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EXISTENCE AND UNIQUENESS OF A RENORMALIZED SOLUTION OF A MULTIVALUED FOURIER PROBLEM WITH L^1-DATA

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Abstract. In this manuscript, we are concerned with the existence and uniqueness of the renormalized solution of a nonlinear elliptic problem $\beta(u) - \operatorname{div} \mathbf{a}(x, Du) \ni f$ in Ω, coupled with a Fourier boundary conditions $\mathbf{a}(x, Du).\eta + \lambda u = g$ on $\partial\Omega$, where f and g are functions of $L^1(\Omega)$ and $L^1(\partial\Omega)$ respectively. The functional setting involves Lebesgue and Sobolev spaces with variable exponent. Some a-priori estimates are used to obtain our results.

1. Introduction

The literature contains a wide field of research on Fourier-type problems involving L^1-data. Many of these studies deal with non-homogeneous boundary conditions of the Fourier type. The aim of this paper is to investigate the nonlinear elliptic boundary value problem

$$P^{\beta}_{f,g} \quad \begin{cases} \beta(u) - \operatorname{div} \mathbf{a}(x, Du) \ni f & \text{in } \Omega, \\ \\ \mathbf{a}(x, Du).\eta + \lambda u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ is a bounded domain with boundary $\partial\Omega$, η is the unit outward normal vector on $\partial\Omega$. The function f is in $L^1(\Omega)$, g is in $L^1(\partial\Omega)$ and $(\lambda > 0)$. The graph $\beta = \partial j$ is maximal monotone defined on $\mathbb{R}^2$, with the additional constraint that $0 \in \beta(0)$. The vector field $\mathbf{a}$ is a Carathodory function such that $\mathbf{a}(x, .)$ is continuous for almost everywhere x in Ω and measurable on Ω for every ξ in $\mathbb{R}^N$ and subject to followings conditions:

(H1) (coercivity) There exist constants $\alpha > 0$ for every $x \in \Omega$ and all $\xi \in \mathbb{R}^N$,

$$\mathbf{a}(x, \xi).\xi \geq \alpha |\xi|^{p(\cdot)}.$$

(H2) (monotonicity) For all $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$ there holds

$$(\mathbf{a}(x, \xi) - \mathbf{a}(x, \eta)).(\xi - \eta) > 0.$$

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(H3) (growth) There is a constant $\Lambda > 0$ and a function $K \in L^{p'(\cdot)}(\Omega)$, such that, for all $(x,\xi) \in \Omega \times \mathbb{R}^N$,

$$|\mathbf{a}(x,\xi)| \leq \Lambda(K(x) + |\xi|^{p(\cdot)-1}).$$

Recently, (cf.[10]) the same type of problem with a Fourier boundary condition, with the function $b : \mathbb{R} \to \mathbb{R}$ continuous, nondecreasing and surjective, instead of a maximal monotone graph β, is studied, and the existence and uniqueness of entropy solutions have been proved in the L^1-setting. The exponent that appears in **(H1)** and **(H3)** depends on the variable x that is we seek solutions to the problem in variables exponent space. The motivation for working in these spaces arises from their application in modeling electrorheological and thermorheological fluids (as cited in [14]), as well as in image restoration (as cited in [4]). When dealing with a problem where the right-hand side is in L^1, Di Perna-Lions introduced the concept of a renormalized solution, which can be applied when the exponent p is constant. However, if p depends on x, the notion proposed by Wittbold and Zimmermann (as cited in [16]) can be adopted and applied to our particular scenario. The rest of the paper is organized as follows: In section 2, we recall the basic proprieties of Lebesgue and Sobolev spaces with variable exponent and our main result is stated and proven in section 3.

2. Preliminaries

Let us compile some fundamental properties of Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and generalized Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ here for convenience. Let us begin to defined the following set

$$C^+(\overline{\Omega}) = \{p : \overline{\Omega} \to \mathbb{R}^+ : p \textit{ is continuous and such that } 1 < p_- \leq p_+ < \infty\},$$

where

$$p_- = \min_{x\in\overline{\Omega}} p(\cdot) \quad \textit{and} \quad p_+ = \max_{x\in\overline{\Omega}} p(\cdot).$$

For any $p(\cdot) \in C^+(\overline{\Omega})$, we recall the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ as

$$L^{p(\cdot)}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable: } \int_\Omega |f|^{p(\cdot)}dx < +\infty\},$$

endowed with the Luxembourg norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0, \int_\Omega \left|\frac{f}{\lambda}\right|^{p(\cdot)} dx \leq 1\}.$$

The space $\left(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)}\right)$ is a separable Banach space. Moreover $L^{p(\cdot)}(\Omega)$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$, where $1/p(\cdot) + 1/p'(\cdot) = 1$. Next (see [7]), we recall the two inequalities below:

(i) For any $f \in L^{p(\cdot)}(\Omega)$ and all $g \in L^{p'(\cdot)}(\Omega)$, then we have the Hölder inequality:

$$\left|\int_\Omega fg dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.1}$$

(ii) If $p(\cdot) \leq q(\cdot)$ a.e. in Ω, and $|\Omega| < \infty$, then for all $f \in L^{p(\cdot)}$,

$$\|f\|_{p(\cdot)} \leq (1 + |\Omega|)\|f\|_{q(\cdot)} \tag{2.2}$$

We define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega) := \{f \in L^{p(\cdot)}(\Omega) : |Df| \in L^{p(\cdot)}(\Omega)^N\}$$

with the norm,

$$\|f\|_{1,p(\cdot)} := \|f\|_{p(\cdot)} + \|Df\|_{p(\cdot)}, \quad \forall f \in W^{1,p(\cdot)}(\Omega).$$

The Banach space $\left(W^{1,p(\cdot)}(\Omega), \|f\|_{1,p(\cdot)}\right)$ is both separable and reflexive. The modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$, defined as $\rho_{p(\cdot)}(f) = \int_\Omega |f|^{p(\cdot)}dx$, plays a crucial role in manipulating the generalized Lebesgue and Sobolev spaces. The subsequent result is of great importance and will be useful for our purposes.

Proposition 2.1. . *(see.* [6]*) If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*

(i) $\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|^{p_-}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p_+}_{p(\cdot)}$;

(ii) $\|u\|_{p(\cdot)} < 1 \Longrightarrow \|u\|^{p_+}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p_-}_{p(\cdot)}$;

(iii) $\|u\|_{p(\cdot)} < 1$ *(resp.* $= 1; > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ *(resp.* $= 1; > 1$*);*

(iv) $\|u_n\|_{p(\cdot)} \to 0$ *(resp.* $\to +\infty) \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to 0$*(resp.* $\to +\infty$*);*

(v) $\rho_{p(\cdot)}\left(\frac{u}{\|u\|_{p(\cdot)}}\right) = 1$.

Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the function $\rho_{1,p(\cdot)}(u)$ as follows:

$$\rho_{1,p(\cdot)}(u) := \int_\Omega |u|^{p(\cdot)}, dx + \int_\Omega |\nabla u|^{p(\cdot)}, dx.$$

The following proposition, which can be found in [17], holds:

Proposition 2.2. *If $u \in W^{1,p(\cdot)}(\Omega)$, then the following properties hold:*

(i) $\|u\|_{1,p(\cdot)} > 1 \Longrightarrow \|u\|^{p_-}_{1,p(\cdot)} \leq \rho_{1,p(\cdot)}(u) \leq \|u\|^{p_+}_{1,p(\cdot)}$;

(ii) $\|u\|_{1,p(\cdot)} < 1 \Longrightarrow \|u\|^{p_+}_{1,p(\cdot)} \leq \rho_{1,p(\cdot)}(u) \leq \|u\|^{p_-}_{1,p(\cdot)}$;

(iii) $\|u\|_{1,p(\cdot)} < 1$ *(resp.* $= 1; > 1) \Longleftrightarrow \rho_{1,p(\cdot)}(u) < 1$ *(resp.* $= 1; > 1$*).*

For additional properties of variable exponents, we direct the reader to the work of Kovacik and Rakosnik in [7]. We bring to mind two inequalities: the Poincar-type inequality and the Poincar-Sobolev type inequality.

Lemma 2.3. *(cf.* [13]*) There exists a constant $C_1' > 0$ for every $u \in W^{1,1}(\Omega)$, we have*

$$\int_\Omega |u|dx \leq C_1' \left(\int_\Omega |Du|dx + \int_{\partial\Omega} |u|d\sigma\right) \qquad \textit{(Poincaré's type inequality)} \tag{2.3}$$

and there exists a constant $C_2' > 0$ for every $u \in W^{1,q}(\Omega)$, $1 < q < N$, we have

$$\left(\int_\Omega |u|^{q^*} dx\right)^{\frac{q}{q^*}} \leq C_2' \left(\int_\Omega |Du|^q dx + \left(\int_{\partial\Omega} |u|d\sigma\right)^q\right) \qquad \textit{(Poincaré-Sobolev type inequality).} \tag{2.4}$$

We will use the following notations in the rest of this paper: Let A be a measurable subset of $\mathbb{R}^N$. Its N-dimensional Lebesgue measure and its characteristic function are denoted by $|A|$ and χ_A, respectively. The positive and negative parts of r are defined as $r^+ = \max(r, 0)$ and $r^- = (-r)^+$. We denote $\text{sign}_0(r)$ as 1, 0, or -1, depending on whether $r > 0$, $r = 0$, or $r \leq 0$. Similarly, $\text{sign}_0^+(r)$ is denoted as 1 or 0, depending on whether $r > 0$ or $r \leq 0$. Furthermore, we will use the notations $u \wedge v = \min(u, v)$ and $u \vee v = \max(u, v)$. Throughout the paper, T_k denotes the truncation function at height $k > 0$ defined by $T_k(r) = \max\{-k, \min(k, r)\}$ for all $r \in \mathbb{R}$ and also define the continuous function $h_n : \mathbb{R} \to \mathbb{R}$ by $h_n(r) = \min((n + 1 - |r|)^+, 1)$ for all $n \in \mathbb{R}$.
For any monotone graph γ in $\mathbb{R} \times \mathbb{R}$ and $\epsilon > 0$, we denote by γ_ϵ the Yosida approximation of γ, given by $\gamma_\epsilon = \epsilon\Big(I - \Big(I + \frac{1}{\epsilon}\gamma\Big)^{-1}\Big)$. Note that γ_ϵ is maximal monotone and Lipschitz.
We recall the definition of the main section γ_0 of γ:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s) & \text{if } \gamma(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap Dom(\gamma) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap Dom(\gamma) = \emptyset. \end{cases}$$

Before to give the definition of renormalized solution, let us define the gradient of measurable functions whose truncates have finite energy (see [2]).

Lemma 2.4. *For every $u \in \mathcal{T}^{1,1}_{loc}(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$DT_k(u) = v\chi_{\{|u|<k\}} \text{ a.e. in } \Omega.$$

Furthermore, $u \in W^{1,1}_{loc}(\Omega)$ if and only if $v \in L^1_{loc}(\Omega)$, and then $v \equiv Du$ in the usual weak sense.

We also define the space

$$\mathcal{T}^{1,p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{measurable} \quad \text{such that}, T_k(u) \in W^{1,p(\cdot)}(\Omega), \forall k > 0\}.$$

On the other hand, we define $\mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,p(\cdot)}(\Omega)$ satisfying the following conditions:
(i) $u_n \to u$ a.e. in Ω,
(ii) $DT_k(u_n) \to DT_k(u)$ in $L^1(\Omega)$ for any $k > 0$,
(iii) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.

The function v is the trace of u in the generalized sense (see [2]). In the sequel the trace of $u \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(\cdot)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\phi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \phi) \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ and $tr(u - \phi) = tr(u) - tr(\phi)$.

3. Main Results

Definition 3.1. *A renormalized solution to $P^{\beta}_{f,g}$ is a pair of functions (u, b) satisfying the following conditions:*
(i) $u : \Omega \to \mathbb{R}$ is measurable and finite a.e. in Ω, $tr(u) \in L^1(\partial\Omega)$, $b \in L^1(\Omega)$ and $b(x) \in \beta(u(x))$ for a.e. x in Ω,

(ii) for each $k > 0$, $T_k(u) \in W^{1,p(\cdot)}(\Omega)$ *and*
(iii)

$$\int_\Omega bh(u)\xi dx + \int_\Omega \mathbf{a}(x, Du).D(h(u)\xi)dx + \lambda \int_{\partial\Omega} uh(u)\xi d\sigma = \int_\Omega fh(u)\xi dx + \int_{\partial\Omega} gh(u)\xi d\sigma, \tag{3.1}$$

holds for all $h \in C_c^1(\mathbb{R})$ *and* $\xi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.
Moreover,

$$\lim_{k\to\infty} \int_{\{k<|u|<k+1\}} |Du|^{p(\cdot)}\, dx = 0. \tag{3.2}$$

Remark. *If* (u, b) *is a renormalized solution of problem* $P^\beta_{f,g}$ *then, it satisfying the following entropy formulation:*

$$\int_\Omega \mathbf{a}(x, Du).DT_k(u - \xi)\, dx \leq \int_\Omega (f - b)\, T_k(u - \xi)\, dx + \int_{\partial\Omega} (g - \lambda u)\, T_k(u - \xi) d\sigma, \tag{3.3}$$

for all $\xi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *such that* $\xi(x) \in \beta(u(x))$ *for a.e.* x *in* Ω.

Remark. *Each term in equation* (3.1) *is clearly well-defined, and the condition expressed in equation* (3.2) *is a necessity in the context of renormalized solutions. Additionally, it provides further details about* Du.

Proposition 3.2 presented below establishes the connection between the concepts of renormalized and entropy solution.

Proposition 3.2. *(see* [11]*) Let's* $f \in L^1(\Omega)$ *and* $g \in L^1(\partial\Omega)$. *Under assumptions* ***(H1)*** *–* ***(H3)****, renormalized solution and entropy solution of problem* $P^\beta_{f,g}$ *are equivalent.*

The following result is the most significant one that we establish

Theorem 3.3. *Assume that* ***(H1)*** *–* ***(H3)*** *hold. Then, the problem* $P^\beta_{f,g}$ *has a unique renormalized solution.*

Proof. To prove the existence of renormalized solution, we use approximate methods for the multi-step proof. Firstly, we prove, for bounded data $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial\Omega)$, the existence of a weak solution u of the elliptic problem with additional strongly monotone perturbation. Secondly, for L^1-data, we approximate f and g by $f_{\mu,\nu} = (f \wedge \mu) \vee (-\nu)$ and $g_{\mu,\nu} = (g \wedge \mu) \vee (-\nu)$ respectively, non decreasing in μ, non increasing in ν such that $\|f_{\mu,\nu}\|_1 \leq \|f\|_1$ and $\|g_{\mu,\nu}\|_1 \leq \|g\|_1$. Our third step is to establish the uniqueness of a renormalized solution to $P^\beta_{f,g}$.

3.1. Existence and uniqueness results for L^∞-data.

Proposition 3.4. *For* $f \in L^\infty(\Omega)$ *and* $g \in L^\infty(\partial\Omega)$*, there exists at least one renormalized solution* (u, b) *of* $P^\beta_{f,g}$.

Proof. Our approach involves approximation through the use of a non-decreasing function $s \mapsto |s|^{p(\cdot)-2}s$ and a minimization technique. We will prove certain preliminary estimates and convergence results that will enable us to pass to the limit

Step 1: Approximate solution for L^∞-data.
Consider the following penalized approached problem for $\epsilon > 0$

$$P_{f,g}^{\beta_\epsilon}\begin{cases} \beta_\epsilon(T_{1/\epsilon}(u_\epsilon)) + \epsilon|u_\epsilon|^{p(\cdot)-2}u_\epsilon - \operatorname{div} \mathbf{a}(x, Du_\epsilon) = f & \text{in} \quad \Omega, \\ \mathbf{a}(x, Du_\epsilon).\eta + \lambda u_\epsilon = g & \text{on} \quad \partial\Omega, \end{cases}$$

with β_ϵ, the Yosida approximation of β as defined in section 3.

Proposition 3.5. *For every $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$ there exists at least a weak solution*
$u_\epsilon \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\partial\Omega)$ of $P_{f,g}^{\beta_\epsilon}$.

Proof. (of Proposition 3.5). The operators A_ϵ is defined for every $\epsilon > 0$ from $W^{1,p(\cdot)}(\Omega)$ to $\left(W^{1,p(\cdot)}(\Omega)\right)^*$ as follows:

$$\langle A_\epsilon u_\epsilon, \xi\rangle = \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))\xi dx + \epsilon \int_\Omega |u_\epsilon|^{p(\cdot)-2}u_\epsilon \xi dx + \lambda \int_{\partial\Omega} u_\epsilon\, \xi d\sigma + \int_\Omega \mathbf{a}(x, Du_\epsilon).D\xi dx.$$

$\langle ., .\rangle$ is the duality bracket between $W^{1,p(\cdot)}(\Omega)$ and its dual space $\left(W^{1,p(\cdot)}(\Omega)\right)^*$.
We assert that A_ϵ is surjective through this lemma.

Lemma 3.6. *The operator A_ϵ is bounded, coercive and verifies the (M)-property.*

Proof. The proof is divided into several claims.

Claim 1: A_ϵ is bounded.
To do this let's take $u_\epsilon \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\partial\Omega)$,

$$\langle A_\epsilon u_\epsilon, u_\epsilon\rangle = \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))u_\epsilon dx + \epsilon \int_\Omega |u_\epsilon|^{p(\cdot)-2}u_\epsilon u_\epsilon dx + \lambda \int_{\partial\Omega} |u_\epsilon|^2 d\sigma + \int_\Omega \mathbf{a}(x, Du_\epsilon).Du_\epsilon dx.$$

Then,

$$|\langle A_\epsilon u_\epsilon, u_\epsilon\rangle| \le \int_\Omega |\beta_\epsilon(T_{1/\epsilon}(u_\epsilon))u_\epsilon| dx + \epsilon \int_\Omega |u_\epsilon|^{p(\cdot)} dx + \lambda \int_{\partial\Omega} |u_\epsilon|^2 d\sigma + \int_\Omega |\mathbf{a}(x, Du_\epsilon).Du_\epsilon| dx. \quad (3.4)$$

As $\beta_\epsilon(T_{1/\epsilon}(u_\epsilon))$ is bounded in $L^{p'(\cdot)}(\Omega)$, then there exist a constant $C_1 > 0$ such that by using Hölder's inequality, we get

$$\int_\Omega |\beta_\epsilon(T_{1/\epsilon}(u_\epsilon))u_\epsilon| dx \le \|\beta_\epsilon(T_{1/\epsilon}(u_\epsilon))\|_{p'(\cdot)} \|u_\epsilon\|_{p(\cdot)} \le C_1 \|u_\epsilon\|_{1,p(\cdot)}. \quad (3.5)$$

The same the Hölder type inequality implies that

$$\epsilon \int_\Omega |u_\epsilon|^{p(\cdot)} dx \le \epsilon \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) |\Omega|^{\frac{1}{p'_-}} \|u_\epsilon\|_{p(\cdot)} \le C_2 \|u_\epsilon\|_{1,p(\cdot)} \quad (3.6)$$

and

$$\begin{aligned} \lambda \int_{\partial\Omega} |u_\epsilon|^2 d\sigma &\le \lambda \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u_\epsilon\|_{p'(\cdot)} \|u_\epsilon\|_{p(\cdot)} \\ &\le C_3 \|u_\epsilon\|_{1,p(\cdot)}. \end{aligned} \quad (3.7)$$

Furthermore, by applying Hölder type inequality and using the growth condition **(H3)**, we can deduce the expression of the last term in inequality (3.4) as follows:

$$\begin{aligned}\int_\Omega |\mathbf{a}(x,Du_\epsilon).Du_\epsilon| dx &\leq \Lambda\left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\|K\|_{p'(\cdot)}\,\|Du_\epsilon\|_{p(\cdot)} + \Lambda\|Du_\epsilon\|_{p(\cdot)} \\ &\leq C_4\|u_\epsilon\|_{1,p(\cdot)}. \end{aligned} \tag{3.8}$$

Gathering (3.5)-(3.8) in (3.4), it follows that there exists a positive constant C, which is dependent on C_1, C_2, C_3, and C_4,

$$|\langle A_\epsilon u_\epsilon, u_\epsilon\rangle| \leq C\|u_\epsilon\|_{1,p(\cdot)},$$

so that A_ϵ is bounded.

Claim 2: The operator A_ϵ is coercive.

$$\begin{aligned}\langle A_\epsilon u_\epsilon, u_\epsilon\rangle &= \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))u_\epsilon dx + \lambda\int_{\partial\Omega}|u_\epsilon|^2 d\sigma \\ &\quad +\epsilon\int_\Omega |u_\epsilon|^{p(\cdot)}dx + \int_\Omega \mathbf{a}(x,Du_\epsilon).Du_\epsilon dx.\end{aligned}$$

Thanks to the monotonicity of $\beta_\epsilon\circ T_{1/\epsilon}$, we use the hypothesis **(H1)** to obtain

$$\begin{aligned}\langle A_\epsilon u_\epsilon, u_\epsilon\rangle &\geq \int_\Omega \mathbf{a}(x,Du_\epsilon).Du_\epsilon dx + \epsilon\int_\Omega |u_\epsilon|^{p(\cdot)}dx \\ &\geq \alpha\int_\Omega |Du_\epsilon|^{p(\cdot)}dx + \epsilon\int_\Omega |u_\epsilon|^{p(\cdot)}dx \\ &\geq \min(\alpha,\epsilon)\left(\int_\Omega |Du_\epsilon|^{p(\cdot)}dx + \int_\Omega |u_\epsilon|^{p(\cdot)}dx\right) \\ &\geq \min(\alpha,\epsilon)\|u_\epsilon\|_{1,p(\cdot)}^{p(\cdot)}.\end{aligned} \tag{3.9}$$

It come from (3.9) that

$$\frac{\langle A_\epsilon u_\epsilon, u_\epsilon\rangle}{\|u_\epsilon\|_{1,p(\cdot)}} \geq \min(\alpha,\epsilon)\|u_\epsilon\|_{1,p(\cdot)}^{p(\cdot)-1}, \tag{3.10}$$

and since $p(\cdot)>1$, $\dfrac{\langle A_\epsilon u_\epsilon, u_\epsilon\rangle}{\|u_\epsilon\|_{1,p(\cdot)}} \to +\infty$ as $\|u_\epsilon\|_{1,p(\cdot)}\to\infty$. Then A_ϵ is coercive.

We recall the notion of operator of type (M).

Definition 3.7. *(Definition 8.3 [8]) Let X be a reflexive Banach space. A bounded operator $\mathcal{B}$ from X to its dual X' is type (M) if*

$$\left.\begin{aligned} &u_n \rightharpoonup u \text{ in } X \\ &\mathcal{B}u_n \rightharpoonup \chi \text{ in } X' \\ &\lim_{n\to\infty}\langle \mathcal{B}u_n, u_n\rangle \leq \langle\chi, u\rangle \end{aligned}\right\} \Rightarrow \chi = \mathcal{B}u.$$

Claim 3: A_ϵ is type (M).
Let us define $A_\epsilon = \mathcal{A}_{\epsilon,1} + \mathcal{A}_{\epsilon,2}$. According to [15], if $\mathcal{A}_{\epsilon,1}$ is type (M) and $\mathcal{A}_{\epsilon,2}$ is monotone, weakly continuous then A_ϵ is type (M).

$$\begin{aligned}\langle A_\epsilon u, \xi\rangle &= \int_\Omega \mathbf{a}(x,Du).D\xi dx + \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u))\xi dx + \epsilon\int_\Omega |u|^{p(\cdot)-2}u\xi dx + \lambda\int_{\partial\Omega} u\,\xi d\sigma \\ &= \int_\Omega \mathbf{a}(x,Du).D\xi dx + \langle \mathcal{A}_{\epsilon,2}u, \xi\rangle.\end{aligned}$$

(a): For the monotony of $\mathcal{A}_{\epsilon,2}$,
we have

$$\langle \mathcal{A}_{\epsilon,2}u, \xi\rangle = \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u))\xi dx + \epsilon\int_\Omega |u|^{p(\cdot)-2}u\xi dx + \lambda\int_{\partial\Omega} u\,\xi d\sigma.$$

We have for u and v belonging to $W^{1,p(\cdot)}(\Omega)$,

$$\begin{aligned}\langle \mathcal{A}_{\epsilon,2}u - \mathcal{A}_{\epsilon,2}v, u-v\rangle &= \langle \mathcal{A}_{\epsilon,2}u, u-v\rangle - \langle \mathcal{A}_{\epsilon,2}v, u-v\rangle \\ &= \int_\Omega (\beta_\epsilon(T_{1/\epsilon}(u)) - \beta_\epsilon(T_{1/\epsilon}(v)))(u-v)dx \\ &\quad +\epsilon\int_\Omega (|u|^{p(\cdot)-2}u - |v|^{p(\cdot)-2}v)(u-v)dx \\ &\quad +\lambda\int_{\partial\Omega} |u-v|^2 d\sigma. \end{aligned} \tag{3.11}$$

From the monotony of $v \mapsto |v|^{p(\cdot)-2}v$, we deduce that

$$0 \le \epsilon\int_\Omega (|u|^{p(\cdot)-2}u - |v|^{p(\cdot)-2}v)(u-v)dx.$$

From the monotony of $\beta_\epsilon \circ T_{1/\epsilon}$, we conclude that

$$\langle \mathcal{A}_{\epsilon,2}u - \mathcal{A}_{\epsilon,2}v, u-v\rangle \ge 0. \tag{3.12}$$

(b): We prove that for all $\epsilon > 0$, the operator $\mathcal{A}_{\epsilon,2}$ is weakly continuous, that is, for all sequences $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p(\cdot)}(\Omega)$, we have $\mathcal{A}_{\epsilon,2}u_n \rightharpoonup \mathcal{A}_{\epsilon,2}u$ as $n \to \infty$.
For all $\xi \in W^{1,p(\cdot)}(\Omega)$, we have

$$\langle A_{\epsilon,2}u_n, \xi\rangle = \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_n))\xi dx + \epsilon\int_\Omega |u_n|^{p(\cdot)-2}u_n\xi dx + \int_{\partial\Omega} u_n\,\xi d\sigma. \tag{3.13}$$

We have $|\beta_\epsilon(T_{1/\epsilon}(u_n))\xi| \le \max(|\beta_\epsilon(1/\epsilon)|, |\beta_\epsilon(-1/\epsilon)|)|\xi| \in L^{p(\cdot)}(\Omega)$.
Let $(u_n) \subset W^{1,p(\cdot)}(\Omega)$ be converging weakly to some $u \in W^{1,p(\cdot)}(\Omega)$. Then $u_n \to u$ strongly in $L^{p(\cdot)}(\Omega)$. Thus, $\exists M > 0, |u_n| \le M$, so $|u_n|^{p(\cdot)-1}|\xi| \le \max(M^{p_--1}, M^{p_+-1})|\xi| \in L^{p(\cdot)}(\Omega)$.
Using the Lebesgue dominated convergence theorem, we can passing to the limit in (3.13) we obtain $\lim_{n\to+\infty}\langle A_{\epsilon,2}u_n, \xi\rangle = \langle A_{\epsilon,2}u, \xi\rangle$. We conclude that $\mathcal{A}_{\epsilon,2}u_n \to \mathcal{A}_{\epsilon,2}u$ as n goes to ∞.

For all $u, v \in W^{1,p(\cdot)}(\Omega)$ we have

$$\langle \mathcal{A}_{\epsilon,1}u - \mathcal{A}_{\epsilon,1}v, u-v\rangle = \int_\Omega \big(a(x,Du) - a(x,Dv)\big)(u-v)dx.$$

As the integral is non-negative and **a** satisfies the monotonicity condition **(H2)**, it follows that $A_{\epsilon,1}$ is monotone. Thanks to the growth condition **(H3)** on **a**, it follows that $A_{\epsilon,1}$ is hemi-continuous. We can conclude that $A_{\epsilon,1}$ is pseudo-monotone, thus type (M). Since $A_{\epsilon,1}$ of type (M) and $A_{\epsilon,2}$ is both monotone and weakly continuous, we can infer that the operator A_ϵ is also of type (M). This ends the proof of Lemma 3.6. □

By applying a well-known theorem for monotone operators (see, for instance, [9]), it follows that A_ϵ is surjective and hence that $P^{\beta_\epsilon}_{f,g}$ has at least one weak solution $u_\epsilon \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\partial\Omega)$ i.e.

$$\int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))\xi dx + \epsilon \int_\Omega |u_\epsilon|^{p(\cdot)-2} u_\epsilon \xi dx + \lambda \int_{\partial\Omega} u_\epsilon \, \xi d\sigma + \int_\Omega \mathbf{a}(x, Du_\epsilon).D\xi dx = \int_\Omega f\xi dx + \int_{\partial\Omega} g\xi d\sigma, \quad \xi \in W^{1,p(\cdot)}(\Omega). \tag{3.14}$$

□

Through a comparison principle, we demonstrate the uniqueness of solutions u_ϵ to problem $P^{\beta_\epsilon}_{f,g}$, where the right-hand sides $f \in L^\infty(\Omega)$. This principle will play an important role in the next.

Proposition 3.8. *Suppose $u_\epsilon, \tilde{u_\epsilon} \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\partial\Omega)$ are two weak solutions of $P^{\beta_\epsilon}_{f,g}$ and $P^{\beta\epsilon}_{\tilde{f},\tilde{g}}$, respectively, for fixed $\epsilon > 0$ $f, \tilde{f} \in L^\infty(\Omega)$ and $g, \tilde{g} \in L^\infty(\partial\Omega)$. The comparison principle below holds*

$$\epsilon \int_\Omega \left(|u_\epsilon|^{p(\cdot)-2} u_\epsilon - |\tilde{u}_\epsilon|^{p(\cdot)-2}\tilde{u_\epsilon}\right)^+ + \lambda \int_{\partial\Omega} (u_\epsilon - \tilde{u}_\epsilon)^+ d\sigma \leq \int_\Omega \big(f - \tilde{f}\big) sign_0^+(u_\epsilon - \tilde{u_\epsilon}) + \int_{\partial\Omega} \big(g - \tilde{g}\big) sign_0^+(u_\epsilon - \tilde{u_\epsilon}). \tag{3.15}$$

Proof. Consider two weak solutions, denoted by u_ϵ and $\tilde{u_\epsilon}$, of two different equations $P^{\beta_\epsilon}_{f,g}$ and $P^{\beta\epsilon}_{\tilde{f},\tilde{g}}$, respectively. Let k be a positive constant and define $\xi = \frac{1}{k} T_k(u_\epsilon - \tilde{u_\epsilon})^+$ as a test function in (3.14). Substituting the to equation writen in u_ϵ and $\tilde{u_\epsilon}$, we have:

$$T_1 + T_2 + T_3 + T_4 = T_5 + T_6, \tag{3.16}$$

with

$$\begin{aligned}
T_1 &= \int_\Omega \big(\beta_\epsilon(T_{1/\epsilon}(u_\epsilon)) - \beta_\epsilon(T_{1/\epsilon}(\tilde{u_\epsilon}))\big) \frac{1}{k} T_k(u_\epsilon - \tilde{u_\epsilon})^+ dx, \\
T_2 &= \epsilon \int_\Omega \left(|u_\epsilon|^{p(\cdot)-2} u_\epsilon - |\tilde{u_\epsilon}|^{p(\cdot)-2}\tilde{u_\epsilon}\right) \frac{1}{k} T_k(u_\epsilon - \tilde{u_\epsilon})^+ dx, \\
T_3 &= \lambda \int_{\partial\Omega} \big(u_\epsilon - \tilde{u_\epsilon}\big) \frac{1}{k} T_k(u_\epsilon - \tilde{u_\epsilon})^+ \, d\sigma, \\
T_4 &= \frac{1}{k} \int_U \big(a(x, Du_\epsilon) - a(x, D\tilde{u_\epsilon})\big) D(u_\epsilon - \tilde{u_\epsilon})^+ \, dx, \\
T_5 &= \int_\Omega \big(f - \tilde{f}\big) \frac{1}{k} T_k(u_\epsilon - \tilde{u_\epsilon})^+ dx, \\
T_6 &= \int_{\partial\Omega} \big(g - \tilde{g}\big) \frac{1}{k} T_k(u_\epsilon - \tilde{u_\epsilon})^+ \, d\sigma,
\end{aligned}$$

with $U = \{0 < u_\epsilon - \tilde{u_\epsilon} < k\}$. By taking the limit as $k \to 0$ in (3.16), we obtain (3.15). □

Step 2: L^∞-a priori estimate.

Lemma 3.9. *Assume **(H1)** – **(H3)** and $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$. Let u_ϵ be a weak solution of problem $P^{\beta_\epsilon}_{f,g}$, then for all $k > 0$*

$$\int_\Omega |Du_\epsilon|^{p(\cdot)} dx \leq Ck(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}) \tag{3.17}$$

and

$$\int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))\, T_k(u_\epsilon) dx \leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}) \tag{3.18}$$

hold for all $0 < \epsilon \leq 1$, where C is a positive constant. Moreover,

$$\|u_\epsilon\|_{L^1(\partial\Omega)} \leq \frac{\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}}{\lambda} \tag{3.19}$$

holds for all $0 < \epsilon \leq 1$.

Proof. Let us proves (3.17). By taking $\xi = T_k(u_\epsilon)$ as test function in (3.14), we have

$$\int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))\, T_k(u_\epsilon) dx + \epsilon \int_\Omega |u_\epsilon|^{p(\cdot)-2} u_\epsilon\, T_k(u_\epsilon) dx + \lambda \int_{\partial\Omega} u_\epsilon\, T_k(u_\epsilon) d\sigma$$
$$+ \int_\Omega \mathbf{a}(x, Du_\epsilon).Du_\epsilon dx = \int_\Omega f\, T_k(u_\epsilon) dx + \int_{\partial\Omega} g\, T_k(u_\epsilon) d\sigma. \tag{3.20}$$

Notice that the three first terms of (3.20) are nonnegative. Then, using Höder type inequality we can estimate the right-hand side of (3.20) as follows

$$\begin{aligned}\int_\Omega f\, T_k(u_\epsilon) dx + \int_{\partial\Omega} g\, T_k(u_\epsilon) d\sigma &\leq \|f\|_{L^1(\Omega)} \|T_k(u_\epsilon)\|_{L^\infty(\Omega)} + \|g\|_{L^1(\partial\Omega)} \|T_k(u_\epsilon)\|_{L^\infty(\partial\Omega)} \\ &\leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}). \end{aligned} \tag{3.21}$$

Thanks to **(H1)** we get

$$\alpha \int_\Omega |Du|^{p(\cdot)} dx \leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}), \tag{3.22}$$

which implies (3.17).
Next, from (3.20) and (3.21) we deduce (3.18). Using equations (3.20) and (3.21) once more, we can infer that

$$\int_{\partial\Omega} u_\epsilon\, T_k(u_\epsilon) d\sigma \leq \frac{k}{\lambda}(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}). \tag{3.23}$$

By dividing (3.23) by $k > 0$ and then letting $k \to 0$, we obtain

$$\int_{\partial\Omega} |u_\epsilon| d\sigma \leq \frac{1}{\lambda}(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}), \tag{3.24}$$

which gives (3.19). □

Lemma 3.10. *The sequence $\left\{\beta_\epsilon(T_{1/\epsilon}(u_\epsilon)\right\}_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

Proof. Divide the inequality (3.18) by k, we get

$$\int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))\, \frac{1}{k} T_k(u_\epsilon) dx \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}, \tag{3.25}$$

which implies, when $k \to 0$,

$$\int_\Omega |\beta_\epsilon(T_{1/\epsilon}(u_\epsilon))|dx \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}. \tag{3.26}$$

□

Lemma 3.11. *Assume **(H1)** − **(H3)** and $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$. Let u_ϵ be a weak solution of problem $P^{\beta_\epsilon}_{f,g}$, then for all $k > 0$,*

$$\int_\Omega |DT_k(u_\epsilon)|^{p_-} dx \le const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)(k+1), \tag{3.27}$$

and

$$\int_{\partial\Omega} |T_k(u_\epsilon)|d\sigma \le \frac{\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}}{\lambda}, \tag{3.28}$$

where $const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)$ is a positive constant depending on $\|f\|_{L^1(\Omega)}$, $\|g\|_{L^1(\partial\Omega)}$ and Ω.

Proof. We first prove (3.27). Observe that

$$\begin{aligned}\int_\Omega |DT_k(u_\epsilon)|^{p_-} dx &= \int_{\{|DT_k(u_\epsilon)|>1\}} |DT_k(u_\epsilon)|^{p_-} dx + \int_{\{|DT_k(u_\epsilon)|\le 1\}} |DT_k(u_\epsilon)|^{p_-} dx \\ &\le \int_{\{|DT_k(u_\epsilon)|>1\}} |DT_k(u_\epsilon)|^{p(\cdot)} dx + |\Omega| \\ &\le \int_\Omega |DT_k(u_\epsilon)|^{p(\cdot)} dx + |\Omega|. \end{aligned} \tag{3.29}$$

By the above inequalities (3.29) and thanks to (3.17), we obtain

$$\int_\Omega |DT_k(u_\epsilon)|^{p_-} dx \le const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)(k+1). \tag{3.30}$$

To prove (3.28), note that $|T_k(u_\epsilon)| \le |u_\epsilon|$, then according to (3.19), we obtain the desired result. □

The following result is necessary for our purposes.

Lemma 3.12. *Assume that **(H1)** − **(H3)** hold, $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial\Omega)$. Let u_ϵ be a weak solution of $P^{\beta_\epsilon}_{f,g}$. For k large enough, we have*

$$|\{|u_\epsilon| > k\}| \le \frac{const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, p_-, (p_-)^*, \Omega)}{k^\alpha} \tag{3.31}$$

and

$$\begin{aligned}|\{|Du_\epsilon| > k\}| \le& \frac{const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)(k+1)}{k^{p_-}} \\ &+ \frac{const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, p_-, (p_-)^*, \Omega)}{k^\alpha}, \end{aligned} \tag{3.32}$$

where $\frac{1}{(p_-)^*} = \frac{1}{p_-} - \frac{1}{N}$, $\alpha = (p_-)^*(1 - \frac{1}{p_-})$.

Proof. (of Lemma 3.12) To begin with, we establish (3.31). Subsequently, we can deduce the following from inequality (3.27):

$$\int_\Omega |DT_k(u_\epsilon)|^{p_-} dx \le kK_1. \tag{3.33}$$

Here, $K_1 > 0$ depending on $\|f\|_1, \|g\|_{L^1(\partial\Omega)}$ and $|\Omega|$. Now, using Poincaré-Sobolev type inequality (2.4), there exists a constant $K_2 > 0$ depending on Ω such that

$$\left(\int_\Omega |T_k(u_\epsilon)|^{(p_-)^*} dx\right)^{\frac{p_-}{(p_-)^*}} \leq K_2\left(\int_\Omega |DT_k(u_\epsilon)|^{p_-} dx + \left(\int_{\partial\Omega} |T_k(u_\epsilon)| d\sigma\right)^{p_-}\right). \tag{3.34}$$

After using Holder's inequality on the last term present on the right hand side of equation (3.34), and considering the inequality (3.28), it becomes clear that

$$\left(\int_{\partial\Omega} |T_k(u_\epsilon)| d\sigma\right)^{p_-} \leq K_3 k, \tag{3.35}$$

where $K_3 > 0$ depending on $\|f\|_1, \|g\|_{L^1(\partial\Omega)}$, p_-,λ, $|\Omega|$ and $|\partial\Omega|$. From (3.33),(3.34) and (3.35), we deduce that for any $k \geq 1$,

$$\left(\int_\Omega |T_k(u_\epsilon)|^{(p_-)^*} dx\right)^{\frac{p_-}{(p_-)^*}} \leq K_4 k, \tag{3.36}$$

where $K_4 > 0$ depending on $\|f\|_1, \|g\|_{L^1(\partial\Omega)}$, p_-, $(p_-)^*$, λ, $|\Omega|$ and $meas(\partial\Omega)$. From (3.36) we derive that

$$\int_\Omega |T_k(u_\epsilon)|^{(p_-)^*} dx \leq K_5 k^{\frac{(p_-)^*}{p_-}}, \tag{3.37}$$

where K_5 is a positive constant depending on $\|f\|_1, \|g\|_{L^1(\partial\Omega)}$, p_-, $(p_-)^*$, λ, $|\Omega|$ and $meas(\partial\Omega)$. Relation (3.37) implies that

$$\int_{\{|u_\epsilon|>k\}} |T_k(u_\epsilon)|^{(p_-)^*} dx \leq K_5 k^{\frac{(p_-)^*}{p_-}}, \tag{3.38}$$

which is equal to

$$k^{(p_-)^*}|\{|u_\epsilon| > k\}| \leq K_5 k^{\frac{(p_-)^*}{p_-}}. \tag{3.39}$$

From (3.39) we have

$$|\{|u_\epsilon| > k\}| \leq K_5 k^{(p_-)^*(\frac{1}{p_-}-1)}, \tag{3.40}$$

We have obtained the desired result (3.31). Our next task is to show the estimate (3.32). Let k and θ be positive parameters. We have

$$\Phi(k,\theta) = meas\{|Du_\epsilon|^{p_-} > \theta, |u_\epsilon| > k\}.$$

Using (3.31) and for a sufficiently large value of $k > 0$, we get

$$\Phi(k,0) \leq \frac{const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, p_-, (p_-)^*, \Omega)}{k^\alpha}. \tag{3.41}$$

Since the function $\theta \mapsto \Phi(k,\theta)$ is non-increasing, we have the inequality $\Phi(0,\theta) \leq \Phi(0,s)$ for any $k,\theta > 0$ and $0 \leq s \leq \theta$,

$$\begin{aligned}
\Phi(0,\theta) &= |\{|Du_\epsilon|^{p_-} > \theta\}| = \frac{1}{\theta}\int_0^\theta \Phi(0,\theta)ds \leq \frac{1}{\theta}\int_0^\theta \Phi(0,s)ds \\
&\leq \frac{1}{\theta}\int_0^\theta \Phi(k,s)ds + \frac{1}{\theta}\int_0^\theta \big(\Phi(0,s)-\Phi(k,s)\big)ds \\
&\leq \frac{1}{\theta}\int_0^\theta \Phi(k,0)ds + \frac{1}{\theta}\int_0^\theta \big(\Phi(0,s)-\Phi(k,s)\big)ds \\
&\leq \Phi(k,0) + \frac{1}{\theta}\int_0^\theta \big(\Phi(0,s)-\Phi(k,s)\big)ds. \qquad (3.42)
\end{aligned}$$

Hence,

$$|\{|Du_\epsilon|^{p_-} > \theta\}| \leq |\{|u_\epsilon| > k\}| + \frac{1}{\theta}\int_0^\theta \big(\Phi(0,s)-\Phi(k,s)\big)ds. \qquad (3.43)$$

Finally, we can express $\Phi(0,s)-\Phi(k,s)$ as $meas(|Du_\epsilon|^{p_-} > s, |u_\epsilon| \leq k)$. Therefore, we obtain

$$\int_0^\infty \big(\Phi(0,s)-\Phi(k,s)\big)ds = \int_{\{|u_\epsilon|\leq k\}} |Du_\epsilon|^{p_-}dx. \qquad (3.44)$$

From

$$\int_\Omega |DT_k(u_\epsilon)|^{p_-}dx \leq const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)(k+1),$$

we deduce that

$$\int_{\{|u_\epsilon|\leq k\}} |Du_\epsilon|^{p_-}dx \leq const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)(k+1). \qquad (3.45)$$

By combining (3.44) and (3.45), we obtain the following

$$\int_0^\infty \big(\Phi(0,s)-\Phi(k,s)\big)ds \leq const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)(k+1). \qquad (3.46)$$

From (3.43), (3.46), and (3.31), we can infer that

$$\begin{aligned}
|\{|Du_\epsilon|^{p_-} > \theta\}| \leq &\frac{const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, \Omega)(k+1)}{\theta} \\
&+ \frac{const(\|f\|_{L^1(\Omega)}, \|g\|_{L^1(\partial\Omega)}, p_-, (p_-)^*, \Omega)}{k^\alpha}. \qquad (3.47)
\end{aligned}$$

An optimal choice can be determined by minimizing this inequality with respect to θ. It is given by $\theta = k^{p_-}$, up to a multiplicative constant. Substituting this optimal choice into the inequality leads to (3.32). Hence, we have completed the proof of Lemma 3.12. □

Step 3: Basic convergence results.

The convergence results below follow from the a priori estimates stated in Lemma 3.9.

Lemma 3.13. *For any $k > 0$, when ϵ tends to 0,*
(i) $T_k(u_\epsilon) \to T_k(u)$ *in* $L^{p_-}(\Omega)$ *and a.e. in* Ω, $DT_k(u_\epsilon) \longrightarrow DT_k(u)$ *in* $(L^{p(\cdot)}(\Omega))^N$,
(ii) $a(x, DT_k(u_\epsilon)) \rightharpoonup a(x, DT_k(u))$ *in* $(L^{p'(x)}(\Omega))^N$,
(iii) u_ϵ *converges to some function* v *a.e. in* $\partial\Omega$.

Proof. (**i**) For $k > 0$, the sequence $(DT_k(u_\epsilon))_{\epsilon>0}$ is bounded in $L^{p(\cdot)}(\Omega)$, thus, the sequence $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Therefore, we can extract a subsequence, still denoted $(T_k(u_\epsilon))_{\epsilon>0}$ for every $k > 0$, $(T_k(u_\epsilon))_{\epsilon>0}$ converges weakly to σ_k in $W^{1,p(\cdot)}(\Omega)$ and also that $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly to σ_k in $L^{p_-}(\Omega)$. We show that the sequence $(u_\epsilon)_{\epsilon>0}$ converges to some function u in measure. To this end, we prove that u_ϵ is a Cauchy sequence in measure. Let $s > 0$ and $k > 0$ be fixed. Define $E_\nu = \{|u_\nu| > k\}$, $E_\mu = \{|u_\mu| > k\}$ and $E_{\nu,\mu} = \{|T_k(u_\nu) - T_k(u_\mu)| > s\}$. Note that $\{|u_\nu - u_\mu| > s\} \subset E_\nu \cup E_\mu \cup E_{\nu,\mu}$, and thus,

$$meas(\{|u_\nu - u_\mu| > s\}) \leq meas(E_\nu) + meas(E_\mu) + meas(E_{\nu,\mu}). \tag{3.48}$$

Let $\eta > 0$, using the previous inequality, we choose $k = k(\eta)$ such that

$$meas(E_\nu) \leq \frac{\eta}{3} \quad \text{and} \quad meas(E_\mu) \leq \frac{\eta}{3}. \tag{3.49}$$

As $T_k(u_\epsilon)$ converges strongly in $L^{p_-}(\Omega)$, it is a Cauchy sequence in $L^{p_-}(\Omega)$. Hence,

$$\forall s > 0,\, \eta > 0,\, \exists \nu_0 = \nu_0(s,\eta) \quad such \quad that \quad \forall\, \nu, \mu \geq \nu_0(s,\eta),$$

$$\Big(\int_\Omega |T_k(u_\nu) - T_k(u_\mu)|^{p(\cdot)} dx\Big)^{\frac{1}{p_-}} \leq \Big(\frac{\eta s^{p_-}}{3}\Big)^{\frac{1}{p_-}}.$$

So,

$$\forall \nu \geq \nu_0,\, \forall \mu \geq \nu_0,\, meas(E_{\nu,\mu}) \leq \frac{1}{s^{p_-}} \int_\Omega (|T_k(u_\nu) - T_k(u_\mu)|)^{p(\cdot)} dx \leq \big(\frac{\eta}{3}\big). \tag{3.50}$$

From (3.48)-(3.50) we deduce that

$$meas(\{|u_\nu - u_\mu| > s\}) \leq \eta, \tag{3.51}$$

for all $\nu, \mu \geq \nu_0(s,\eta)$. The inequality (3.51) shows that $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure, which implies the existence of a measurable function u such that $u\epsilon \to u$ in measure. Then, $(u_\epsilon)_{\epsilon>0}$ converges almost everywhere to some measurable function u. We can then extract a subsequence still denoted $(u_\epsilon)_{\epsilon>0}$ such that $u_\epsilon \to u$ a.e. in Ω. As for $k > 0$, T_k is continuous, then $T_k(u_\epsilon) \to T_k(u)$ a.e. in Ω and $\sigma_k = T_k(u)$ a.e. in Ω. Using similar arguments as in the proof of Lemma 3.7 in [11], we deduce that $DT_k(u_\epsilon) \to DT_k(u)$ strongly in $(L^{p(\cdot)}(\Omega))^N$ as $\epsilon \to 0$.

Proof of (ii): Thanks to **(H3)** and (3.17), the sequence $(\mathbf{a}(x, DT_k(u_\epsilon))_{\epsilon>0}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, so there exists a sub-sequence, still denoted $(\mathbf{a}(x, DT_k(u_\epsilon))_{\epsilon>0}$, such that $\mathbf{a}(x, DT_k(u_\epsilon)) \rightharpoonup \Phi_k$ in $(L^{p'(\cdot)}(\Omega))^N$ as $\epsilon \to 0$. It remains to prove that $\operatorname{div} \varPhi_k = \operatorname{div} \mathbf{a}(x, DT_k(u))$. Let us first show that the following inequality holds for all $k > 0$,

$$\limsup_{\epsilon\to 0} \int_\Omega \mathbf{a}(x, DT_k(u_\epsilon)).D\,(T_k(u_\epsilon) - T_k(u))\, dx \leq 0. \tag{3.52}$$

Indeed, let $k, \epsilon > 0$. Using $\xi = h_n(u_\epsilon)\,(T_k(u_\epsilon) - T_k(u))$ as a test function in (3.14) leads to

$$\int_\Omega h_n(u_\epsilon)\mathbf{a}(x, DT_k(u_\epsilon)).D(T_k(u_\epsilon) - T_k(u))dx = A_{k,n,\epsilon} + B_{k,n,\epsilon} + C_{k,n,\epsilon} + D_{k,n,\epsilon} + E_{k,n,\epsilon} + F_{k,n,\epsilon}, \tag{3.53}$$

where

$$\begin{aligned}
A_{k,n,\epsilon} &= \int_\Omega f h_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)dx,\\
B_{k,n,\epsilon} &= \int_{\partial\Omega} g h_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)d\sigma,\\
C_{k,n,\epsilon} &= -\int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon)h_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big),\\
D_{k,n,\epsilon} &= -\epsilon\int_\Omega |u_\epsilon|^{p(\cdot)-2}u_\epsilon(T_k(u_\epsilon)-T_k(u))h_n(u_\epsilon)dx,\\
E_{k,n,\epsilon} &= -\lambda\int_{\partial\Omega} u_\epsilon\,(T_k(u_\epsilon)-T_k(u))h_n(u_\epsilon)dx,\\
F_{k,n,\epsilon} &= -\int_\Omega h_n'(u_\epsilon)\mathbf{a}(x,DT_k(u_\epsilon)).Du_\epsilon\big(T_k(u_\epsilon)-T_k(u)\big)dx.
\end{aligned}$$

We examine the behavior of the each terms in (3.53) as $\epsilon\to 0$ then $n\to\infty$, respectively. By observing that $|fh_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)|\le 2k|f|\in L^1(\Omega)$, $|gh_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)|\le 2k|g|\in L^1(\partial\Omega)$, and $|u_\epsilon h_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)|\le 2kC\in L^1(\partial\Omega)$, we can apply the Lebesgue dominated convergence theorem to conclude

$$\lim_{\epsilon\downarrow 0} A_{k,n,\epsilon}=\lim_{\epsilon\downarrow 0}\int_\Omega f h_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)dx=0, \tag{3.54}$$

$$\lim_{\epsilon\downarrow 0} B_{k,n,\epsilon}=\lim_{\epsilon\downarrow 0}\int_{\partial\Omega} g h_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)d\sigma=0, \tag{3.55}$$

$$\lim_{\epsilon\downarrow 0} E_{k,n,\epsilon}=-\lim_{\epsilon\to 0}\lambda\int_{\partial\Omega} u_\epsilon\,(T_k(u_\epsilon)-T_k(u))h_n(u_\epsilon)dx=0. \tag{3.56}$$

Next, we focus on term $C_{k,n,\epsilon}$. Due to Lemma 3.10 and the convergence of sequence $T_k(u_\epsilon)-T_k(u)$ to zero almost everywhere in Ω and in $L^\infty(\Omega)$ weak-$*$ as ϵ goes to zero, Lebesgue Dominated Convergence Theorem leads to

$$\lim_{\epsilon\downarrow 0} C_{k,n,\epsilon}=-\limsup_{\epsilon\downarrow 0}\int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon)h_n(u_\epsilon)\big(T_k(u_\epsilon)-T_k(u)\big)dx=0. \tag{3.57}$$

Moreover, one has $\Big||u_\epsilon|^{p(\cdot)-2}u_\epsilon(T_k(u_\epsilon)-T_k(u))h_n(u_\epsilon)\Big|\le 2kC$ then we can deduce that

$$\lim_{\epsilon\downarrow 0} D_{k,n,\epsilon}=-\lim_{\epsilon\downarrow 0}\epsilon\int_\Omega |u_\epsilon|^{p(\cdot)-2}u_\epsilon(T_k(u_\epsilon)-T_k(u))h_n(u_\epsilon)dx=0. \tag{3.58}$$

Considering the term $F_{k,n,\epsilon}$, we have

$$|F_{k,n,\epsilon}|\le 2k\int_{\{n<|u_\epsilon|<n+1\}}\mathbf{a}(x,Du_\epsilon)Du_\epsilon dx.$$

We prove that

$$\limsup_{n\to\infty}\limsup_{\epsilon\to 0}\int_{\{n<|u_\epsilon|<n+1\}}\mathbf{a}(x,Du_\epsilon)Du_\epsilon dx\le 0. \tag{3.59}$$

To prove (3.59), we take $\xi = T_1(u_\epsilon - T_n(u_\epsilon))$ as test function in (3.14) to obtain

$$\begin{aligned}\int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))T_1(u_\epsilon - T_n(u_\epsilon))dx + \epsilon\int_\Omega |u_\epsilon|^{p(\cdot)-2}u_\epsilon\, T_1(u_\epsilon - T_n(u_\epsilon))dx \\ + \lambda\int_{\partial\Omega} u_\epsilon\, T_1(u_\epsilon - T_n(u_\epsilon))d\sigma + \int_\Omega \mathbf{a}(x, Du_\epsilon).DT_1(u_\epsilon - T_n(u_\epsilon))dx \\ = \int_\Omega f\, T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\partial\Omega} gT_1(u_\epsilon - T_n(u_\epsilon))\, d\sigma. \quad (3.60)\end{aligned}$$

Since the three first terms of (3.60) are non-negative, then we deduce that

$$\int_{\{n<|u_\epsilon|<n+1\}} \mathbf{a}(x, Du_\epsilon)Du_\epsilon dx \le \int_\Omega f\, T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\partial\Omega} gT_1(u_\epsilon - T_n(u_\epsilon))\, d\sigma. \quad (3.61)$$

By the Lebesgue Dominated Convergence Theorem, one sees that

$$\lim_{\epsilon\to 0}\Big(\int_\Omega f\, T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\partial\Omega} gT_1(u_\epsilon - T_n(u_\epsilon))\, d\sigma\Big) = \int_\Omega f\, T_1(u - T_n(u))dx + \int_{\partial\Omega} gT_1(u - T_n(u))\, d\sigma.$$

Again, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\Big(\int_\Omega f\, T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\partial\Omega} gT_1(u_\epsilon - T_n(u_\epsilon))\, d\sigma\Big) = 0. \quad (3.62)$$

Passing to the limit as $\epsilon \to 0$ and to the limit as $n \to \infty$ in (3.61) and using (3.62), we deduce (3.59). Using the peceding result, we obtain

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty} F_{k,n,\epsilon} \le 0. \quad (3.63)$$

Combining (3.54)-(3.63) and letting $n \to \infty$, we obtain (3.52).
Using the MintyBrowders arguments, we identify $\mathbf{a}(x, DT_k(u))$ with Φ_k. To do it, let $\varphi \in \mathcal{D}(\Omega)$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Using (3.52) and assumption **(H2)** we get

$$\begin{aligned}\lambda\int_\Omega \Phi_k D\varphi dx &= \lim_{\epsilon\to 0}\int_\Omega \lambda\mathbf{a}(x, DT_k(u_\epsilon))D\varphi dx \\ &\ge \limsup_{\epsilon\to 0}\int_\Omega \mathbf{a}(x, DT_k(u_\epsilon))D(T_k(u_\epsilon) - T_k(u) + \lambda\varphi)dx, \\ &\ge \limsup_{\epsilon\to 0}\int_\Omega \mathbf{a}(x, D[T_k(u) - \lambda\varphi])D(T_k(u_\epsilon) - T_k(u) + \lambda\varphi)dx, \\ &\ge \lambda\int_\Omega \mathbf{a}(x, D[T_k(u) - \lambda\varphi])D\varphi. \quad (3.64)\end{aligned}$$

Dividing by $\lambda < 0$ and by $\lambda > 0$, and passing to the limit with $\lambda \to 0$, we obtain

$$\int_\Omega \Phi_k D\varphi dx = \int_\Omega \mathbf{a}(x, DT_k(u))D\varphi dx, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Consequently, we have $\mathbf{a}(x, DT_k(u)) = \Phi_k$ a.e. in Ω. This leads us to the conclusion that

$$\mathbf{a}(x, DT_k(u_\epsilon)) \rightharpoonup \mathbf{a}(x, DT_k(u)) \quad \text{weakly in} \quad (L^{p'(x)}(\Omega))^N. \quad (3.65)$$

It remains to show **(iii).** Using Lemma 3.11, we obtain thanks to the Hölder inequality and the Poincaré-Sobolev type inequality,

$$\int_\Omega |T_k(u_\epsilon)|dx \le (meas(\Omega))^{\frac{1}{((p_-)^*)'}}\,\big(Ck\big)^{\frac{1}{p_-}}, \quad (3.66)$$

and

$$\int_{\Omega} |DT_k(u_\epsilon)|dx \le (meas(\Omega))^{\frac{1}{(p_-)'}} \big(Ck\big)^{\frac{1}{p_-}}. \tag{3.67}$$

By applying Fatou's Lemma, we can take the limit in (3.66) and (3.67) as $\epsilon \to 0$ to obtain

$$\int_{\Omega} |T_k(u)|dx \le (meas(\Omega))^{\frac{1}{((p_-)^*)'}} \big(Ck\big)^{\frac{1}{p_-}} \tag{3.68}$$

and

$$\int_{\Omega} |DT_k(u)|dx \le (meas(\Omega))^{\frac{1}{(p_-)'}} \big(Ck\big)^{\frac{1}{p_-}}, \tag{3.69}$$

for all $k \ge 1$.
For any $k > 0$, let

$$A_k := \{x \in \partial\Omega : |T_k(u(x))| < k\} \ and \ B := \partial\Omega \setminus \bigcup_{k>0} A_k.$$

Then

$$\begin{aligned} meas(B) = \frac{1}{k}\int_B |T_k(u)|dx &\le \frac{1}{k}\int_{\partial\Omega} |T_k(u)|dx \\ &\le \frac{M_1}{k}\|T_k(u)\|_{W^{1,1}(\Omega)} \\ &\le \frac{M_1}{k}\Big(\|T_k(u)\|_{L^1(\Omega)} + \|DT_k(u)\|_{L^1(\Omega)}\Big) \\ &\le \frac{M_2}{k^{1-\frac{1}{p_-}}}. \end{aligned}$$

As $p_- > 1$, by letting $k \to +\infty$ we deduce that $meas(B) = 0$. Define on $\partial\Omega$, the function v by

$$v(x) := T_k(u(x)) \ if \ x \in A_k.$$

We take $x \in \partial\Omega \setminus (\bar{B} \cup B)$, then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_\epsilon(x) - v(x) = (u_\epsilon(x) - T_k(u_\epsilon(x))) + (T_k(u_\epsilon(x)) - T_k(u(x))).$$

Since $x \in A_k$ we have $|T_k(u(x))| < k$ and so $|T_k(u_\epsilon(x))| < k$, from which we deduce that $|u_\epsilon(x)| < k$. Therefore

$$u_\epsilon(x) - v(x) = (T_k(u_\epsilon(x)) - T_k(u(x))) \to 0 \quad \epsilon \to 0.$$

This means that u_ϵ converges to v a.e. on $\partial\Omega$, which ends the proof of Lemma 3.13. □

Lemma 3.14. *For all $h \in C_c^1(\mathbb{R})$ and $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$,*

$$D[h(u_\epsilon)\varphi] \to D[h(u)\varphi] \quad \textit{strongly in} \quad (L^{p(\cdot)}(\Omega))^N, \quad \textit{as } \epsilon \to 0.$$

Proof. The proof of this statement is analogous to the proof of Lemma 3.8 in ([12]). □

Step 4: Passage to the limit: Testing (3.14) by $h_n(u_\epsilon)h(u)\xi$ where $h \in C_c^1(\mathbb{R})$ and $\xi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we can write

$$I^1_{\epsilon,n} + I^2_{\epsilon,n} + I^3_{\epsilon,n} + I^4_{\epsilon,n} = I^5_{\epsilon,n} + I^6_{\epsilon,n} \tag{3.70}$$

with

$$\begin{aligned}
I^1_{\epsilon,n} &= \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))h_n(u_\epsilon)h(u)\xi dx,\\
I^2_{\epsilon,n} &= \epsilon \int_\Omega |u_\epsilon|^{p(\cdot)-2}u_\epsilon h_n(u_\epsilon)h(u)\xi dx,\\
I^3_{\epsilon,n} &= \lambda \int_{\partial\Omega} u_\epsilon\, h_n(u_\epsilon)h(u)\xi\, d\sigma,\\
I^4_{\epsilon,n} &= \int_\Omega \mathbf{a}(x, Du_\epsilon).D[h_n(u_\epsilon)h(u)\xi]dx,\\
I^5_{\epsilon,n} &= \int_\Omega f h_n(u_\epsilon)h(u)\xi dx,\\
I^6_{\epsilon,n} &= \int_{\partial\Omega} g h_n(u_\epsilon)h(u)\xi\, d\sigma.
\end{aligned}$$

Let's go to the limit in (3.70), when $\epsilon \to 0$ then $n \to \infty$.

Item 1: Passing to the limit as $\epsilon \downarrow 0$

By Lebesgue dominated convergence Theorem, we see that

$$\lim_{\epsilon\downarrow 0} I^2_{\epsilon,n} = \lim_{\epsilon\downarrow 0} \epsilon \int_\Omega |u_\epsilon|^{p(\cdot)-2}u_\epsilon h_n(u_\epsilon)h(u)\xi dx = 0, \tag{3.71}$$

$$\lim_{\epsilon\downarrow 0} I^5_{\epsilon,n} = \lim_{\epsilon\downarrow 0} \int_\Omega f h_n(u_\epsilon)h(u)\xi dx = \int_\Omega f h_n(u)h(u)\xi dx := I^5_n, \tag{3.72}$$

$$\lim_{\epsilon\downarrow 0} I^3_{\epsilon,l} = \lim_{\epsilon\downarrow 0} \lambda \int_{\partial\Omega} u_\epsilon\, h_n(u_\epsilon)h(u)\xi\, d\sigma = \lambda \int_{\partial\Omega} u\, h_n(u)h(u)\xi\, d\sigma := I^3_n \tag{3.73}$$

and

$$\lim_{\epsilon\downarrow 0} I^6_{\epsilon,n} = \lim_{\epsilon\downarrow 0} \int_{\partial\Omega} g h_n(u_\epsilon)h(u)\xi d\sigma = \int_{\partial\Omega} g h_n(u)h(u)\xi d\sigma := I^6_n. \tag{3.74}$$

Using (3.65) and according to Lemma 3.14, we deduce that

$$\lim_{\epsilon\to 0} I^4_{\epsilon,l} = \lim_{\epsilon\to 0} \int_\Omega \mathbf{a}(x, Du_\epsilon).D[h_n(u_\epsilon)h(u)\xi]dx = \int_\Omega \mathbf{a}(x, Du).D[h_n(u)h(u)\xi]dx := I^4_n. \tag{3.75}$$

Now, we are concerning with the term $I^1_{\epsilon,n}$.

By (3.26), $\beta_\epsilon(T_{1/\epsilon}(u_\epsilon))$ is uniformly bounded in $L^1(\Omega)$. It follows that there exists b such that

$$\beta_\epsilon(T_{1/\epsilon}(u_\epsilon)) \rightharpoonup^* b \text{ in } L^\infty(\Omega). \tag{3.76}$$

We deduce that

$$\lim_{\epsilon\to 0} I^1_{\epsilon,n} = \lim_{\epsilon\to 0} \int_\Omega \beta_\epsilon(T_{1/\epsilon}(u_\epsilon))h_n(u_\epsilon)h(u)\xi dx = \int_\Omega b h_n(u)h(u)\xi dx := I^1_n. \tag{3.77}$$

Item 2: Passage to the limit with $l \longrightarrow +\infty$

Combining (3.70) with (3.71)-(3.77) we find

$$I^1_n + I^3_n + I^4_n = I^5_n + I^6_n. \tag{3.78}$$

Choosing $j > 0$ such that supp $h \subset [-j, j]$, we can replace u by $T_j(u)$ in I_n^1, I_n^3 and I_n^4. It follows that

$$\lim_{n\to+\infty} I_n^1 = \int_\Omega b\, h(u)\, \xi\, dx, \tag{3.79}$$

$$\lim_{n\to+\infty} I_n^3 = \lambda \int_{\partial\Omega} u\, h(u)\xi\, d\sigma, \tag{3.80}$$

$$\lim_{n\to+\infty} I_n^4 = \int_\Omega \mathbf{a}(x, Du).D[h(u)\xi]\, dx, \tag{3.81}$$

$$\lim_{n\to+\infty} I_n^5 = \int_\Omega f\, h(u)\, \xi\, dx, \tag{3.82}$$

$$\lim_{n\to+\infty} I_n^6 = \int_{\partial\Omega} g\, h(u)\, \xi\, d\sigma. \tag{3.83}$$

Combining (3.70) with (3.79)-(3.83) we obtain

$$\int_\Omega b\, h(u)\, \xi\, dx + \lambda \int_{\partial\Omega} u\, h(u)\xi\, d\sigma + \int_\Omega \mathbf{a}(x, Du).D[h(u)\xi]\, dx = \int_\Omega f\, h(u)\, \xi\, dx + \int_{\partial\Omega} g\, h(u)\, \xi\, d\sigma, \tag{3.84}$$

for all $h \in C_c^1(\mathbb{R})$ and $\xi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Step 5: subdifferential argument.
Consider a maximal monotony graph β, there exists a convex, lower semicontinuous, proper function $j : \mathbb{R} \to [0, \infty]$ such that $\beta(r) = \partial j(r)$ for all $r \in \mathbb{R}$, almost everywhere in Ω. The function j_ϵ has the following properties:
(i) for $\epsilon > 0$, j_ϵ is convex and differentiable. Also, $\beta_\epsilon(r) = \partial j_\epsilon(r)$ for $r \in \mathbb{R}$ and a.e. in Ω.
(ii) $\lim\limits_{\epsilon\to 0} j_\epsilon(r) = j(r)$. It follows from (i) that

$$j_\epsilon(r) \geq j_\epsilon(T_{1/\epsilon}(u_\epsilon)) + (r - T_{1/\epsilon}(u_\epsilon))\beta_\epsilon(T_{1/\epsilon}(u_\epsilon)) \tag{3.85}$$

holds for all $r \in \mathbb{R}$ and a.e. in Ω. Suppose we have a measurable subset A of Ω, and let χ_A be its characteristic function. Let us fix ϵ_0 and multiply (3.85) by the function $h_n(u_\epsilon)\chi_A$. Next, we integrate this resulting expression over the set A. Using (ii), we arrive at

$$\begin{aligned}\int_A j_\epsilon(r) h_n(u_\epsilon) dx \geq& \int_A j_{\epsilon_0}(T_{n+1}(u_\epsilon)) h_n(u_\epsilon) dx \\ &+ (r - T_{n+1}(u_\epsilon)) h_n(u_\epsilon)\beta_\epsilon(T_{1/\epsilon}(u_\epsilon))\end{aligned} \tag{3.86}$$

for all $r \in \mathbb{R}$ and $0 < \epsilon < \epsilon_0$. By stretching $\epsilon_0 \to 0$ and $l \to \infty$, we obtain

$$\int_A j(r) dx \geq \int_A j(u) dx + b(r - u).$$

Since A is arbitrarily chosen, we deduce from preceding inequality that

$$j(r) \geq j(u)) + b(r - u), \tag{3.87}$$

for $r \in \mathbb{R}$ and for $x \in \Omega$, $u(x) \in D(\beta(u(x)))$ and $b(x) \in \beta(u(x))$ a.e. in Ω.
To end the proof of Proposition 3.4, it remains to show that u satisfies the renormalized condition (3.2). Thanks to (3.59) and using assumptions **(H1)** and **(H2)**,

it follows that

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\int_{\{n<|u_\epsilon|<n+1\}}|Du_\epsilon|^{p(\cdot)}dx=0. \tag{3.88}$$

Thanks to Lemma 3.13,

$$Du_\epsilon \longrightarrow Du \text{ strongly in } (L^{p(\cdot)}(\Omega))^N,$$

which is equivalent to say

$$\lim_{\epsilon\to 0}\int_\Omega |Du_\epsilon - Du|^{p(\cdot)}dx=0$$

Consequently, Lebesgue generalized convergence theorem implies that

$$\int_\Omega |Du_\epsilon|^{p(\cdot)}dx \to \int_\Omega |Du|^{p(\cdot)}dx \text{ as } \epsilon\to 0.$$

Then,

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\int_{\{n<|u_\epsilon|<n+1\}}|Du_\epsilon|^{p(\cdot)}dx=\lim_{n\to\infty}\int_{\{n<|u|<n+1\}}|Du|^{p(\cdot)}dx=0. \tag{3.89}$$

□

Next, we demonstrate that the renormalized solution of $P^\beta_{f,g}$, when $f\in L^\infty(\Omega)$ and $g\in L^\infty(\partial\Omega)$, can be considered as an extension of the weak solution concept.

Proposition 3.15. *Let (u,b) be a renormalized solution to $P^\beta_{f,g}$ for $f\in L^\infty(\Omega)$ and $g\in L^\infty(\partial\Omega)$. Then $u\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\partial\Omega)$ and thus, in particular u is a weak solution to $P^\beta_{f,g}$.*

Proof. The proof of Proposition 3.15 is similar to that of Proposition 5.2 in [16]. □

3.2. Existence results for L^1-data.

To prove the existence of renormalized solutions for L^1-data, the primary approach is to examine the problem through its approximated version.

$$P(b_{\mu,\nu},f_{\mu,\nu},g_{\mu,\nu})\ \begin{cases} b_{\mu,\nu}-\operatorname{div}\mathbf{a}(x,Du_{\mu,\nu})=f_{\mu,\nu} & \text{in}\quad \Omega,\\ \mathbf{a}(x,Du_{\mu,\nu}).\eta+\lambda u_{\mu,\nu}=g_{\mu,\nu} & \text{on}\quad \partial\Omega,\end{cases}$$

where $f_{\mu,\nu}$ and $g_{\mu,\nu}$ are some bi-monotones sequence defined by $f_{\mu,\nu}=(f\wedge\mu)\vee(-\nu)$ and $g_{\mu,\nu}=(g\wedge\mu)\vee(-\nu)$ respectively, non decreasing in μ, non increasing in ν such that $\|f_{\mu,\nu}\|_1\le\|f\|_1$ and $\|g_{\mu,\nu}\|_1\le\|g\|_1$.

From Proposition 3.4, it follows that for all $\mu,\nu\in\mathbb{N}$ there exists $u_{\mu,\nu}\in W^{1,p(\cdot)}(\Omega)$, $b_{\mu,\nu}\in L^\infty(\Omega)$, such that $(u_{\mu,\nu},b_{\mu,\nu})$ is a renormalized solution of $P(b_{\mu,\nu},f_{\mu,\nu},g_{\mu,\nu})$. Choose $h(u_{\mu,\nu})\xi$ as test function, we have

$$\begin{aligned}&\int_\Omega b_{\mu,\nu}h(u_{\mu,\nu})\xi dx+\lambda\int_{\partial\Omega}u_{\mu,\nu}h(u_{\mu,\nu})\xi d\sigma\\ &\quad+\int_\Omega \mathbf{a}(x,Du_{\mu,\nu}).D(h(u_{\mu,\nu})\xi)dx=\int_\Omega f_{\mu,\nu}h(u_{\mu,\nu})\xi dx+\int_{\partial\Omega}g_{\mu,\nu}h(u_{\mu,\nu})\xi d\sigma.\end{aligned} \tag{3.90}$$

holds for all $h\in C^1_c(\mathbb{R})$ and $\xi\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$. We will now present some a priori estimates that will be essential for the remainder of the study.

Lemma 3.16. *Let $(u_{\mu,\nu}, b_{\mu,\nu})$ be a renormalized solution of $P(b_{\mu,\nu}, f_{\mu,\nu}, g_{\mu,\nu})$. Then, for any $k > 0$ and $\mu, \nu \in \mathbb{N}$, we have*

$$\int_\Omega |DT_k(u_{\mu,\nu})|^{p(\cdot)} dx \leq \frac{k}{\alpha}\big(\|f\|_1 + \|g\|_1\big), \tag{3.91}$$

$$\int_\Omega |b_{\mu,\nu}| dx \leq \|f\|_1 + \|g\|_1, \tag{3.92}$$

$$\int_{\partial\Omega} |u_{\mu,\nu}| d\sigma \leq \frac{1}{\lambda}\big(\|f\|_1 + \|g\|_1\big), \tag{3.93}$$

hold for all $\mu, \nu \in \mathbb{N}$.

Proof. The proof of Lemma 3.16 follows the same lines as the proof of Lemma 3.9. □

In order to pass to the limit as $\mu, \nu \to \infty$, in the approximate problem $P(b_{\mu,\nu}, f_{\mu,\nu}, g_{\mu,\nu})$, the strong convergence of $u_{\mu,\nu}$ in $L^1(\Omega)$ is necessary. Thus, we need the following lemma.

Lemma 3.17. *For any $\mu, \nu \in \mathbb{N}$, we have*

$$u_{\mu,\nu+1} \leq u_{\mu,\nu} \leq u_{\mu+1,\nu} \quad \textit{a.e. in } \Omega, \tag{3.94}$$

$$u_{\mu,\nu+1} \leq u_{\mu,\nu} \leq u_{\mu+1,\nu} \quad \textit{a.e. in } \partial\Omega \tag{3.95}$$

and

$$b_{\mu,\nu+1} \leq b_{\mu,\nu} \leq b_{\mu+1,\nu} \quad \textit{a.e. in } \Omega. \tag{3.96}$$

Proof. Since $f_{\mu,\nu}$ and $g_{\mu,\nu}$ are increasing in μ and decreasing in ν, then for all $\mu, \nu > 0$,

$$f_{\mu,\nu+1} \leq f_{\mu,\nu} \leq f_{\mu+1,\nu} \textit{ and } g_{\mu,\nu+1} \leq g_{\mu,\nu} \leq g_{\mu+1,\nu}.$$

From Proposition 3.8 it follows that for all $\epsilon > 0$,

$$u^\epsilon_{\mu,\nu+1} \leq u^\epsilon_{\mu,\nu} \leq u^\epsilon_{\mu+1,\nu} \quad \text{a.e. in } \Omega \tag{3.97}$$

and

$$u^\epsilon_{\mu,\nu+1} \leq u^\epsilon_{\mu,\nu} \leq u^\epsilon_{\mu+1,\nu} \quad \text{a.e. in } \partial\Omega. \tag{3.98}$$

Then, passing the limit with $\epsilon \to 0$ in (3.97)-(3.98), yield (3.94)-(3.95). Setting $b_\epsilon := \beta_\epsilon\big(T_{\frac{1}{\epsilon}}(u_\epsilon)\big)$, using (3.94), the monotonicity of $\beta_\epsilon \circ T_{\frac{1}{\epsilon}}$ and the fact that $\beta_\epsilon(T_{1/\epsilon}(u_\epsilon)) \rightharpoonup^* b$ in $L^\infty(\Omega)$ we get

$$b_{\mu,\nu+1} \leq b_{\mu,\nu} \leq b_{\mu+1,\nu} \quad \text{a.e. in } \Omega. \tag{3.99}$$

□

Remark. *By (3.99) and (3.92), for any $\nu \in \mathbb{N}$ there exists $b_\nu \in L^1(\Omega)$ such that $b_{\mu,\nu} \to b_\nu$ as $\mu \to +\infty$ in $L^1(\Omega)$ and a.e. in Ω and $b \in L^1(\Omega)$, such that $b_\nu \to b$ as $\nu \to +\infty$ in $L^1(\Omega)$.*
According to (3.94), we can infer that the sequence $(u_{\mu,\nu})_\mu$ is monotone increasing. Thus, for any $\nu \in \mathbb{N}$, $u_{\mu,\nu} \to u_\nu$ almost everywhere in Ω, where $u_\nu : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Using (3.94) again, we can conclude that the sequence $(u_\nu)_\nu$ is monotone decreasing. Consequently, $u_\nu \to u$, where $u : \Omega \to \overline{\mathbb{R}}$ is a measurable function. In consequence, we can write:

$$u_{\mu,\nu} \uparrow_\mu u_\nu \downarrow_\nu u \quad \textit{strongly in } L^1(\Omega), \tag{3.100}$$

and

$$b_{\mu,\nu} \uparrow_\mu b_\nu \downarrow_\nu b \quad \textit{weakly in } L^1(\Omega). \tag{3.101}$$

To show that u is finite almost everywhere, we give an estimate of the sets of $u_{\mu,\nu}$ at different levels, as shown below.

Lemma 3.18. *For $\mu, \nu \in \mathbb{N}$, let $(u_{\mu,\nu}, b_{\mu,\nu})$ be a renormalized solutions of $P(b_{\mu,\nu}, f_{\mu,\nu}, g_{\mu,\nu})$. Then, there exists a constant $C > 0$, not depending on $\mu, \nu \in \mathbb{N}$ such that*

$$|\{|u_{\mu,\nu}| \geq l\}| \leq C\, l^{-(p_- -1)}, \tag{3.102}$$

$$\lim_{\nu\to+\infty} \lim_{m\to+\infty} |\{|u_{\mu,\nu}| \geq l\}| = |\{|u| \geq l\}| \leq C\, l^{-(p_- -1)} \leq C\, l^{-(p_- -1)} \tag{3.103}$$

for all $l \geq 1$ and

$$b \in \beta(u) \ \textit{a.e. in } \Omega. \tag{3.104}$$

Proof. The demonstration of Lemma 3.18 proceeds similarly to that of Lemma 6.2 in [16]. □

On the basis of these estimates, the following results are given.

Lemma 3.19. *For $\mu, \nu \in \mathbb{N}$, let $(u_{\mu,\nu}, b_{\mu,\nu})$ be a renormalized solutions of $P(b_{\mu,\nu}, f_{\mu,\nu}, g_{\mu,\nu})$. There exists a subsequence $(\mu(\nu))_\nu$ such that setting $b_\nu := b_{\mu(\nu),\nu}$, $f_\nu := f_{\mu(\nu),\nu}$ and $u_\nu := u_{\mu(\nu),\nu}$, we have,*

(i) $u_\nu \to u$ *almost everywhere in* Ω,

(ii) $T_k(u_\nu) \to T_k(u)$ *in* $L^{p(\cdot)}(\Omega)$ *and a.e. in* Ω

(iii) $DT_k(u_\nu) \rightharpoonup DT_k(u)$ *in* $\left(L^{p(\cdot)}(\Omega)\right)^N$,

(iv) u_ν *converges to* v *a.e. on* $\partial\Omega$.

Proof. For the proof of **(i)**, we proceed in the same way as in Lemma 3.13. For **(ii)**, **(iii)**, observe that from (3.91), the sequence $(T_k(u_\nu))_{\nu\in\mathbb{N}}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. We can extract from this sequence a subsequence still denoted $(T_k(u_\nu))_{\nu\in\mathbb{N}}$ which converges weakly to $T_k(u)$ in $W^{1,p(\cdot)}(\Omega)$, strongly in $L^{p(\cdot)}(\Omega)$ and a.e. in Ω when $\nu \to +\infty$. Since $\{DT_k(u_\nu)\}_{\nu\in\mathbb{N}}$ is bounded in $(L^{p(\cdot)}(\Omega))^N$ and converges in measure to $DT_k(u)$, then $DT_k(u_\nu) \to DT_k(u)$ strongly in $L^1(\Omega)$ (See[2] Lemma 6.1). Moreover $u_\nu \to u$ a.e. in Ω. Consequently $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$.
The proof of **(iv)** is similar to the proof of **(iii)** Lemma 3.13 for a sequence indexed by ν. □

Now we have to prove the pseudomonotony argument.

Lemma 3.20. *There exists a field $\Phi_k \in \left(L^{p'(\cdot)}(\Omega)\right)^N$ satisfying*

$$a(x, DT_k(u_\nu)) \rightharpoonup \Phi_k \ \textit{in} \left(L^{p'(\cdot)}(\Omega)\right)^N$$

and

$$\operatorname{div} \mathit{\Phi}_k = \operatorname{div} \mathbf{a}(x, DT_k(u)).$$

Proof. The sequence $(\mathbf{a}(x, DT_k(u_\nu))_{\nu\in\mathbb{N}}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$ by **(H3)**. Then, there exists $\Phi_k \in (L^{p'(\cdot)}(\Omega)^N)$ such that $\mathbf{a}(x, DT_k(u_\nu)) \rightharpoonup \Phi_k$ in $(L^{p'(\cdot)}(\Omega)^N)$ as $\nu \to \infty$. It remains to prove that $\operatorname{div} \Phi_k = \operatorname{div} \mathbf{a}(x, DT_k(u))$.
We claim that

$$\lim_{\nu\to\infty} \sup \int_\Omega \mathbf{a}(x, DT_k(u_\nu)).D(T_k(u_\nu) - T_k(u))dx \leq 0. \tag{3.105}$$

The proof of (3.105) is the same as that shown in **(ii)** of Lemma 3.13. Indeed, the choice of the admissible test function $h_n(u_\nu)(T_k(u_\nu) - T_k(u))$ in $P(b_{\mu,\nu}, f_{\mu,\nu}, g_{\mu,\nu})$ and a study of the behavior of each term when $\nu \to \infty$ allows us to obtain the desired result i.e. (3.105). Now, our goal is to prove that

$$\operatorname{div} \Phi_k = \operatorname{div} \mathbf{a}(x, DT_k(u)) \text{ in } \mathcal{D}'(\Omega) \text{ for all } k > 0. \tag{3.106}$$

Indeed, for $\Phi_k \in \mathcal{D}'(\Omega)$, $\Phi_k \geq 0$, $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
\lambda \int_\Omega \Phi_k D\varphi dx &= \lim_{\nu\to 0} \int_\Omega \lambda \mathbf{a}(x, DT_k(u_\nu)) D\varphi dx \\
&\geq \lim_{\nu\to 0} \sup \int_\Omega \mathbf{a}(x, DT_k(u_\nu)) D(T_k(u_\nu) - T_k(u) + \lambda\varphi) dx, \\
&\geq \lim_{\nu\to 0} \sup \int_\Omega \mathbf{a}(x, D[T_k(u) - \lambda\varphi]) D(T_k(u_\nu) - T_k(u) + \lambda\varphi) dx, \\
&\geq \lambda \int_\Omega \mathbf{a}(x, D[T_k(u) - \lambda\varphi]) D\varphi\, dx.
\end{aligned} \tag{3.107}$$

Dividing by $\lambda < 0$ and by $\lambda > 0$, and passing to the limit with $\lambda \to 0$, we obtain

$$\int_\Omega \Phi_k D\varphi dx = \int_\Omega \mathbf{a}(x, DT_k(u)) D\varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence $\mathbf{a}(x, DT_k(u)) = \Phi_k$ a.e. in Ω.
We conclude that

$$\mathbf{a}(x, DT_k(u_\nu)) \rightharpoonup \mathbf{a}(x, DT_k(u)) \quad \text{weakly in} \quad (L^{p'(\cdot)}(\Omega))^N.$$

□

To conclude with the proof of existence part, we pass to the limit in (3.90) as $\mu, \nu \to +\infty$. So, testing (3.90) by $h_n(u_\nu)h(u)\xi$ where $h \in C_c^1(\mathbb{R})$ and $\xi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ we write

$$I_{\nu,l}^1 + I_{\nu,l}^2 + I_{\nu,l}^4 = I_{\nu,l}^4 + I_{\nu,l}^5, \tag{3.108}$$

where

$$\begin{aligned}
I_{\nu,l}^1 &= \int_\Omega b_\nu\, h_n(u_\nu)\, h(u)\, \xi dx, \\
I_{\nu,l}^2 &= \lambda \int_{\partial\Omega} u_\nu\, h_n(u_\nu)\, h(u) \xi d\sigma, \\
I_{\nu,l}^3 &= \int_\Omega a(x, Du_\nu).D[h_n(u_\nu)\, h(u)\, \xi] dx, \\
I_{\nu,l}^4 &= \int_\Omega f_\nu\, h_n(u_\nu)\, h(u)\, \xi dx, \\
I_{\nu,l}^5 &= \int_{\partial\Omega} g_\nu\, h_n(u_\nu)\, h(u) \xi d\sigma.
\end{aligned}$$

We aim too take the limit of (3.108) as $\nu \to \infty$, and then $n \to \infty$.

Step 1: passing to the limit with $\nu \to +\infty$
The convergence results of Lemma 3.19 allow us to conclude that

$$\lim_{n\to\infty} I^1_{\nu,l} = \int_\Omega b\, h_n(u)\, h(u)\, \xi\, dx := I^1_l, \tag{3.109}$$

$$\lim_{\nu\to\infty} I^2_{\nu,l} = \lambda \int_{\partial\Omega} u_\nu\, h_n(u_\nu)\, h(u) \xi d\sigma := I^2_l, \tag{3.110}$$

$$\lim_{\nu\to\infty} I^4_{\nu,l} = \int_\Omega f\, h_n(u)\, h(u)\, \xi\, dx := I^4_l, \tag{3.111}$$

$$\lim_{\nu\to\infty} I^5_{\nu,l} = \int_{\partial\Omega} g\, h_n(u)\, h(u)\, \xi d\sigma := I^5_l. \tag{3.112}$$

With similar arguments as the precedent,

$$\lim_{\nu\to\infty} I^3_{\nu,l} = \int_\Omega a(x, Du).D[h_n(u_\nu)\, h(u)\, \xi]\, dx := I^3_n. \tag{3.113}$$

Step 2: passage to the limit with $n \to +\infty$
By combining (3.109) with (3.110)-(3.113), we obtain for all $n \geq 1$

$$I^1_n + I^2_n + I^3_l = I^4_n + I^5_n. \tag{3.114}$$

Choosing $j > 0$ such that $supp(h) \subset [-j, j]$, then $T_j(u) = u$ on $supp(h)$. This allows us to substitute u with $T_j(u)$ in I^1_l, I^2_l and I^3_l. Thus, we have:

$$\lim_{n\to\infty} I^1_n = \int_\Omega b\, h(u)\, \xi\, dx, \tag{3.115}$$

$$\lim_{n\to\infty} I^2_n = \lambda \int_{\partial\Omega} u\, h(u)\, \xi dx, \tag{3.116}$$

$$\lim_{n\to\infty} I^3_n = \int_\Omega a(x, Du).D[h(u)\, \xi]\, dx, \tag{3.117}$$

$$\lim_{n\to\infty} I^4_n = \int_\Omega f\, h(u)\, \xi\, dx, \tag{3.118}$$

$$\lim_{n\to\infty} I^5_n = \int_{\partial\Omega} g\, h(u)\, \xi dx. \tag{3.119}$$

Gathering (3.115)-(3.119), we obtain (3.1).

3.3. **Proof of uniqueness for L^1-data.**

To prove Theorem 3.3 and establish the uniqueness of renormalized solutions for the problem $P^\beta_{f,g}$ with $L^1(\Omega)$-data, we consider two renormalized solutions (u, b) and $(\tilde{u}, \tilde{b})$, where $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$, and let γ and k be two positive real numbers. We use $\xi = T_\gamma(\tilde{u})$ as the test function for the solution (u, b) and $\xi = T_\gamma(u)$ as the test function for the solution $(\tilde{u}, \tilde{b})$ in the entropy inequality (3.3), lead to:

$$\int_\Omega b\,T_k(u-T_\gamma(\tilde u))\,dx+\lambda\int_{\partial\Omega}u_1\,T_k(u-T_\gamma(\tilde u))\,d\sigma+\int_\Omega \mathbf{a}(x,Du).DT_k(u-T_\gamma(\tilde u))\,dx \leq \int_\Omega f\,T_k(u-T_\gamma(\tilde u))\,dx+\int_{\partial\Omega}g\,T_k(u-T_\gamma(\tilde u))\,d\sigma \tag{3.120}$$

and

$$\int_\Omega \tilde b\,T_k(\tilde u-T_\gamma(u))dx+\lambda\int_{\partial\Omega}u_2\,T_k(\tilde u-T_\gamma(u))d\sigma+\int_\Omega \mathbf{a}(x,D\tilde u).DT_k(\tilde u-T_\gamma(u))\,dx \leq \int_\Omega f\,T_k(\tilde u-T_\gamma(u))dx+\int_{\partial\Omega}g\,T_k(\tilde u-T_\gamma(u))d\sigma. \tag{3.121}$$

Adding inequalities (3.120) and (3.121), we get

$$K_{\gamma,k}+L_{\gamma,k}+I_{\gamma,k}\leq\int_\Omega f\,(T_k(u-T_\gamma(\tilde u)+T_k(\tilde u-T_\gamma(u))dx + \int_{\partial\Omega}g\,(T_k(u-T_\gamma(\tilde u)+T_k(\tilde u-T_\gamma(u))d\sigma. \tag{3.122}$$

where

$$I_{\gamma,k}:=\int_\Omega \mathbf{a}(x,Du)\,.DT_k(u-T_\gamma(\tilde u))dx+\int_\Omega a(x,D\tilde u)\big).DT_k(\tilde u-T_\gamma(u))\,dx,$$
$$K_{\gamma,k}:=\int_\Omega b\,T_k(u-T_\gamma(\tilde u))dx+\int_\Omega \tilde b\,T_k(\tilde u-T_\gamma(u))\,dx,$$
$$L_{\gamma,k}:=\lambda\int_{\partial\Omega}u_1T_k(u-T_\gamma(\tilde u))d\sigma+\lambda\int_{\partial\Omega}u_2T_k(\tilde u-T_\gamma(u))\,d\sigma.$$

Our goal is to take the limit as $\gamma\to+\infty$ for k fixed and then $k\to+\infty$ in (3.122). Since $T_k(u-T_\gamma(\tilde u))+T_k(\tilde u-T_\gamma(u))=0$ in $\{|u|\leq\gamma,|\tilde u|\leq\gamma\}$, it is possible to estimate the two integrals on the right-hand side of equation (3.122) by

$$\left|\int_\Omega f\left(T_k(u-T_\gamma(\tilde u))+T_k(\tilde u-T_\gamma(u))\right)dx\right|\leq 2k\left(\int_{\{|u|>\gamma\}}|f|\,dx+\int_{\{|\tilde u|>\gamma\}}|f|\,dx\right)$$

and

$$\left|\int_{\partial\Omega} g\left(T_k(u-T_\gamma(\tilde u))+T_k(\tilde u-T_\gamma(u))\right)d\sigma\right|\leq 2k\left(\int_{\{|u|>\gamma\}}|g|\,d\sigma+\int_{\{|\tilde u|>\gamma\}}|g|\,d\sigma\right).$$

By the Lebesgue converge dominated Theorem and using the fact that, $meas(\{|u_i|>h\})\to0$ when $\gamma\to+\infty$ (for $i=1,2$), it follows that

$$\lim_{\gamma\to+\infty}\int_\Omega f\left(T_k(u-T_\gamma(\tilde u))+T_k(\tilde u-T_\gamma(u))\right)dx=0 \tag{3.123}$$

and

$$\lim_{\gamma\to+\infty}\int_\Omega g\left(T_k(u-T_\gamma(\tilde u))+T_k(\tilde u-T_\gamma(u))\right)d\sigma=0. \tag{3.124}$$

For the two first terms of (3.122), notice that, $T_k(u-T_\gamma(\tilde u))$ and $T_k(\tilde u-T_\gamma(u))$ tends respectively to $T_k(u-\tilde u)$ and $T_k(\tilde u-u)$ when h goes to infinity. Obviously,

$$|b\,T_k(u-T_\gamma(\tilde u))|\leq k|b|\in L^1(\Omega)$$

and

$$|\tilde{b}\, T_k(\tilde{u} - T_\gamma(u))| \leq k|\tilde{b}| \in L^1(\Omega).$$

Then by the Lebesgue's dominated convergence theorem, it yields

$$\lim_{\gamma \to +\infty} \left(\int_\Omega b\, T_k(u - T_\gamma(\tilde{u}))\, dx + \int_\Omega \tilde{b}\, T_k(\tilde{u} - T_\gamma(u)) dx \right) = \int_\Omega (b - \tilde{b})\, T_k(u - \tilde{u})\, dx. \tag{3.125}$$

By the same way we obtain

$$\lim_{\gamma \to +\infty} \lambda \left(\int_{\partial\Omega} u\, T_k(u - T_\gamma(\tilde{u}))\, dx + \int_{\partial\Omega} \tilde{u}\, T_k(\tilde{u} - T_\gamma(u)) \right) = \lambda \int_{\partial\Omega} (u - \tilde{u})\, T_k(u - \tilde{u})\, dx. \tag{3.126}$$

To deal with $I_{\gamma,k}$, we can use the same manage as in [10].
Define

$$\omega_1 = \{|u - \tilde{u}| \leq k, |\tilde{u}| \leq \gamma\}, \qquad \omega_2 = \omega_1 \cap \{|u| \leq \gamma\}\ and \quad \omega_3 = \omega_1 \cap \{|u| > \gamma\}.$$

We start with the first term of $I_{\gamma,k}$. By **(H1)**, we have

$$\begin{aligned} \int_\Omega \mathbf{a}(x, Du)\, .DT_k(u - T_\gamma(\tilde{u}))dx &= \int_{\{|u - T_\gamma(\tilde{u})| \leq k\}} \mathbf{a}(x, Du)\, .D(u - T_\gamma(\tilde{u}))dx \\ &\geq \int_{\omega_2} \mathbf{a}(x, Du).D(u - \tilde{u})\, dx - \int_{\omega_3} \mathbf{a}(x, Du).D\tilde{u}\, dx \end{aligned} \tag{3.127}$$

According to growth conditions **(H3)** and the Hölder inequality, the last integral in (3.127) converges to 0 as $\gamma \to \infty$. Consequently,

$$\int_{\{|u - T_\gamma(\tilde{u})| \leq k\}} \mathbf{a}(x, Du)\, .D(u - T_\gamma(\tilde{u}))dx \geq \int_{\omega_2} \mathbf{a}(x, Du).D(u - \tilde{u})\, dx. \tag{3.128}$$

By the same technique, we treat the last term of $I_{h,k}$ to obtain

$$\int_{\{|u - T_\gamma(\tilde{u})| \leq k\}} \mathbf{a}(x, D\tilde{u})\, .D(\tilde{u} - T_\gamma(u))dx \geq -\int_{\omega_2} \mathbf{a}(x, D\tilde{u}).D(u - \tilde{u})\, dx. \tag{3.129}$$

Combining (3.125),(3.126), (3.128) and (3.129), it follows from (3.122) that

$$\begin{aligned} &\int_\Omega (b - \tilde{b})\, T_k(u - \tilde{u})dx + \lambda \int_{\partial\Omega} (u_1 - u_2)\, T_k(u - \tilde{u})d\sigma \\ &\qquad + \int_{\{|u - \tilde{u}| \leq k\}} \big(\mathbf{a}(x, Du) - \mathbf{a}(x, D\tilde{u})\big).(Du - D\tilde{u})\, dx \leq 0. \end{aligned} \tag{3.130}$$

Since $b \in \partial j(u)$, $\tilde{b} \in \partial j(\tilde{u})$ and the sub-gradient of a convex function j is monotone then
$\langle \partial j(u) - \partial j(\tilde{u}), u - \tilde{u} \rangle \geq 0$. Moreover, thanks to assumption **(H2)** and the fact that $(u - \tilde{u})T_k(u - \tilde{u}) \geq 0$, it follows that

$$\int_\Omega \big(b - \tilde{b}\big) T_k(u - \tilde{u})\, dx \geq 0, \tag{3.131}$$

$$\lambda \int_{\partial\Omega} (u_1 - u_2)\, T_k(u - \tilde{u})\, d\sigma \geq 0, \tag{3.132}$$

$$\int_{\{|u - \tilde{u}| \leq k\}} \big(\mathbf{a}(x, Du) - \mathbf{a}(x, D\tilde{u})\big).(Du - D\tilde{u})\, dx \geq 0. \tag{3.133}$$

Consequently, we deduce from (3.130)

$$\int_\Omega \big(b-\tilde{b}\big)T_k(u-\tilde{u})\,dx = 0, \tag{3.134}$$

$$\lambda\int_{\partial\Omega}(u_1-u_2)\,T_k(u-\tilde{u})\,d\sigma = 0, \tag{3.135}$$

$$\int_{\{|u-\tilde{u}|\le k\}}\big(\mathbf{a}(x,Du)-\mathbf{a}(x,D\tilde{u})\big).(Du-D\tilde{u})\,dx = 0. \tag{3.136}$$

Since $\mathbf{a}$ is strictly monotone, it can be deduced from (3.136) that $Du = D\tilde{u}$ almost everywhere in Ω. This implies the existence of a constant c such that $u-\tilde{u} = c$ almost everywhere in Ω.
By taking the limit of equation (3.134) as k tends to zero, we can derive the following result:

$$\begin{aligned}\lim_{k\to 0}\int_\Omega \big(b-\tilde{b}\big)\frac{1}{k}T_k(u-\tilde{u})dx &= \int_\Omega \big(b-\tilde{b}\big)sign_0(u-\tilde{u})dx\\ &= \int_\Omega |b-\tilde{b}|\,dx = 0. \end{aligned}\tag{3.137}$$

To summarize, using equation (3.137), we can conclude that $b=\tilde{b}$ almost everywhere in Ω.

$$u-\tilde{u} = c \quad\text{and}\quad b=\tilde{b}\quad \text{a.e. in } \Omega. \tag{3.138}$$

From (3.135), we obtain

$$\lim_{k\to 0}\lambda\int_{\partial\Omega}\big(u-\tilde{u}\big)\frac{1}{k}T_k(u-\tilde{u})d\sigma = \lambda\int_{\partial\Omega}|u-\tilde{u}|d\sigma = 0$$

this leads to

$$u-\tilde{u} = 0 \quad \text{a.e. in} \quad \partial\Omega, \tag{3.139}$$

hence $c=0$ and $u=\tilde{u}$ almost everywhere in Ω.
Finally,

$$\begin{cases} u=\tilde{u} & \text{a.e. in } \Omega,\\ b=\tilde{b} & \text{a.e. in } \Omega,\end{cases} \tag{3.140}$$

so the proof of Theorem 3.3 is achieved. □

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