

THE EXISTENCE AND UNIQUENESS OF CONDITIONAL EXPECTATION FOR INTEGRANDS WITH EXTENDED REAL VALUES THROUGH MINIMAL ASSUMPTIONS

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ABSTRACT. EVSTIGNEEV initially conceived the concept of establishing the existence and uniqueness of conditional expectation for integrands with extended real values. The present paper introduces a distinctive perspective by establishing the existence and uniqueness of the conditional expectation under minimal assumptions and without imposing a priori regularity conditions on the integrands, in contrast to previous works.

1. INTRODUCTION

It was I. V. EVSTIGNEEV [1] who first had the idea of obtaining the existence and uniqueness of conditional expectation for integrands with extended real values defined on the product of measurable spaces. The lack of clear explanations for certain points in his proof, coupled with insufficient conditions to guarantee uniqueness, generated controversy surrounding his paper. This issue was further addressed by A. DERRAS [2] and C. CASTAING F. EZZAKI [3], specifically examining positive integrands defined on the product of spaces, where the second space is Souslinien. Thus, it was crucial to clarify the situation, leading us to present a proof of existence utilizing the EVSTIGNEEV method, supported by different compelling arguments. For uniqueness, we give characteristic conditions in the case where the second space satisfies projection-selection properties (VON NEUMANN & R. J. AUMANN type) for the sub- σ -field. We conclude this study by examining the scenario of a valued integrand in a separable reflexive Banach space.

Our objective is to demonstrate the existence and uniqueness of the conditional expectation (under minimal assumptions), even in the absence of a priori regularity conditions imposed on the integrands. (Contrary to the works of L. THIBAUT [4, 5] and V. JALBY [6]).

The remainder of the paper is organized as follows; In Section 2, some preliminaries on the \mathbb{P} -discrete closure of decomposable are shown. In Sections 3, we

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established the existence and uniqueness of the conditional expectation for integrand with extended real values, without imposing a priori regularity conditions on the integrands. In Sections 4, we treated the vector case. Section 5 is intended to justify sections 3 and 4.

2. PRELIMINARIES ON THE \mathbb{P} -DISCRETE CLOSURE OF DECOMPOSABLE

Let $(\Omega, \mathbb{B}, \mathbb{P})$ be a probabilistic space and (\mathbb{X}, \mathbb{T}) a measurable space, we note $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ the set of maps defined on Ω with values in \mathbb{X} which are (\mathbb{B}, \mathbb{T}) -measurable; we equip $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ with the \mathbb{P} -discrete topology (corresponding to the topology induced by the discrete convergence), this is the topology induced by the following semi-metric: $d_{\mathbb{P}} : (u, v) \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B}) \times \mathcal{L}_{\mathbb{X}}^0(\mathbb{B}) \rightarrow \mathbb{P}(\{u \neq v\})$.

A subset \mathcal{D} of $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ is a decomposable (respectively σ -decomposable) if it verifies: $\forall (u, v) \in \mathcal{D}^2, \forall B \in \mathbb{B} : u1_B + v1_{B^c} \in \mathcal{D}$, (Respectively for all $(u_n)_n \in \mathcal{D}^{\mathbb{N}}$, and all $(B_n)_n$ a \mathbb{B} -partition of Ω ; $\sum_n u_n 1_{B_n} \in \mathcal{D}$).

The \mathbb{P} -discrete closure $\overline{\mathcal{D}}$ of \mathcal{D} can be characterized in different ways as shown by the following result:

Lemma 2.1. *Let \mathcal{D} be a non-empty decomposable of $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$, we consider the following sets:*

$$\mathcal{D}_{ess} := \{u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})/\Omega \text{ is the essential union of } \{u = v\} \text{ when } v \in \mathcal{D}\}$$

$$\mathcal{D}^{\sigma} := \{u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})/u = \sum_n u_n 1_{B_n} \text{ with } (u_n)_n \in \mathcal{D}^{\mathbb{N}} \text{ and } (B_n)_n \text{ a } \mathbb{B}\text{-partition of } \Omega\}$$

Then $\overline{\mathcal{D}} = \mathcal{D}_{ess} = \mathcal{D}^{\sigma}$.

Remark. \mathcal{D}^{σ} is the set of maps which coincide \mathbb{P} -almost surely with an element of the σ -decomposable generated by \mathcal{D} . In particular when \mathcal{D} is the decomposable generated by a non-empty subset \mathcal{K} of $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$, then any element u of \mathcal{D} coincides \mathbb{P} -almost surely with a map of the type $\sum_n u_n 1_{B_n}$ where $(u_n)_n$ is a sequence of \mathcal{K} and $(B_n)_n$ is a \mathbb{B} -partition of Ω .

Proof of Lemma 2.1:

We show the following inclusions: $\mathcal{D}_{ess} \subseteq \overline{\mathcal{D}} \subseteq \mathcal{D}^{\sigma} \subseteq \mathcal{D}_{ess}$.

(1) Let us show first that, $\mathcal{D}_{ess} \subseteq \overline{\mathcal{D}}$:

If $\Omega = \bigcup_{v \in \mathcal{D}_{ess}} \{u = v\}$, then there exists a sequence $(v_n)_n$ of elements of \mathcal{D} such that: $\Omega \setminus \bigcup_n \{u = v_n\}$ has zero probability.

Let's pose: $B_0 = \{u = v_0\} \cup (\Omega \setminus \bigcup_n \{u = v_n\})$, $u_0 = v_0$ and for $n \geq 1$,

$B_n = \{u = v_n\} \setminus \bigcup_{k < n} \{u = v_k\}$ and $u_n = \sum_{k < n} v_k 1_{B_k} + v_n 1_{(\bigcup_{j < n} B_j)^c}$, then $(B_n)_n$ is a \mathbb{B} -partition of Ω . As \mathcal{D} is decomposable, u_n is still an element of \mathcal{D} and by construction, we obtain:

$$\{u = u_{n_k}\} \subseteq (\bigcup_{k \leq n} B_k)^c \cup (\Omega \setminus \bigcup_{k \leq n} \{u = u_{n_k}\})$$

then, $\mathbb{P}(\{u = u_{n_k}\}) \leq 1 - \mathbb{P}(\bigcup_{k \leq n} B_k) + 1 - \mathbb{P}(\bigcup_{k \leq n} \{u = u_{n_k}\}) \rightarrow 0$ when $n \rightarrow +\infty$, hence, the sequence $(u_n)_n$ converges \mathbb{P} -discretely to u , which shows the inclusion.

(2) Let us show that, $\overline{\mathcal{D}} \subseteq \mathcal{D}^{\sigma}$:

Let $u \in \overline{\mathcal{D}}$; there exists a sequence $(u_n)_n$ of elements of \mathcal{D} such as:

$\mathbb{P}(\{u = u_{n_k}\}) \rightarrow 0$ when $n \rightarrow +\infty$. We can then extract a subsequence (u_{n_k}) satisfies $\mathbb{P}(\{u = u_{n_k}\}) \leq \frac{1}{2^k}$ for all integer k .

Let: $\Omega_m = \bigcap_{n \geq k} \{u = u_{n_k}\}$ for all integer m , then $B_m = \Omega_m \setminus \Omega_{(m-1)}$ for $m \geq 1$ and $B_0 = \Omega_0$. By construction, the B_m are disjoint two by two, moreover, on B_m , u and u_{n_m} coincide, consequently: $u = \sum_m u_{n_m} 1_{B_m} + u 1_{\Omega \setminus \bigcup_m B_m}$. It suffices then to

verify that $\Omega \setminus \bigcup_m B_m$ has zero probability. Since $\mathbb{P}(\Omega \setminus \bigcup_m B_m) = \mathbb{P}(\Omega \setminus \bigcup_m^\uparrow B_m) = \inf_m^\downarrow \mathbb{P}((\Omega_m)^c)$, with $\mathbb{P}((\Omega_m)^c) \leq \sum_{k \geq n} \mathbb{P}\{u \neq u_{n_k}\} \leq \sum_{k \geq m} \frac{1}{2^k} = \frac{1}{2^{(m-1)}}$. Hence the result.

(3) Let us show now that, $\mathcal{D}^\sigma \subseteq \mathcal{D}_{ess}$:

If u coincide \mathbb{P} -almost surely with a function of the type $\sum_n u_n 1_{B_n}$, with $(u_n)_n \in \mathcal{D}^{\mathbb{N}}$, then we have: $\Omega = \bigcup_n B_n \subseteq \bigcup_n \{u = u_n\} \subseteq \bigcup_{v \in \mathcal{D}_{ess}} \{u = v\}$ \mathbb{P} as, hence the result. \square

For a set $\Gamma \in \mathbb{B} \otimes \mathbb{T}$, we denote by $\mathcal{L}_\Gamma^0(B)$ the set of \mathbb{B} -measurable \mathbb{P} -selections of Γ ; recall that u is a \mathbb{P} -selection of Γ when: $\{\omega \in \Omega / (\omega, u(\omega)) \notin \Gamma\}$ has zero probability.

From the previous lemma, it is clear that: $\mathcal{D} := \mathcal{L}_\Sigma^0(B)$ is a \mathbb{P} -discretely closed decomposable for all $\Sigma \in \mathbb{B} \otimes \mathbb{T}$ because \mathcal{D} is σ -decomposable. However, a \mathbb{P} -discretely closed decomposable \mathbb{D} is not necessarily the set of \mathbb{P} -selections of a $\Sigma \in \mathbb{B} \otimes \mathbb{T}$, it suffices to consider for example:

$$\mathcal{D} = \{u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B}) / u = v \text{ } \mathbb{P}\text{as where } v \text{ is at most countable image}\}$$

The following two results give elementary examples of \mathbb{P} -discretely closed decomposable, which are representable in the previous sense.

Lemma 2.2. *Let \mathcal{D}_1 and \mathcal{D}_2 be non-empty decomposables of $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ such as: $\overline{\mathcal{D}_1} = \mathcal{L}_{\Sigma_1}^0(\mathbb{B})$ and $\overline{\mathcal{D}_2} = \mathcal{L}_{\Sigma_2}^0(\mathbb{B})$, with Σ_1 and Σ_2 belonging to $\mathbb{B} \otimes \mathbb{T}$. The \mathbb{P} -discrete closure of the decomposable generated by $\mathcal{D}_1 \cup \mathcal{D}_2$ coincides with $\mathcal{L}_{\Sigma_1 \cup \Sigma_2}^0(\mathbb{B})$.*

Proof of Lemma 2.2:

$\mathcal{L}_{\Sigma_1 \cup \Sigma_2}^0(\mathbb{B})$ is a discretely closed \mathbb{P} -decomposable which contains $\mathcal{L}_{\Sigma_1}^0(\mathbb{B})$, $\mathcal{L}_{\Sigma_2}^0(\mathbb{B})$ and $\mathcal{D}_1 \cup \mathcal{D}_2$ as well as the closure of the decomposable generated by the latter.

Conversely, let $u \in \mathcal{L}_{\Sigma_1 \cup \Sigma_2}^0(\mathbb{B})$.

Consider the following set: $B_1 := \{\omega \in \Omega / (\omega, u(\omega)) \in \Sigma_1\}$ with u_1 and u_2 are two arbitrary elements of \mathcal{D}_1 and \mathcal{D}_2 respectively.

$U_1 := u 1_{B_1} + u_1 1_{(B_1)^c}$ is a \mathbb{B} -measurable \mathbb{P} -selection of Σ_1 and $U_2 := u 1_{B_1} + u_2 1_{(B_1)^c}$ is a \mathbb{B} -measurable \mathbb{P} -selection of Σ_2 ; by hypothesis, there exists a sequence (v_n^1) of elements of \mathcal{D}_1 and a sequence (v_n^2) of elements of \mathcal{D}_2 converging \mathbb{P} -discretely with U_1 and U_2 respectively.

Let $v_n = v_n^1 1_{B_1} + v_n^2 1_{(B_1)^c}$. By construction v_n is an element of the decomposable generated by $\mathcal{D}_1 \cup \mathcal{D}_2$ and it suffices to check that (v_n) converges \mathbb{P} -discretely to u which results from the following inclusions:

$$\begin{aligned} \{v_n \neq u\} &= (\{v_n^1 \neq u\} \cap B_1) \dot{\cup} (\{v_n^2 \neq u\} \cap (B_1)^c) \\ &= (\{v_n^1 \neq U_1\} \cap B_1) \dot{\cup} (\{v_n^2 \neq U_2\} \cap (B_1)^c) \\ &\subseteq \{v_n^1 \neq U_1\} \cup \{v_n^2 \neq U_2\}. \end{aligned}$$

Since, $\mathbb{P}(\{v_n \neq u\}) \leq \mathbb{P}(\{v_n^1 \neq U_1\}) + \mathbb{P}(\{v_n^2 \neq U_2\}) \rightarrow 0$ when $n \rightarrow +\infty$, then $\mathbb{P}(\{v_n \neq u\}) \rightarrow 0$ when $n \rightarrow +\infty$. Hence the result.

Lemma 2.3. *Let $g : \Omega \times \mathbb{X} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ $\mathbb{B} \otimes \mathbb{T}$ -measurable. The set of \mathbb{B} -measurable \mathbb{P} -selections of the domain of g is equal to the \mathbb{P} -discrete closure of the domain of its integral functional as soon as the latter is non-empty.*

Recall that the integral functional associated with g is defined by:

$$I_g : u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B}) \rightarrow \int_{\Omega} g(\omega, u(\omega)) d\mathbb{P}(\omega)$$

We will note $\Lambda_g^{\mathbb{B}}$ its domain, i.e. $\Lambda_g^{\mathbb{B}} = \{u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B}) / I_g < +\infty\}$.

Remark. The following inclusion $\overline{\Lambda_g^{\mathbb{B}}} \subseteq \mathcal{L}_{dom(g)}^0(\mathbb{B})$ is always true, but it can be strict if $\Lambda_g^{\mathbb{B}}$ is empty; it suffices for example to consider g independent of $x : (\omega, x) \in \Omega \times \mathbb{X} \rightarrow g(\omega, x) = \alpha(\omega)$, where α is \mathbb{B} -measurable with finite positive values \mathbb{P} -almost surely and it is not integrable.

Proof of Lemma 2.3:

$\mathcal{L}_{dom(g)}^0(\mathbb{B})$ is a \mathbb{P} -discretely closed decomposable, it contains $\Lambda_g^{\mathbb{B}}$ and then its \mathbb{P} -almost closure.

Conversely, if u is a \mathbb{B} -measurable \mathbb{P} -selection of the domain of g then, $N := \Omega \setminus \bigcup_n^{\uparrow} \{g(u) \leq n\}$ has zero probability, so consider the following \mathbb{B} -partition $(B_n)_n$ defined by: for $n \geq 1$, $B_n = \{(n-1) < g(u) \leq n\}$ and $B_0 = \{g(u) \leq 0\} \cup N$. If u_{∞} is an arbitrary element of $\Lambda_g^{\mathbb{B}}$ then, $u = \sum_n (u1_{B_n} + u_{\infty}1_{(B_n)^c})1_{B_n}$. According to Lemma 3.1, it suffices to check that $u1_{B_n} + u_{\infty}1_{(B_n)^c}$ is an element of $\Lambda_g^{\mathbb{B}}$ for all integer n :

$$\begin{aligned} \int_{\Omega} g(u1_{B_n} + u_{\infty}1_{(B_n)^c})d\mathbb{P} &= \int_{\Omega} g(u)1_{B_n}d\mathbb{P} + \int_{\Omega} g(u_{\infty})1_{(B_n)^c}d\mathbb{P} \\ &\leq n + \int_{\Omega} g(u_{\infty})d\mathbb{P} < +\infty \end{aligned}$$

3. CONDITIONAL EXPECTATION FOR INTEGRAND WITH EXTENDED REAL VALUES

Let $(\Omega, \mathbb{A}, \mathbb{P})$ be a probability space, \mathbb{B} is a sub- σ -field of (\mathbb{X}, \mathbb{T}) a measurable space. For all $f : \Omega \times \mathbb{X} \rightarrow \overline{\mathbb{R}} \mathbb{A} \otimes \mathbb{T}$ -measurable ($\overline{\mathbb{R}}$ is equipped with its Borel σ -field) and for all $u : \Omega \rightarrow \mathbb{X}$ (\mathbb{A}, \mathbb{T})-measurable we denote by $f(u)$ the function \mathbb{A} -measurable defined by: $f(u) : \omega \in \Omega \rightarrow f(\omega, u(\omega))$ and by $\Lambda_f^{\mathbb{A}}$ (respectively $\Lambda_f^{\mathbb{B}}$) the set of maps $u : \Omega \rightarrow \mathbb{X}$ (\mathbb{A}, \mathbb{T})-measurable (respectively (\mathbb{B}, \mathbb{T}) -measurable) such as $f(u)$ are \mathbb{P} -integrable.

When f is positive, $\Lambda_f^{\mathbb{A}}$ is the domain of the integral functional associated with f and it is a decomposable of $\mathcal{L}_{\mathbb{X}}^0(\mathbb{A})$.

3.1. Definition of the conditional expectation. .

Let $f : \Omega \times \mathbb{X} \rightarrow \overline{\mathbb{R}} \mathbb{A} \otimes \mathbb{T}$ -measurable. For all $u \in \mathcal{L}_{\mathbb{X}}^0$, the positive part $(f(u))^+$ (respectively negative $(f(u))^-$) of $f(u)$ admits a unique (for \mathbb{P} -almost surely equality) \mathbb{B} -conditional expectation $\mathbb{E}^{\mathbb{B}}((f(u))^+)$ (respectively $\mathbb{E}^{\mathbb{B}}((f(u))^-)$) ([7, section I-2.9]).

In particular, if $u \in \Lambda_{f^+}^{\mathbb{B}} \cup \Lambda_{f^-}^{\mathbb{B}}$ then at least one of the functions $(f(u))^+$ and $(f(u))^-$ is \mathbb{P} -integrable and $\mathbb{E}^{\mathbb{B}}((f(u))^+) - \mathbb{E}^{\mathbb{B}}((f(u))^-)$ is the unique (for \mathbb{P} -almost surely equality) \mathbb{B} -conditional expectation of $f(u)$.

We call conditional expectation of f any map $g : \Omega \times \mathbb{X} \rightarrow \overline{\mathbb{R}} \mathbb{B} \otimes \mathbb{T}$ -measurable such as: $\forall u \in \Lambda_{f^+}^{\mathbb{B}} \cup \Lambda_{f^-}^{\mathbb{B}} : g(u) = \mathbb{E}^{\mathbb{B}}(f(u))\mathbb{P}$ as.

Lemma 3.1. *If $(f_n)_n$ is an increasing sequence of $\mathbb{A} \otimes \mathbb{T}$ -measurable functions with extended positive real values such that, for all n , f_n admits a conditional expectation g_n then $\sup_n(g_n)$ is a conditional expectation of $\sup_n(f_n)$.*

Remark. *If in all generality, we can verify that $\sup_n g_n$ is a conditional expectation of $\sup_n f_n$ it seems on the other hand that without the projection theorems (see (H) paragraph 2.2) we cannot a priori choose the sequence $(g_n)_n$ increasing.*

Proof of Lemma 3.1:

Let $u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$. According to ([7], section 12-10), we have:

$$\mathbb{E}^{\mathbb{B}}(\sup_n^{\uparrow}(f_n)) = \sup_n^{\uparrow} \mathbb{E}^{\mathbb{B}}(f_n) \text{ Pas}$$

For each integer n , $\mathbb{E}^{\mathbb{B}}(f_n(u))$ and $g_n(u)$ coincide \mathbb{P} -almost surely; the countable union of negligible being still negligible, we deduce that $\mathbb{E}^{\mathbb{B}}(\sup_n^{\uparrow}(f_n(u)))$ and $\sup_n g_n(u)$ coincide \mathbb{P} -almost surely.

3.2. Existence of conditional expectation.**Theorem 3.2.** (of existence)

Any function defined on $\Omega \times \mathbb{X}$ with extended real values $\mathbb{A} \otimes \mathbb{T}$ -measurable admits a conditional expectation with respect to any sub- σ -field of \mathbb{A} . [1]

Remark. The existence of a conditional expectation for functions with positive values is directly deduced from the following result.

Lemma 3.3. Let \mathcal{H} be a convex cone of extended-valued positive functions defined on a set \mathbb{E} satisfying the following properties:

- (1) \mathcal{H} contains the constants.
- (2) \mathcal{H} is stable under comparative bounded differences, i.e. for all $f, g \in \mathcal{H}$
 $f \leq g$ and g is bounded $\implies g - f \in \mathcal{H}$
- (3) \mathcal{H} is stable by countable sup-increasing: $\forall (f_n)_n^{\uparrow} \in \mathcal{H} : \sup_n^{\uparrow}(f_n) \in \mathcal{H}$.

If \mathcal{F} is a non-empty \cap -stable subset of $\mathcal{P}(\mathbb{E})$ such as \mathcal{H} contains $\{1_F : F \in \mathcal{F}\}$, then \mathcal{H} contains all measurable extended positive-valued functions for the σ -field generated by \mathcal{F} .

Remark. This result is the adaptation to the convex cone of functions with positive extended real values of the method used by M. METIVIER [11] in the space of bounded functions. We will see in the appendix that this is the particular case (characteristic function) of a functional result obtained by combining the methods of M. METIVIER and P. A. MEYER.

Proof of Theorem 3.2:

We apply Lemma 3.3 to $\mathbb{E} = \Omega \times \mathbb{X}$, \mathcal{H} the set of all $\mathbb{A} \otimes \mathbb{T}$ -measurable positive functions admitting conditional expectations with respect to the sub- σ -field \mathbb{B} and the set: $\mathcal{F} = \{A \times B : A \in \mathbb{A}, B \in \mathbb{T}\}$.

It is clear that \mathcal{H} is a convex cone by comparative and sup-increasing bounded differences according to Lemma 3.1 and the set: $\{1_{A \times B} : A \in \mathbb{A}, B \in \mathbb{T}\}$ is contained in \mathcal{H} because $1_{A \times B} = 1_A \cdot 1_B$ admits $\mathbb{E}^{\mathbb{B}}(1_A) \cdot 1_B$ as conditional expectation. Moreover \mathcal{F} is a \cap -stable generating subset of $\mathbb{A} \otimes \mathbb{T}$; from Lemma 3.3, we then deduce that \mathcal{H} contains all positive functions $\mathbb{A} \otimes \mathbb{T}$ -measurable.

In the case of a function of arbitrary sign, let g and h be conditional expectations f^+ and f^- and ϕ an arbitrary $\mathbb{B} \otimes \mathbb{T}$ -measurable function. The function ψ defined by:

$$\psi : (\omega, x) \rightarrow f(\omega, u(\omega)) = \begin{cases} g(\omega, x) - h(\omega, x) & ; \text{ if } (\omega, x) \in \text{dom}g \cup \text{dom}h \\ \psi(\omega, x) & ; \text{ otherwise.} \end{cases}$$

is a conditional expectation of f , indeed let $u \in \Lambda_{f^+}^{\mathbb{B}} \cup \Lambda_{f^-}^{\mathbb{B}}$ then $u \in \Lambda_g^{\mathbb{B}} \cup \Lambda_h^{\mathbb{B}}$ and is therefore \mathbb{P} -selection of $\text{dom}g \cup \text{dom}h$.

$$\psi(u) = g(u) - h(u) \stackrel{\text{Pas}}{=} \mathbb{E}^{\mathbb{B}}(f^+(u)) - \mathbb{E}^{\mathbb{B}}(f^-(u)) \stackrel{\text{Pas}}{=} \mathbb{E}^{\mathbb{B}}[f^+(u) - f^-(u)] \stackrel{\text{Pas}}{=} \mathbb{E}^{\mathbb{B}}(f(u)).$$

It clearly appears in the proof of the previous theorem that the choice of ϕ is totally arbitrary; consequently if the projection on Ω of $(\text{dom}g \cup \text{dom}h)^c$ is not \mathbb{P} -negligible, the uniqueness (for \mathbb{P} -almost surely equality) of the conditional expectation fails; contrary to what I. V. Evstigneev asserts ([1]; proposal 1, p. 517) the condition:

$$(H) : \left(\begin{array}{c} \text{for all set } \Sigma \in \mathbb{B} \otimes \mathbb{T}, \Omega_\Sigma := \text{proj}_\Omega \Sigma \in \tilde{B} \\ \text{and there exists} \\ \xi \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B}) \text{ such as } \{\omega \in \Omega_\Sigma / (\omega, \xi(\omega)) \notin \Sigma\} \text{ is } \mathbb{P} - \text{negligible.} \end{array} \right)$$

(Where \tilde{B} denotes the \mathbb{P} -completed σ -field of \mathbb{B}) is not sufficient for the conditional expectation to be determined for \mathbb{P} -almost surely equality.

3.3. Uniqueness of the conditional expectation. In what follows, we assume that the considered sets satisfy the projection hypotheses (H).

Theorem 3.4. (of uniqueness)

Let $f : \Omega \times \mathbb{X} \rightarrow \overline{\mathbb{R}} \mathbb{A} \otimes \mathbb{T}$ -measurable such as the sets $\Lambda_{f^-}^{\mathbb{B}}$ and $\Lambda_{f^+}^{\mathbb{B}}$ are nonempty. A necessary and sufficient condition for f to have a unique conditional expectation for \mathbb{P} -almost surely equality is that the decomposable generated by $\Lambda_{f^-}^{\mathbb{B}} \cup \Lambda_{f^+}^{\mathbb{B}}$ be \mathbb{P} -discretely dense in $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$.

In the case of uniqueness, we will denote $\mathbb{E}^{\mathbb{B}}(f)$ an arbitrary representative of the class of conditional expectations of f .

Remark. .

- (1) According to theorem 3.4, we can say that: $\mathbb{E}^{\mathbb{B}}(f) = \mathbb{E}^{\mathbb{B}}(f^+) - \mathbb{E}^{\mathbb{B}}(f^-)$.
- (2) The union of the decomposable $\Lambda_{f^+}^{\mathbb{B}}$ and $\Lambda_{f^-}^{\mathbb{B}}$ is not necessarily decomposable.
- (3) If f is positive \mathbb{P} -almost surely, $\Lambda_{f^-}^{\mathbb{B}} = \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ and by the previous theorem, it is immediate that any positive $\mathbb{A} \otimes \mathbb{T}$ -measurable function admits a unique conditional expectation $\mathbb{E}^{\mathbb{B}}(f)$ for \mathbb{P} -almost sure equality.

Corollary 3.5. Let $f : \Omega \times \mathbb{X} \rightarrow \overline{\mathbb{R}} \mathbb{A} \otimes \mathbb{T}$ -measurable such as $\Lambda_{f^+}^{\mathbb{B}}$ is nonempty. A necessary and sufficient condition for f to have a unique proper conditional expectation is that $\Lambda_{f^-}^{\mathbb{B}}$ be \mathbb{P} -discretely dense in $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$.

Proof of Corollary 3.5:

If $\Lambda_{f^-}^{\mathbb{B}}$ is \mathbb{P} -discretely dense in $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$, then we have the same thing for $\Lambda_{\mathbb{E}^{\mathbb{B}}f^-}^{\mathbb{B}}$ and for $\Lambda_{f^-}^{\mathbb{B}} \cup \Lambda_{f^+}^{\mathbb{B}}$, f therefore admits a unique conditional expectation $\mathbb{E}^{\mathbb{B}}(f)$ with: $\mathbb{E}^{\mathbb{B}}(f^-) = (\mathbb{E}^{\mathbb{B}}(f))^-$ \mathbb{P} -as and $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ coincides with $\mathcal{L}_{\text{dom}\Lambda_{\mathbb{E}^{\mathbb{B}}f^-}^{\mathbb{B}}}^0(\mathbb{B})$ from Lemma 2.3,

we deduce that $\text{proj}_\Omega\{\mathbb{E}^{\mathbb{B}}(f) = -\infty\}$ has zero probability.

Conversely, if f admits a unique proper conditional expectation then $\mathbb{E}^{\mathbb{B}}(f^-)$ and $(\mathbb{E}^{\mathbb{B}}(f))^-$ coincide \mathbb{P} -almost surely, therefore, $\mathbb{E}^{\mathbb{B}}(f^-)$ is \mathbb{P} -almost surely real-valued and $\Lambda_{f^-}^{\mathbb{B}} = \Lambda_{\mathbb{E}^{\mathbb{B}}f^-}^{\mathbb{B}}$ is \mathbb{P} -discretely dense in $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ by Lemma 2.3.

Proof of Theorem 3.4:

Sufficient condition of Theorem 3.4: If g and h are two conditional expectations of f , then by definition they satisfy for all $u \in \Lambda_{f^-}^{\mathbb{B}} \cup \Lambda_{f^+}^{\mathbb{B}}$:

$$g(u) = \mathbb{E}^{\mathbb{B}}f(u) = h(u) \text{ } \mathbb{P}\text{-as.}$$

Their coincidence is then immediately deduced from the following result, which will be useful to us later.

Lemma 3.6. *Let \mathcal{K} be a subset whose generated decomposable is \mathbb{P} -discretely dense in $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$. If g and h are two functions defined on $\Omega \times \mathbb{X}$, $\mathbb{B} \otimes \mathbb{T}$ -measurable such as $g(u)$ and $h(u)$ coincide \mathbb{P} -almost surely for all $u \in \mathcal{K}$, then g and h coincide \mathbb{P} -almost surely.*

Proof of Lemma 3.6:

According to Lemma 3.1, any $u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ coincides \mathbb{P} -almost surely with an application of the form: $\sum_n u_n 1_{B_n}$, where $(u_n) \in \mathcal{K}^{\mathbb{N}}$ and (B_n) is a \mathbb{B} -partition of Ω . By hypothesis, g and h verify for any integer: $g(u_n) = h(u_n)$ \mathbb{P} -a.s, hence the following equalities: $g(u) = \sum_n g(u_n) 1_{B_n} = \sum_n h(u_n) 1_{B_n} = h(u)$ \mathbb{P} -a.s, are satisfied out of a \mathbb{P} -negligible and this for all $u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$.

As $\{g \neq h\}$ belongs to $\mathbb{B} \otimes \mathbb{T}$, we deduce from hypothesis (H) that its projection on Ω has zero probability. This ensures the \mathbb{P} -almost sure coincidence of g and h .

Necessary condition of Theorem 3.4: Let g be a conditional expectation of f^+ and h a conditional expectation of f^- . If ϕ is an arbitrary $\mathbb{B} \otimes \mathbb{T}$ -mesarable function, we have seen that:

$$\psi : (\omega, x) \rightarrow f(\omega, u(\omega)) = \begin{cases} g(\omega, x) - h(\omega, x) & ; \text{ if } (\omega, x) \in \text{dom}g \cup \text{dom}h \\ \psi(\omega, x) & ; \text{ otherwise.} \end{cases}$$

is a conditional expectation of f . Since f admits a unique conditional expectation, $\text{dom}g \cup \text{dom}h$ coincides \mathbb{P} -almost surely with $\Omega \times \mathbb{X}$. Given Lemma 2.3, we have the following equalities: $\mathcal{L}_{\text{dom}g}^0(\mathbb{B}) = \overline{\Lambda_{f^+}^{\mathbb{B}}}$ and $\mathcal{L}_{\text{dom}h}^0(\mathbb{B}) = \overline{\Lambda_{f^-}^{\mathbb{B}}}$. From Lemma 2.2, we conclude that the \mathbb{P} -discrete closure of the decomposable generated by $\Lambda_{f^-}^{\mathbb{B}} \cup \Lambda_{f^+}^{\mathbb{B}}$ coincides with $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$.

Proposition 3.7. *Let $\Gamma \in \mathbb{A} \otimes \mathbb{T}$ and $\mathbb{F}^{\mathbb{B}}\Gamma = \{\mathbb{E}^{\mathbb{B}}(1_{\Gamma^c}) = 0\}$, the indicator of $\mathbb{F}^{\mathbb{B}}\Gamma$ is the conditional expectation of the indicator of Γ and $\mathbb{F}^{\mathbb{B}}\Gamma$ is the essential union of the elements \sum of $\mathbb{B} \otimes \mathbb{T}$ contained \mathbb{P} -almost surely in Γ .*

Remark. *Contrary to the case treated by A. TRUFFERT in [8], paragraph 1.8. p. 135 (see also [9], chap. 20), it seems that we cannot assert here that $\mathbb{F}^{\mathbb{B}}\Gamma$ is contained \mathbb{P} -almost surely in Γ for lack of knowing how to verify that the essential union is a countable union of sets of the family.*

Proof of Proposition 3.7:

As $\delta_{\Gamma} = \sup_n^{\uparrow} n 1_{\Gamma^c}$, then $\mathbb{E}^{\mathbb{B}}(\delta_{\Gamma}) = \sup_n^{\uparrow} n \mathbb{E}^{\mathbb{B}}(1_{\Gamma^c}) = \delta_{\{\mathbb{E}^{\mathbb{B}}(1_{\Gamma^c})=0\}}$ \mathbb{P} -a.s, which proves the 1st statement.

Let $\sum \in \mathbb{B} \otimes \mathbb{T}$ contained \mathbb{P} -almost surely in Λ , suppose that \sum doesnt contained \mathbb{P} -almost surely in $\mathbb{F}^{\mathbb{B}}\Gamma$, then the probability of $\text{proj}_{\Omega} \sum \mathbb{F}^{\mathbb{B}}\Gamma$ is strictly positive, and by hypothesis (H), there exists $B \in \mathbb{B}$ with strictly positive probability and $u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ such as: $\forall \omega \in B : (\omega, u(\omega)) \in \sum \setminus \mathbb{F}^{\mathbb{B}}\Gamma$ which implies: $\delta_{\mathbb{F}^{\mathbb{B}}\Gamma}(u) d\mathbb{P} = +\infty$ and $\int_B \delta_{\Gamma}(u) d\mathbb{P} = 0$, which is in contradiction with the fact that $\delta_{\mathbb{F}^{\mathbb{B}}\Gamma}$ is a conditional expectation of δ_{Γ} .

Finally, let $\sum_0 \in \mathbb{B} \otimes \mathbb{T}$ containing \mathbb{P} -almost surely in Γ , let us show by contradiction that it still contains $\mathbb{F}^{\mathbb{B}}\Gamma$.

If $\text{proj}_{\Omega} \sum_0 \mathbb{F}^{\mathbb{B}}\Gamma$ has non-zero probability, then, there exists $B_0 \in \mathbb{B}$ and $u_0 \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ such as: $\forall \omega \in B_0 : (\omega, u_0(\omega)) \in \mathbb{F}^{\mathbb{B}}\Gamma \setminus \sum_0$ (*).

$Gr(u_0) \cap B_0 \times \mathbb{X}$ belongs to $\mathbb{B} \otimes \mathbb{T}$ and is continuous \mathbb{P} -almost surely in Γ because: $\int_{B_0} \delta_{\Gamma}(u_0) d\mathbb{P} = \int_{B_0} \delta_{\mathbb{F}^{\mathbb{B}}\Gamma}(u_0) d\mathbb{P} = 0$. we deduce by hypothesis on \sum_0 that it is continuous \mathbb{P} -almost surely in the latter, which is in contradiction with (*).

4. CONDITIONAL EXPECTATION OF A VECTOR INTEGRAND

In this paragraph, we consider $(\Omega, \mathbb{A}, \mathbb{X})$ a probabilistic space, \mathbb{X} a metrizable sublinian space equipped with its Borel σ -field $\mathbb{B}_{\mathbb{X}}$ and \mathbb{Y} a reflexive Banach space separable from dual \mathbb{Y}' and from Borel $\mathbb{B}_{\mathbb{Y}}$.

Now let \mathbb{B} be a sub- σ -field of \mathbb{A} . We recall that if v is a map of Ω with values in $(\mathbb{A}, \mathbb{B}_{\mathbb{Y}})$ -measurable and Bochner-integrated, there exists a unique (\mathbb{P} -almost surely) map $\mathbb{E}^{\mathbb{B}}(v)$ ($\mathbb{B}, \mathbb{B}_{\mathbb{Y}}$)-measurable such as: $\forall B \in \mathbb{B} : \int_B v d\mathbb{P} = \int_B \mathbb{E}^{\mathbb{B}}(v) d\mathbb{P}$ (See [10] Theorem 5.4). $\mathbb{E}^{\mathbb{B}}(v)$ is called the conditional expectation of v with respect to \mathbb{B} .

Let $\Phi : \Omega \times \mathbb{X} \rightarrow \mathbb{Y}$ be a vector \mathbb{A} -integrand, i.e. $(\mathbb{A} \otimes \mathbb{B}_{\mathbb{X}}, \mathbb{B}_{\mathbb{Y}})$ -measurable map. We call conditional expectation of Φ with respect to \mathbb{B} any \mathbb{B} -integrand: $\Psi : \Omega \times \mathbb{X} \rightarrow \mathbb{Y}$ such that $\Psi(u)$ coincides \mathbb{P} -almost surely with $\mathbb{E}^{\mathbb{B}}(\Phi(u))$ for all $u \in \Lambda_{\Phi}^{\mathbb{B}}$, where $\Lambda_{\Phi}^{\mathbb{B}}$ denotes the set of maps $u \in \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$ such that $\Phi(u)$ are Bochner-integrands. Note that this set is a decomposable of $\mathcal{L}_{\mathbb{X}}^0$. We define on $\Omega \times \mathbb{X}$ the following real-valued \mathbb{A} -integrands: $\|\Phi\| : (\omega, x) \in \Omega \times \mathbb{X} \rightarrow \|\Phi(\omega, x)\|_{\mathbb{Y}}$ and for all $y' \in \mathbb{Y}'$ $\langle \Phi, y' \rangle : (\omega, x) \in \Omega \times \mathbb{X} \rightarrow \langle \Phi(\omega, x), y' \rangle$.

Theorem 4.1. *Let Φ be a vector \mathbb{A} -integrand such that $\Lambda_{\Phi}^{\mathbb{B}}$ is \mathbb{P} -discretely dense in $\mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$.*

Φ admits a unique (for \mathbb{P} -almost surely equality) conditional expectation $\mathbb{E}^{\mathbb{B}}(\Phi)$ with respect to \mathbb{B} which satisfies the following properties:

- (1) $\|\mathbb{E}^{\mathbb{B}}(\Phi)\| \leq \mathbb{E}^{\mathbb{B}}\|\Phi\| \quad \mathbb{P}as$
- (2) $\forall y' \in \mathbb{Y}' : \langle \mathbb{E}^{\mathbb{B}}(\Phi), y' \rangle = \mathbb{E}^{\mathbb{B}}(\langle \Phi, y' \rangle) \quad \mathbb{P}as$

Proof of Theorem 4.1:

Let $f_{\Phi} : (\omega, x, y') \in \Omega \times \mathbb{X} \times \mathbb{Y}' \rightarrow \langle \Phi(\omega, x), y' \rangle \in \mathbb{R}$ a $(\mathbb{A} \otimes \mathbb{B}_{\mathbb{X}} \otimes \mathbb{B}_{\mathbb{Y}'}, \mathbb{B}_{\mathbb{R}})$ -measurable and for all $u \in \Lambda_{\Phi}^{\mathbb{B}}$ and all $v' \in \mathcal{L}_{\mathbb{Y}'}^{\infty}(\mathbb{B})$, $f(\omega, x)$ is \mathbb{P} -integrable, consequently $\Lambda_f^{\mathbb{B}}$ contains $\Lambda_{\Phi}^{\mathbb{B}} \times \mathcal{L}_{\mathbb{Y}'}^{\infty}(\mathbb{B})$, the latter being a \mathbb{P} -discretely dense decomposable in $\mathcal{L}_{\mathbb{X} \times \mathbb{Y}'}^0(\mathbb{B})$, according to Theorem 3.2 and Theorem 3.4, f_{Φ} admits a unique (for \mathbb{P} -almost surely equality) conditional expectation $\mathbb{E}^{\mathbb{B}}(f_{\Phi})$ with respect to \mathbb{B} .

Let us start by establishing the following property:

$$\forall \omega \in \Omega, \forall x \in \mathbb{X}, \quad y' \in \mathbb{Y}' \rightarrow \mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, x, y')) \text{ is continuous linear.}$$

For linearity, we proceed in the same way as A. TRUFFERT ([8], Proposition 1.6.2), considering the three maps defined on $\Omega \times [0, 1] \times \mathbb{X} \times \mathbb{Y}' \times \mathbb{Y}'$ as follows:

$$g(\omega, \alpha, x, y'_1, y'_2) = \alpha y'_1 + (1 - \alpha) y'_2$$

$$h(\omega, \alpha, x, y'_1, y'_2) = \alpha f_{\Phi}(\omega, x, y'_1) + (1 - \alpha) f_{\Phi}(\omega, x, y'_2)$$

$$i(\omega, \alpha, x, y'_1, y'_2) = \alpha \mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, x, y'_1)) + (1 - \alpha) \mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, x, y'_2))$$

The linearity of $\mathbb{E}^{\mathbb{B}}(f_{\Phi})$ (respectively f_{Φ}) with respect to the last variable then results in the almost sure equality of the integrands $(\mathbb{E}^{\mathbb{B}} f_{\Phi})og$ and i (respectively $f_{\Phi}og$ and h). The hypotheses guarantee the uniqueness of the conditional expectations of $(\mathbb{E}^{\mathbb{B}} f_{\Phi})og$ and h , consequently they coincide \mathbb{P} -almost surely with $(\mathbb{E}^{\mathbb{B}} f_{\Phi})og$ and i respectively, hence the equality almost sure of these.

The positive integrand $\|\Phi\|$ admits a unique conditional expectation with respect to \mathbb{B} which we will note ψ for simplicity. According to lemma 2.3, ψ is \mathbb{P} -almost surely with real values because: $\mathcal{L}_{dom\psi}^0(\mathbb{B}) = \Lambda_{\psi}^{\mathbb{B}} = \Lambda_{\|\Phi\|}^{\mathbb{B}} = \mathcal{L}_{\mathbb{X}}^0(\mathbb{B})$.

To verify the continuity of $\mathbb{E}^{\mathbb{B}}(f_{\Phi})$ with respect to the last variable, it suffices to establish the following inequality: $\forall \omega \in \Omega, \forall x \in \mathbb{X}, \forall y' \in \mathbb{Y}', \|y'\| \leq 1$ then,

$\mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, x, y')) \leq \psi(\omega, x)$. We proceed by contradiction assuming that the projection of:

$$\Sigma := \{(\omega, x, y') \in \Omega \times \mathbb{X} \times \mathbb{Y}' : \|y'\| \leq 1 \text{ and } \mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, x, y')) > \psi(\omega, x)\}$$

has strictly positive probability; thanks to the selection theorems due to AUMANN [15], there exists $B \in \mathbb{B}$ with strictly positive probability and $(u, v') \in \mathcal{L}_{\mathbb{X} \times \mathbb{Y}'}^0(\mathbb{B})$ such as: $\mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, u(\omega), v'(\omega))) > \psi(\omega, u(\omega))$ and $\forall \omega \in: \|v'(\omega)\| \leq 1$.

Let $B_n = \{\omega \in \Omega : \psi(\omega, u(\omega)) \leq n\}$, as ψ is \mathbb{P} -almost surely with real values, $\Omega \setminus \cup_n^{\uparrow} B_n$ has zero probability. We deduce the existence of an integer n_0 for which $B \cap B_{n_0}$ still has strictly positive probability and $\int_{B \cap B_{n_0}} \psi(u) d\mathbb{P} < +\infty$.

Let u_0 be an arbitrary element of $\Lambda_{\Phi}^{\mathbb{B}}$ and v_0 an element of the unit ball of $\mathcal{L}_{\mathbb{Y}'}^{\infty}(\mathbb{B})$, then: $(\bar{u}, \bar{v}') = (u, v')1_{B \cap B_{n_0}} + (u_0, v'_0)1_{(B \cap B_{n_0})^c} \in \Lambda_{f_{\Phi}}^{\mathbb{B}} \times \mathcal{L}_{\mathbb{Y}'}^{\infty}(\mathbb{B})$.

Because,

$$|f_{\Phi}(\bar{u}, \bar{v}')| \leq \|\Phi(\bar{u})\|$$

$$\text{and } \int \|\Phi\|(\bar{u}) d\mathbb{P} = \int_{B \cap B_{n_0}} \psi(u) d\mathbb{P} + \int_{(B \cap B_{n_0})^c} \|\Phi(u_0)\| d\mathbb{P} < +\infty$$

Consequently, we then obtain:

$$\int_{B \cap B_{n_0}} \langle \Phi(u), v' \rangle d\mathbb{P} = \int_{B \cap B_{n_0}} \mathbb{E}^{\mathbb{B}}(f_{\Phi}(\Phi(u), v')) d\mathbb{P} > \int_{B \cap B_{n_0}} \psi(u) d\mathbb{P} = \int_{B \cap B_{n_0}} \|\Phi(u)\| d\mathbb{P}$$

With $\|v'\| \leq 1$ on $B \cap B_{n_0}$, and therefore a contradiction.

Even if it means replacing $\mathbb{E}^{\mathbb{B}}(f_{\Phi})$ by 0 on $N \times \mathbb{X} \times \mathbb{Y}'$ where N is a \mathbb{P} -negligible, we can assume that: $y' \rightarrow \mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, x, y'))$ is continuous linear for all $(\omega, x) \in \Omega \times \mathbb{X}$. Since \mathbb{Y} is a reflexive Banach, then for all $(\omega, x) \in \Omega \times \mathbb{X}$, $\mathbb{E}^{\mathbb{B}}(f_{\Phi}(\omega, x, \cdot))$ is represented by $\langle \psi(\omega, x), \cdot \rangle$ with $\psi(\omega, x) \in \mathbb{Y}$.

Thus defined on $\Omega \times \mathbb{X}$, ψ is scalarly measurable and therefore $(\mathbb{B} \otimes \mathbb{B}_{\mathbb{X}}, \mathbb{B}_{\mathbb{Y}})$ -measurable given the separability assumption on \mathbb{Y} . To conclude, we then use the following result.

Lemma 4.2. *Under the hypothesis of the previous theorem, a vector \mathbb{B} -integrand ψ is a conditional expectation of Φ if and only if f_{ψ} is a conditional expectation of f_{Φ} .*

Proof of Lemma 4.2:

According to [14], chapter 3.3, we have for all $u \in \Lambda_{\Phi}^{\mathbb{B}}$ and all $v_0 \in \mathcal{L}_{\mathbb{Y}'}^{\infty}(\mathbb{B})$:

$$\langle \mathbb{E}^{\mathbb{B}}\Phi(u), v' \rangle = \mathbb{E}^{\mathbb{B}}(\langle \Phi(u), v' \rangle) \quad \mathbb{P}as$$

If ψ is a conditional expectation of Φ , we then deduce that:

$$f_{\psi}(u, v') = \mathbb{E}^{\mathbb{B}}[(f_{\Phi}(u, v'))] = \mathbb{E}^{\mathbb{B}}(f_{\Phi}(u, v')) \quad \mathbb{P}as$$

As $\Lambda_{\Phi}^{\mathbb{B}} \times \mathcal{L}_{\mathbb{Y}'}^0(\mathbb{B})$ is \mathbb{P} -discretely dense in $\mathbb{L}_{\mathbb{X} \times \mathbb{Y}'}^0(\mathbb{B})$ and according to Lemma 3.6, f_{ψ} and $\mathbb{E}(f_{\Phi})$ coincide \mathbb{P} -almost surely.

Conversely, if f_{ψ} is a conditional expectation of f_{Φ} then for all $u \in \Lambda_{\Phi}^{\mathbb{B}}$ and all $v'_0 \in \mathcal{L}_{\mathbb{Y}'}^{\infty}(\mathbb{B})$: $\langle \psi(u), v' \rangle := \mathbb{E}^{\mathbb{B}}[\langle \Phi(u), v' \rangle] = \langle \mathbb{E}^{\mathbb{B}}(\Phi(u)), v' \rangle$. Hence $\psi(u)$ and $\mathbb{E}^{\mathbb{B}}(\Phi(u))$ coincide \mathbb{P} -almost surely and ψ is a conditional expectation of Φ .

5. APPENDIX:

In this part, we consider a set \mathbb{E} , a family \mathcal{H} of bounded positive real-valued functions defined on the set \mathbb{E} and \mathcal{C} a non-empty subset of \mathcal{H} .

Proposition 5.1. *We suppose that \mathcal{H} is a convex cone containing the positive constants and satisfying the following properties:*

- (1) \mathcal{H} is stable under comparative bounded differences, i.e.

$$\forall f, g \in \mathcal{H}, f \leq g \text{ and } g \text{ is bounded} \implies g - f \in \mathcal{H}.$$

- (2) \mathcal{H} is stable by countable bounded sup-increasing, i.e.

$$\forall (f_n)_n \in \mathcal{H}, \sup_n^\uparrow(f_n) \text{ is bounded} \implies \sup_n^\uparrow(f_n) \in \mathcal{H}.$$

And that \mathcal{C} is a \wedge and \vee -stable convex cone containing the constant function equal to 1. Then \mathcal{H} contains all the measurable extended positive-valued functions for the σ -field generated by \mathcal{C} .

Remark. .

- (1) As any extended real-valued function is the sup-increasing of its truncations, i.e. $f = \sup_n^\uparrow(f \wedge n)$, we easily deduce from the previous proposition that \mathcal{H} contains all positive real-valued (respectively extended) functions if condition (2) of Proposition 5.1. is replaced by:

$$\forall (f_n)_n \in \mathcal{H}, \sup_n^\uparrow(f_n) < +\infty \implies \sup_n^\uparrow(f_n) \in \mathcal{H} \text{ (respectively } \sup_n^\uparrow(f_n \wedge n) \in \mathcal{H}).$$

- (2) The convex cone \mathcal{C} generated by the set of characteristic functions based on a subset \mathcal{F} of $\mathcal{P}(\mathbb{E})$ is a lattice as soon as \mathcal{F} is \cap -stable, since:

$$\sum_{i \in I} \alpha_i 1_{A_i} \wedge \sum_{j \in J} \beta_j 1_{B_j} = \sum_{(i,j) \in I \times J} (\alpha_i \wedge \beta_j) 1_{A_i \cap B_j}$$

$$\text{and } \sum_{i \in I} \alpha_i 1_{A_i} \vee \sum_{j \in J} \beta_j 1_{B_j} = \sum_{(i,j) \in I \times J} (\alpha_i \vee \beta_j) 1_{A_i \cap B_j}.$$

In this case the σ -field generated by the elements of \mathcal{C} is none other than that generated by \mathcal{F} , Lemma 3.3. is therefore easily deduced from the result and the previous remark.

- (3) We will see that the proof does not require the use of Zorn's lemma, unlike the following corollary which generalizes the result of C. DELLACHERIE and P. A. MEYER ([13], Theorem 22.3, page 22).

Corollary 5.2. *If \mathcal{H} is stable by bounded countable sup-increasing and \mathcal{C} is a convex cone stable by compared differences, \wedge -stable containing $\{1\}$ then \mathcal{H} contains all functions with positive real values bounded measurable for the σ -field generated by the elements of \mathcal{C} .*

\mathcal{H} contains all measurable bounded positive real-valued functions for the array generated by the elements of \mathcal{C} .

Proof of Corollary 5.2:

The set of convex cones stable by compared differences, \wedge -stable containing $\mathcal{C} \cup \{1\}$ and contained in \mathcal{H} is an inductive set, Zorn's theorem allows us to choose a maximal element \mathcal{C}_∞ which is stable by bounded countable sup-increasing. Indeed, let $(f_n)_n$ be an increasing sequence of elements of \mathcal{C}_∞ which the $\sup f$ has bounded values, consider \mathcal{C}_f the convex cone stable by compared differences, \wedge -stable generated by $\mathcal{C}_\infty \cup \{f\}$, it suffices to verify that \mathcal{C}_f is still contained in \mathcal{H} .

If $g \in \mathcal{C}_\infty$, we have: $g + f = \sup_n^\uparrow(g + f_n)$ and $g \wedge f = \sup_n^\uparrow(g \wedge f_n)$.

If $g \leq f$, then: $f - g = \sup_n^\uparrow(f_n - g)$ and if $f \leq g$, then: $g - f = g - f_0 - \sup_n^\uparrow(f_n - f_0)$.

Thanks to the properties of \mathcal{H} , we verify that these functions thus constructed are still elements of \mathcal{H} .

It is clear that \mathcal{C}_∞ is \vee -stable since if $f, g \in \mathcal{C}_\infty$ there exists $R > 0$ bounding f and g and $f \vee g = R - (R - f) \wedge (R - g) \in \mathcal{C}_\infty$.

To conclude, we apply Proposition 5.1 to \mathcal{C}_∞ and \mathcal{C} .

Proof of Proposition 5.1:

Let $\sigma(\mathcal{C})$ the σ -field generated by the elements \mathcal{C} . Any measurable function with bounded positive real values is the sup of a sequence of elements of the convex cone generated by $\{1_A/A \in \sigma(\mathcal{C})\}$, it is therefore sufficient, taking into account the fact that \mathcal{H} is a convex cone, to verify that this set is contained in \mathbb{H} , which amounts to showing that $\sigma(\mathcal{C})$ is included in the following set, $\mathcal{D} := \{A \in \mathcal{P}(\mathcal{C})/1_A \in \mathcal{H}\}$.

Let us first show that \mathcal{D} is a Dynkin system:

- (1) \mathcal{D} is stable by compared differences:
 If $A, B \in \mathcal{D}$ with: $A \subseteq B$ then 1_A and $1_B \in \mathcal{H}$ and $1_A \leq 1_B$, consequently:
 $1_{B \setminus A} \leq 1_B - 1_A \in \mathcal{H}$ from (1) and $B \setminus A \in \mathcal{D}$.
- (2) \mathcal{D} is stable under increasing countable unions:
 If $(A_n)_n \in \mathcal{D}^{\mathbb{N}}$ then $1_{\cup_n A_n} = \sup_n 1_{A_n} \in \mathcal{H}$ from (2)
 On the other hand $\sigma(\mathcal{C})$ is generated by:
 $\mathcal{F} = \{\cap_{i \in I} \{f_i \geq t_i\} / f_i \in \mathcal{C}, t_i \geq 0 \text{ for } i \in I \text{ with finite } I\}$ which is \cap -stable and admits E for element, consequently $\sigma(\mathcal{C})$ is none other than the Dynkin system generated by \mathcal{F} and it suffices to conclude to show that \mathcal{F} is contained in \mathcal{D} :

Let I be finite, $f_i \in \mathcal{C}$ and $t_i > 0 (\{f_i \geq 0\} = \mathbb{E})$ for all $i \in I$, we have:

$$f := 1_{\cap_{i \in I} \{f_i \geq t_i\}} = \inf_n 1_{\cap_{i \in I} (1/t_i) f_i \wedge 1} = 1 - \sup_n (1 - (\cap_{i \in I} (1/t_i) f_i \wedge 1)^n).$$

Since \mathcal{C} is a \wedge -stable cone, $\cap_{i \in I} (1/t_i) f_i \wedge 1$ is an element of \mathcal{C} . To verify that $f \in \mathcal{H}$, it suffices to ensure that $h^n \in \mathcal{H}$ as soon as $h \in \mathcal{C}$.

However, since the function $x \in \mathbb{R}^+ \rightarrow x^n$ is convex, it is the sup-countable of continuous affine functions, consequently, h^n is the sup-crescent of a sequence of elements of \mathcal{C} since the latter is a \vee -stable convex cone.

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