stackrel{\mathcal{I}_2}{\to} 0 \quad \text{and} \quad \frac{1}{(j_{s-1} k_{t-1})^{\eta}} \sum_{r,u=1,1}^{j_{s-1},k_{t-1}} |\rho_y(V_{ru}) - \rho_y(V)| \stackrel{\mathcal{I}_2}{\to} 0.$$

Thus, when the above equality is considered, for each $y \in Y$ we have

$$\frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} |\rho_y(V_{mn}) - \rho_y(V)| \stackrel{\mathcal{I}_2}{\to} 0,$$

that is, $V_{mn} \stackrel{N_\theta[I_{W_2}^{\eta}]}{\longrightarrow} V$. Consequently, $C_1[I_{W_2}^{\eta}] \subseteq N_\theta[I_{W_2}^{\eta}]$. □

Theorem 4.16. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. If $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$, then $N_\theta[I_{W_2}^{\eta}] \subseteq C_1[I_{W_2}^{\eta}]$.*

Proof. Let $0 < \eta \leq 1$ be given. Also, we assume that $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$. Then, there exist $M, N > 0$ such that $q_s < M$ and $q_t < N$ for all s and t. Suppose that $V_{mn} \stackrel{N_\theta[I_{W_2}^{\eta}]}{\longrightarrow} V$. Then, for every $\xi > 0$ and each $y \in Y$ we can find $S, T > 0$ and $H > 0$ such that

$$\sup_{\substack{m \geq S \\ n \geq T}} \tau_{mn} < \xi \quad \text{and} \quad \tau_{mn} < H \quad \text{for all} \quad m, n = 1, 2, \ldots$$

where

$$\tau_{st} = \frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} |\rho_y(V_{mn}) - \rho_y(V)|.$$

If i and j are integers satisfying $j_{s-1} < i \leq j_s$ and $k_{t-1} < j \leq k_t$ where $s > S$ and $t > T$, then for each $y \in Y$ we have

$$\frac{1}{(ij)^\eta} \sum_{m,n=1,1}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)|$$

$$\leq \frac{1}{(j_{s-1}k_{t-1})^\eta} \sum_{m,n=1,1}^{j_s,k_t} |\rho_y(V_{mn}) - \rho_y(V)|$$

$$= \frac{1}{(j_{s-1}k_{t-1})^\eta} \Bigg(\sum_{I_{11}} |\rho_y(V_{mn}) - \rho_y(V)| + \sum_{I_{12}} |\rho_y(V_{mn}) - \rho_y(V)|$$

$$+ \sum_{I_{21}} |\rho_y(V_{mn}) - \rho_y(V)| + \sum_{I_{22}} |\rho_y(V_{mn}) - \rho_y(V)|$$

$$+ \cdots + \sum_{I_{st}} |\rho_y(V_{mn}) - \rho_y(V)| \Bigg)$$

$$= \frac{h_{11}^\eta}{(j_{s-1}k_{t-1})^\eta} .\tau_{11} + \frac{h_{12}^\eta}{(j_{s-1}k_{t-1})^\eta} .\tau_{12} + \frac{h_{21}^\eta}{(j_{s-1}k_{t-1})^\eta} .\tau_{21}$$

$$+ \frac{h_{22}^\eta}{(j_{s-1}k_{t-1})^\eta} .\tau_{22} + \cdots + \frac{h_{st}^\eta}{(j_{s-1}k_{t-1})^\eta} \cdot \tau_{st}$$

$$\leq \sum_{m,n=1,1}^{S,T} \frac{h_{mn}}{j_{s-1}k_{t-1}} .\tau_{mn} + \sum_{\substack{m=S+1 \\ n=T+1}}^{s,t} \frac{h_{mn}}{j_{s-1}k_{t-1}} .\tau_{mn}$$

$$\leq \left(\sup_{\substack{m \geq 1 \\ n \geq 1}} \tau_{mn} \right) \frac{j_S k_T}{j_{s-1}k_{t-1}} + \left(\sup_{\substack{m \geq S \\ n \geq T}} \tau_{mn} \right) \frac{(j_s - j_S)(k_t - k_T)}{j_{s-1}k_{t-1}}$$

$$\leq H \frac{j_S k_T}{j_{s-1}k_{t-1}} + \xi \, M \, N.$$

Since $j_{s-1}, k_{t-1} \to \infty$ as $i, j \to \infty$, it follows that for each $y \in Y$

$$\frac{1}{(ij)^\eta} \sum_{m,n=1,1}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)| \overset{\mathcal{I}_2}{\to} 0,$$

that is, $V_{mn} \overset{C_1[I_{W_2}^\eta]}{\longrightarrow} V$. Consequently, $N_\theta[I_{W_2}^\eta] \subseteq C_1[I_{W_2}^\eta]$. □

Theorem 4.17. *Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be a double lacunary sequence. If*

$$1 < \liminf_s q_s^\eta \leq \limsup_s q_s < \infty \quad \textit{and} \quad 1 < \liminf_t q_t^\eta \leq \limsup_t q_t < \infty,$$

then $N_{\theta_2}[I_{W_2}^\eta] = C_1[I_{W_2}^\eta]$.

Proof. This can be obtained from Theorem 4.15 and Theorem 4.16, immediately. □

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