Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 13 Issue 3 (2021), Pages 39-50.

ON CONFORMALLY SYMMETRIC GENERALIZED RICCI-RECURRENT MANIFOLDS WITH APPLICATIONS IN GENERAL RELATIVITY

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Abstract. In this paper, we consider conformally symmetric generalized Ricci-recurrent manifolds. We prove that such a manifold is a quasi-Einstein manifold and study its geometric properties. Also, we obtain several interesting results. Among others, the universal cover of this manifold splits geometrically as L^1xN^{n-1}, where L is a line, $(N^{n-1}, g_{N^{n-1}})$ is Einstein, $\varphi = -\frac{1}{n}r$. Moreover, we demonstrate the applications of the conformally symmetric generalized Ricci-recurrent spacetime with non-zero constant scalar curvature in the theory of general relativity.

1. Introduction

Let (M, g) be an $n-$dimensional Riemannian manifold. If its Riemannian curvature tensor R is parallel with respect to its Levi-Civita connection ∇, namely $\nabla R = 0$, then a manifold (M, g) is called *locally symmetric*. The study of Riemannian symmetric spaces was initiated by Cartan [3]. As is well known, every complete simple connected locally symmetric manifold is a Riemannian symmetric space, namely for each point in M, there exists an involutive isometry s_p is then called *the symmetry at p*. A generalization of the notion of local symmetry is given by the notion of *conformal symmetry*. A manifold (M, g) is said to be conformally symmetric [4] if its Weyl conformal curvature tensor C is a parallel, namely the relation $\nabla C = 0$ holds on (M, g). Derdzinski and Roter [9] have shown that the class of conformally symmetric manifolds contains all locally symmetric as well as conformally flat manifolds of dimension $n \geq 4$. In 1985, Nickerson [14] gave some characterizations of Riemannian manifolds of dimension $n \geq 4$ which are conformal to a Riemannian locally symmetric space but are themselves neither conformally flat nor locally symmetric. Thus, conformal symmetry is a proper generalization of local symmetry.

During the last decade, many authors have studied the notion of locally symmetric manifolds in several ways to a different extent such as some generalized recurrent

2010 *Mathematics Subject Classification.* 53C15, 53C25,53B15, 53B30.
Key words and phrases. Generalized Ricci-recurrent manifold; Quasi-Einstein manifold; Perfect fluid spacetime; Einstein's field equation; Energy-momentum tensor.

Submitted May 25, 2020. Published July 20, 2021.
Communicated by K. Duggal.

manifolds by Shaikh, Roy, and Kundu [19], pseudo-Riemannian manifolds with recurrent concircular curvature tensor by Olszak and Olszak [16], weakly symmetric manifolds by Yilmaz [20], pseudo semiconformally symmetric manifolds by Kim [12], weakly conformally symmetric manifolds by Mantica and Suh [13], almost generalized pseudo-Ricci symmetric manifolds by Baishya [1] etc...

Patterson [17] introduced Ricci-recurrent manifolds. Based on Patterson, a Riemannian manifold (M, g) is called *Ricci-recurrent* if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z),$$

for some 1-form A. Such a type of manifold is denoted by R_n.

A non-flat Riemannian manifold (M, g) is called *a generalized Ricci-recurrent manifold* if its Ricci tensor satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z) \tag{1.1}$$

where A and B are non-zero 1-forms such that

$$g(X, P) = A(X), \qquad g(X, Q) = B(X), \tag{1.2}$$

for all X [7]. The vector fields P and Q defined by (1.2) are called the basic vector fields corresponding to the $1-$forms A and B, respectively. An n-dimensional manifold of this kind is denoted by $(GR)_n$. If B becomes zero, then it reduces to a Ricci recurrent manifold R_n.

A Riemannian manifold or semi-Riemannian manifold (M, g), $\dim M = n > 2$, is said to be an Einstein manifold if the condition

$$S = \frac{r}{n} g$$

holds on M, where r denotes the scalar curvature of (M, g) [2]. Moreover, quasi-Einstein manifolds arose during the study of exact solutions of Einstein's field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. The notion of quasi-Einstein manifold has been first introduced by Deszcz et. all [10]. Based on them, a non-flat Riemannian manifold (M, g), $n > 2$, is said to be a quasi-Einstein manifold if its Ricci tensor is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bw(X)w(Y) \tag{1.3}$$

where a and b are constants and w is a non-zero 1-form, equivalent to the unit vector field ρ, that is, $g(X, \rho) = w(X)$ and $g(\rho, \rho) = 1$. It is to be noted that Chaki and Maity [5] also introduced the notion of quasi-Einstein manifold, considering a, b scalars instead of a, b as constants. In such a case a, b are called associated scalars. w and ρ are called the associated $1-$form and the generator of the manifold, respectively.

A non-flat Riemannian $(M, g), (n > 3)$, is called *a conformally symmetric generalized Ricci-recurrent manifold* if its Weyl conformal tensor

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{&S(Y,Z)X - S(X,Z)Y \\ &+g(Y,Z)LX - g(X,Z)LY\} \\ &+ \frac{r}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right] \end{aligned} \tag{1.4}$$

satisfies

$$\nabla C = 0 \tag{1.5}$$

and its Ricci tensor satisfies the condition (1.1).

In this study, we prove that in conformally symmetric $(GR)_n$, the basic vector fields P and Q are collinear, $Q = \varphi P$, φ is constant, and this manifold is a quasi-Einstein manifold. Moreover, we show that the associated $1-$form of this manifold is covariant-constant, and we conclude that the universal cover of this manifold splits geometrically as $L^1 x N^{n-1}$, where L is a line, $(N^{n-1}, g_{N^{n-1}})$ is Einstein, $\varphi = -\frac{1}{n}r$. We obtain that in a conformally symmetric $(GR)_n$, the basic vector field P is an eigenvector of the Ricci tensor corresponding to the eigenvalue $\frac{r+(n-2)\varphi}{2}$ and is a recurrent vector field. Then, we deal with a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature. We prove that a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature is a quasi-Einstein spacetime. We obtain that in such a spacetime, if the matter distribution is a perfect fluid whose velocity vector is a vector field corresponding to 1-form A of the spacetime, then the energy density σ and the isotropic pressure p are constants and the cosmological constant satisfies the relation $\frac{r}{3} < \lambda < 0$. We show that in such a spacetime, if the matter content is a perfect fluid whose velocity vector is a vector field corresponding to 1-form A of the spacetime, then the acceleration vector of the fluid and the expansion scalar must be equal to zero. Finally, we prove that in a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature obeying Einstein's field equations without a cosmological constant, if the Ricci tensor obeys the timelike convergence condition, then in such a spacetime without pure matter, the pressure of the fluid is positive.

2. Preliminaries

Let L denote the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S, that is, $g(LX, Y) = S(X, Y)$ for any vector fields X, Y. Therefore,

$$g\left((\nabla_X L)Y, Z\right) = \left(\nabla_X S\right)(Y, Z). \tag{2.1}$$

Let $\overline{A}$ be a 1-form defined by

$$\overline{A}(X) = A(LX).$$

Substituting $Y = Z = e_i$ into (1.1) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold for $1 \le i \le n$ and taking summation over i, we achieve

$$dr(X) = rA(X) + nB(X). \tag{2.2}$$

In the local coordinates, (1.1) and (2.2) are respectively expressed as:

$$\nabla_k S_{ij} = A_k S_{ij} + B_k g_{ij} \tag{2.3}$$

and

$$\nabla_k r = A_k r + nB_k. \tag{2.4}$$

Further, contracting (2.3) by g^{jk} , we obtain

$$\nabla_k S_i^k = A^j S_{ij} + B_i. \tag{2.5}$$

Using $\nabla_k S_i^k = \frac{1}{2}\nabla_k r$ and (2.5), we get

$$\nabla_k r = 2A^j S_{ij} + 2B_i. \tag{2.6}$$

Combining (2.4) and (2.6), we achieve

$$A^j S_{ij} = \frac{1}{2}\left(rA_i + (n-2)B_i\right). \tag{2.7}$$

3. Conformally symmetric $(GR)_n$

In this section, we assume that the manifold $(GR)_n$ is conformally symmetric. Then, differentiating (1.4) covariantly and using (1.5) and the well-known formulas

$$\nabla_l R_{ijk} = \nabla_k S_{ij} - \nabla_j S_{ik} \qquad \text{and} \qquad \nabla_l S_j = \frac{1}{2}\nabla_l r,$$

we obtain

$$\left(\nabla_X S\right)(Y,Z) - \left(\nabla_Z S\right)(X,Y) = \frac{1}{2(n-1)}\left[dr(X)g(Y,Z) - dr(Z)g(X,Y)\right] \tag{3.1}$$

[11].

Using (1.1), (2.2) and (3.1), we get

$$\overline{A}(X) = \frac{r}{2}A(X) + \frac{(n-2)}{2}B(X). \tag{3.2}$$

From (1.1) and (3.1), we achieve

$$\begin{aligned} &A(X)S(Y,Z) + B(X)g(Y,Z) - A(Z)S(X,Y) - B(Z)g(X,Y) \\ &= \frac{1}{2(n-1)}\left[dr(X)g(Y,Z) - dr(Z)g(X,Y)\right]. \end{aligned} \tag{3.3}$$

Substituting $Z = P$ into (3.3) and using (1.1) and (2.2), we obtain

$$\begin{aligned} &A(X)\overline{A}(Y) - \frac{r}{2(n-1)}A(X)A(Y) + \frac{(n-2)}{2(n-1)}B(X)A(Y) - B(P)g(X,Y) \\ &+ \frac{1}{2(n-1)}\left[rA(P) + nB(P)\right]g(X,Y) = A(P)S(X,Y). \end{aligned} \tag{3.4}$$

Using (3.2) in (3.4), we get

$$S(X,Y) = \left(\frac{r(n-2)}{2(n-1)}\right)\frac{A(X)A(Y)}{A(P)} + \left(\frac{n(n-2)}{2(n-1)}\right)\frac{B(X)A(Y)}{A(P)} + \left(\frac{1}{2(n-1)}\right)\frac{[rA(P)-(n-2)B(P)]}{A(P)}g(X,Y). \tag{3.5}$$

Since this expression must be symmetric in X, Y, by using (3.5) we obtain

$$\frac{n(n-2)}{2(n-1)A(P)}\left[B(X)A(Y) - B(Y)A(X)\right] = 0. \tag{3.6}$$

This implies that the basic vector fields P and Q in the manifold under consideration are co-directional, namely $Q = \varphi P$, where φ is a non-zero scalar function. Hence

$$B(X) = \varphi A(X), \tag{3.7}$$

for all X.

Now, using (3.7), (1.1) becomes

$$(\nabla_X S)(Y,Z) = [S(Y,Z) + \varphi g(Y,Z)]\, A(X). \tag{3.8}$$

In local coordinates, this equation becomes

$$\nabla_k S_{ij} = [S_{ij} + \varphi g_{ij}]\, A_k. \tag{3.9}$$

Transvecting the above equation by A twice provides the following.

On the left side:

$$A^i A^j \nabla_k S_{ij} = A^i \nabla_k\left(A^j S_{ij}\right) - A^i\left(\nabla_k A^j\right) S_{ij}. \tag{3.10}$$

Considering (3.7), it follows from (2.7) that

$$A^j S_{ij} = \frac{1}{2}\left(r + (n-2)\varphi\right) A_i. \tag{3.11}$$

Substituting (3.11) into (3.10) yields

$$A^i A^j \nabla_k S_{ij} = \frac{1}{2}|A|^2 \nabla_k\left(r + (n-2)\varphi\right), \tag{3.12}$$

where $|A|^2 = A(P)$.

On the right side: Considering (3.11), we have

$$A^i A^j\left(S_{ij} + \varphi g_{ij}\right) A_k = \left(A^i A^j S_{ij} + \varphi|A|^2\right) A_k. \tag{3.13}$$

Substituting (3.11) into the above expression yields

$$A^i A^j\left(S_{ij} + \varphi g_{ij}\right) A_k = \frac{1}{2}\left(r + n\varphi\right)|A|^2 A_k. \tag{3.14}$$

Now, considering (2.4), from (3.7) we obtain

$$\nabla_k r = \left(r + n\varphi\right) A_k. \tag{3.15}$$

From (3.14) and (3.15)

$$A^i A^j\left(S_{ij} + \varphi g_{ij}\right) A_k = \frac{1}{2}\nabla_k r|A|^2. \tag{3.16}$$

Hence, combining (3.12) and (3.16), we obtain

$$\frac{1}{2}|A|^2 \nabla_k\left(r + (n-2)\varphi\right) = \frac{1}{2}\nabla_k r|A|^2. \tag{3.17}$$

From this expression, we achieve

$$(n-2)\nabla_k\varphi = 0, \tag{3.18}$$

which implies that φ is constant.

Hence, we prove the following theorem.

Theorem 3.1. *In a conformally symmetric $(GR)_n$, the basic vector fields P and Q are collinear, $Q = \varphi P$, and φ is constant.*

Considering (3.5), we obtain

$$S(X,Y) = \frac{(r-(n-2)\varphi)}{2(n-1)} g(X,Y) + \left(\frac{(n-2)(r+n\varphi)}{2(n-1)}\right)\frac{A(X)A(Y)}{A(P)}. \tag{3.19}$$

Hence we can write

$$S(X,Y) = \frac{(r-(n-2)\varphi)}{2(n-1)} g(X,Y) + \left(\frac{(n-2)(r+n\varphi)}{2(n-1)}\right)E(X)E(Y)$$

where $E(X) = \frac{A(X)}{\sqrt{A(P)}}$.

By considering (1.3), the above equation implies that such a manifold is quasi-Einstein. Thus we can state the following theorem.

Theorem 3.2. *A conformally symmetric $(GR)_n$ is a quasi-Einstein manifold.*

Now substituting $Y = P$ into (3.19), we achieve

$$S(X,P) = \left(\frac{r+(n-2)\varphi}{2}\right)A(X) = \left(\frac{r+(n-2)\varphi}{2}\right)g(X,P). \tag{3.20}$$

Then the following theorem holds true:

Theorem 3.3. *The basic vector field P is an eigenvector of the Ricci tensor corresponding to the eigenvalue $\frac{r+(n-2)\varphi}{2}$.*

Since φ is constant, we have

$$\nabla_k(r+n\varphi) = (r+n\varphi)A_k. \tag{3.21}$$

In the local coordinates, (3.19) becomes

$$S_{ij} = \frac{(r-(n-2)\varphi)}{2(n-1)} g_{ij} + \left(\frac{(n-2)(r+n\varphi)}{2(n-1)}\right)\frac{A_iA_j}{|A|^2}. \tag{3.22}$$

Considering the covariant derivative of this expression and using (3.21), we obtain

$$\nabla_k S_{ij} = \frac{(r+n\varphi)}{2(n-1)}\left(\frac{A_iA_j}{|A|^2} + g_{ij}\right)A_k + \frac{(n-2)(r+n\varphi)}{2(n-1)}\nabla_k\left(\frac{A_iA_j}{|A|^2}\right). \tag{3.23}$$

Moreover, substituting (3.22) into (3.8), we achieve

$$\nabla_k S_{ij} = \frac{(r+n\varphi)}{2(n-1)}\left(\frac{A_iA_j}{|A|^2} + g_{ij}\right). \tag{3.24}$$

By equating (3.23) and (3.24), we obtain

$$\nabla_k\left(\frac{A_i}{|A|}\frac{A_j}{|A|}\right) = 0. \tag{3.25}$$

Hence, we conclude that $E = \frac{A}{|A|}$ is covariant-constant.

Then we can state the following theorem.

Theorem 3.4. *In a conformally symmetric $(GR)_n$, the associated $1-$form E is covariant-constant, where $E = \frac{A}{|A|}$.*

Corollary 3.5. *Since the $1-$form A is not equal to zero on any open set in a conformally symmetric $(GR)_n$, we show that $E = \frac{A}{|A|}$ is covariant-constant. Therefore, the universal cover of this manifold splits geometrically as $(GR)_n = L^1 x N^{n-1}$, where L^1 is a line. From (3.19), we conclude that*

$$S_{N^{n-1}} = \frac{r-(n-2)\varphi}{2(n-1)} g_{N^{n-1}}$$

on N^{n-1}. Thus, we can say that $\left(N^{n-1}, g_{N^{n-1}}\right)$ is Einstein, and $\varphi = -\frac{r}{n}$.

Substituting (3.7) and (3.20) into (1.1), we obtain

$$(\nabla_X S)(Y,P) = \left(\frac{r+n\varphi}{2}\right) A(X)A(Y). \tag{3.26}$$

The covariant derivative of the Ricci tensor S is

$$(\nabla_X S)(Y,P) = \nabla_X S(Y,P) - S(\nabla_X Y, P) - S(Y, \nabla_X P). \tag{3.27}$$

Moreover, we have

$$\begin{aligned}(\nabla_X A)(Y) &= \nabla_X A(Y) - A(\nabla_X Y) \\ &= \nabla_X g(Y,P) - g(\nabla_X Y, P) \\ &= g(Y, \nabla_X P),\end{aligned} \tag{3.28}$$

for $(\nabla_X g)(Y,P) = 0$, and

$$A(\nabla_X P) = \frac{1}{2} X(A(P)). \tag{3.29}$$

From (3.20) and (3.27), we obtain

$$(\nabla_X S)(Y,P) = \left(\frac{r+(n-2)\varphi}{2}\right)(\nabla_X A)(Y) - S(Y,\nabla_X P). \tag{3.30}$$

Using (3.19), we get

$$S(Y,\nabla_X P) = \left(\frac{r-(n-2)\varphi}{2(n-1)}\right)(\nabla_X A)(Y) + \left(\frac{(n-2)(r+n\varphi)}{2(n-1)A(P)}\right) A(Y)A(\nabla_X P). \tag{3.31}$$

Using (3.26) and (3.31) in (3.30), we have

$$A(X)A(Y) = \frac{(n-2)}{(n-1)}(\nabla_X A)(Y) - \frac{(n-2)}{(n-1)A(P)} A(Y)A(\nabla_X P). \tag{3.32}$$

From (3.28) and (3.29), we achieve

$$g(Y,\nabla_X P)=\frac{(n-1)}{(n-2)}A(X)A(Y)+\frac{1}{2A(P)}X(A(P))A(Y). \tag{3.33}$$

Thus, we get

$$\nabla_X P=\beta(X)P, \tag{3.34}$$

where $\beta(X)=\frac{(n-1)}{(n-2)}A(X)+\frac{1}{2A(P)}X(A(P))$. Hence P is a recurrent vector field. Thus we have proved that the following:

Theorem 3.6. *In a conformally symmetric $(GR)_n$, the basic vector field P is a recurrent vector field.*

Corollary 3.7. *Since equation (3.7) holds, the basic vector field Q is also a recurrent vector field.*

4. Conformally symmetric generalized Ricci-recurrent spacetime

The spacetime of general relativity is regarded as a connected four− dimensional semi-Riemannian manifold (M,g) with Lorentzian metric g with signature $(-,+,+,+)$ and a perfect fluid spacetime is a spacetime whose matter content is a perfect fluid. The geometry of the Lorentzian manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity [8].

In this section, we deal with a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature and having the basic vector P as the timelike vector field of the fluid, that is,

$$A(X)=g(X,P),\quad g(P,P)=-1.$$

Now from (1.1) and (2.1), it follows that

$$\begin{aligned}(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)&=A(X)S(Y,Z)+B(X)g(Y,Z)\\&-A(Z)S(Y,X)-B(Z)g(Y,X)\end{aligned} \tag{4.1}$$

and

$$(\nabla_X L)Z-(\nabla_Z L)X=A(X)LZ-A(Z)LX+B(X)Z-B(Z)X. \tag{4.2}$$

Contracting (4.2), from r =constant, we get

$$2rA(X)+6B(X)=\overline{A}(X). \tag{4.3}$$

Since r is constant, from (2.2), it follows that

$$B(X)=-\frac{r}{4}A(X). \tag{4.4}$$

From (4.3) and (4.4), we have

$$\overline{A}(X)=\frac{r}{2}A(X). \tag{4.5}$$

Moreover, again from r =constant, using (3.1) we get

$$(\nabla_X S)(Y,Z)-(\nabla_Z S)(X,Y)=0. \tag{4.6}$$

Using (4.6) in (4.1), we have

$$A(X)S(Y,Z)+B(X)g(Y,Z)-A(Z)S(Y,X)-B(Z)g(Y,X)=0. \tag{4.7}$$

Substituting $Z=P$ into (4.7) and $g(P,P)=-1$, we obtain

$$A(X)\overline{A}(Y)+S(Y,X)+B(X)A(Y)-B(P)g(Y,X)=0. \tag{4.8}$$

Using (4.4) and (4.5) in (4.8), we get

$$S(X,Y)=B(P)g(X,Y)-\frac{r}{4}A(X)A(Y). \tag{4.9}$$

Thus we have proved that the following.

Theorem 4.1. *A conformally symmetric $(GR)_4$ with non-zero constant scalar curvature is a quasi-Einstein spacetime.*

Since a quasi-Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [6], a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature can be considered as a model of the perfect fluid spacetime in general relativity.

For the perfect fluid spacetime, we have Einstein's field equations with a cosmological constant can be written as

$$S(X,Y)-\frac{1}{2}rg(X,Y)+\lambda g(X,Y)=kT(X,Y), \tag{4.10}$$

where k is the gravitational constant and T is the energy− momentum tensor of type $(0,2)$ given by

$$T(X,Y)=(\sigma+p)A(X)A(Y)+pg(X,Y). \tag{4.11}$$

where σ is the energy density and p is the isotropic pressure of the fluid [15].

Einstein's field equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely, the motion of matter is determined by the metric tensor of space which non-flat.

Now (4.10) can be written as follows:

$$S(X,Y)-\frac{1}{2}rg(X,Y)+\lambda g(X,Y)=k\left[(\sigma+p)A(X)A(Y)+pg(X,Y)\right]. \tag{4.12}$$

Then it follows that

$$S(X,Y)=\left(kp+\frac{r}{2}-\lambda\right)g(X,Y)+k(\sigma+p)A(X)A(Y). \tag{4.13}$$

Let us now substitute $Y=P$ into (4.9). Then, we have

$$S(X,P)=\frac{r}{2}A(X). \tag{4.14}$$

Substituting $Y=P$ into (4.13) gives

$$S(X,P)=\left(\frac{r}{2}-\lambda-k\sigma\right)A(X). \tag{4.15}$$

Using (4.14), (4.15) gives

$$\sigma = -\frac{\lambda}{k}. \tag{4.16}$$

Taking a frame field and contracting (4.13) over X and Y we obtain by using (4.16)

$$p = \frac{1}{3k}(3\lambda - r). \tag{4.17}$$

From (4.16), it follows that σ is constant. Since r is constant, p is also constant. Moreover, from $\sigma > 0$ and (4.16) we have $\lambda < 0$. Again from (4.17), since $p > 0$, we have $\lambda > \frac{r}{3}$.

Hence we can state the following theorem.

Theorem 4.2. *In a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature, if the matter distribution is perfect fluid whose velocity vector is a vector field corresponding to the* 1*-form* A *of the spacetime, then the energy density* σ *and the isotropic pressure* p *are constants and the cosmological constant satisfies the relation* $\frac{r}{3} < \lambda < 0$.

The energy equation and the force equations for a perfect fluid are respectively expressed as :

$$P.\sigma = -(\sigma + p)\,\mathrm{div}\,P, \tag{4.18}$$

and

$$(\sigma + p)\nabla_P P = -\,\mathrm{grad}\,p - (P.p)P \tag{4.19}$$

[15].

Since p and σ are constants and $\sigma + p \neq 0$, it follows from (4.18) and (4.19) that $\mathrm{div}\,P = 0$ and $\nabla_P P = 0$, where $\mathrm{div}\,P$ and $\nabla_P P$ represent the expansion scalar and the acceleration vector, respectively. Thus, we can say that both the expansion scalar and the acceleration vector become zero.

Then we can state the following theorem.

Theorem 4.3. *In a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature, if the matter content is perfect fluid whose velocity vector is a vector field corresponding to the* 1*-form* A *of the spacetime, then the acceleration vector of the fluid and the expansion scalar must be equal to zero.*

5. Einstein's Field Equations without cosmological constant

Let us now assume that a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature is a perfect fluid spacetime without cosmological constant. Einstein's field equations without cosmological constant can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) = k\left[(\sigma + p)A(X)A(Y) + pg(X,Y)\right] \tag{5.1}$$

[15].

Taking a frame field and contracting (5.1) over X and Y, we get

$$r = k(\sigma - 3p). \tag{5.2}$$

Now using (5.2) in (4.14), we obtain

$$S(X,P) = -\frac{k(\sigma+3p)}{2}A(X). \tag{5.3}$$

Substituting $X = P$ into (5.3), we achieve

$$S(P,P) = \frac{k(\sigma+3p)}{2}. \tag{5.4}$$

If the Ricci tensor S of type $(0,2)$ of the spacetime satisfies the condition

$$S(X,X) > 0, \tag{5.5}$$

for every X, then (5.5) is called the timelike convergence condition [18].

Let us now determine the sign of the pressure in such a spacetime without pure matter. Since, here $\lambda = 0$, we obtain from (4.16)

$$\sigma = 0. \tag{5.6}$$

(5.6) and (5.2) yield

$$r = -3pk. \tag{5.7}$$

From (4.11), we have

$$T(P,P) = \sigma = 0. \tag{5.8}$$

Hence, from (5.1), (5.7) and (5.8), we achieve

$$p = \frac{2}{3k}S(P,P). \tag{5.9}$$

Since $S(P,P) > 0$, it follows from (5.9) that $p > 0$.

Then we can state the following theorem.

Theorem 5.1. *In a conformally symmetric generalized Ricci-recurrent spacetime of non-zero constant scalar curvature satisfying Einstein's field equations without cosmological constant, if the Ricci tensor obeys the timelike convergence condition, then in such a spacetime without pure matter, the pressure of the fluid is positive.*

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