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SOLUTION OF SYSTEM OF URYSOHN INTEGRAL EQUATIONS IN α - COMPLETE EXTENDED BRANCIARI b -DISTANCE SPACES

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Abstract. In this paper, an α-complete extended Branciari b-distance space and rational $\alpha - \lambda - JS$ contractive conditions in the underlying spaces are introduced. Further, we establish the coincidence point, common fixed points, and uniqueness of fixed points for two pairs of mappings in new spaces under the aforementioned contractive conditions. We illustrate the work with an example and apply the results to determine the existence of solutions for a system of Urysohn integral equations.

1. Introduction and preliminaries

Denote $\mathbb{R}$:= the set of real numbers, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{N}$:= the set of natural numbers, and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

Numerous authors [14, 15, 16, 17, 11, 3, 19, 20, 22] introduce and generalize the concept of distance within the metric fixed point theory in a variety of ways. Bakhtin [4] defines the concept of b-metric space, which Czerwik employs in [7, 8]. Kamran et al.[19] introduces the concept of extended b-metric space, whereas Branciari [5] extends the metric space and introduce the concept of Branciari distance by replacing the triangle inequality property with a quadrilateral inequality.

Definition 1.1. [19] *Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \to \mathbb{R}_+ \backslash (0, 1)$. A function $\sigma_e : \Xi^2 \to \mathbb{R}_+$ is said to be an extended b-metric (σ_e-metric, for short) if the following conditions are met:*

(eb1) $\sigma_e(\varkappa, \varpi) = 0 \iff \varkappa = \varpi$;
(eb2) $\sigma_e(\varkappa, \varpi) = \sigma_e(\varpi, \varkappa)$;
(eb3) $\sigma_e(\varkappa, \varpi) \leq w(\varkappa, \varpi)[\sigma_e(\varkappa, \upsilon) + \sigma_e(\upsilon, \varpi)]$,

for all $\varkappa, \varpi, \upsilon \in \Xi$. The symbol (Ξ, σ_e) denotes a σ_e-metric space.

Definition 1.2. [5] *Let $\Xi \neq \emptyset$ be a set and let $\sigma_b : \Xi^2 \to \mathbb{R}_+$ such that, for all $\varkappa, \varpi \in \Xi$ and all distinct $u, v \in \Xi \backslash \{\varkappa, \varpi\}$,*

(bd1) $\sigma_b(\varkappa, \varpi) = 0$ *if and only if* $\varkappa = \varpi$;

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(bd2) $\sigma_b(\varkappa, \varpi) = \sigma_b(\varpi, \varkappa)$;
(bd3) $\sigma_b(\varkappa, \varpi) \leq \sigma_b(\varkappa, u) + \sigma_b(u, v) + \sigma_b(v, \varpi)$.

The symbol (Ξ, σ_b) denotes Branciari distance space and abbreviated as "BDS.

Abdeljawad et al. [1] define an extended Branciari b-distance space by combining the extended b-metric and the Branciari distance.

Definition 1.3. [1] *Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \to \mathbb{R}_+\backslash(0,1)$. We say that a function $b_e : \Xi^2 \to \mathbb{R}_+$ is an extended Branciari b-metric (b_e-metric, in short) if it satisfies:*

(B1) $b_e(\varkappa, \varpi) = 0$ *if and only if* $\varkappa = \varpi$,
(B2) $b_e(\varkappa, \varpi) = b_e(\varpi, \varkappa)$,
(B3) $b_e(\varkappa, \varpi) \leq w(\varkappa, \varpi)[b_e(\varkappa, \nu) + b_e(\nu, \varrho) + b_e(\varrho, \varpi)]$

for all $\varkappa, \varpi \in \Xi$ all distinct $\nu, \varrho \in \Xi\backslash\{\varkappa, \varpi\}$. The symbol (Ξ, b_e) denotes an extended Branciari b-distance space (EBbDS, in short). For $w(\varkappa, \varpi) = 1$, (Ξ, b_e) will be called a Branciari b-distance space (BbDS, in short).

Example 1.4. [1]

Let $\Xi = C([0,1],\ \mathbb{R})$ and define $b_e : \Xi^2 \to \mathbb{R}_+$ by $b_e(A,\ B) = \int_0^1 (A(t) - B(t))^2 dt$ with $w(A, B) = |A(t)| + |B(t)| + 2$. Note that $b_e(A,\ B) \geq 0$ for all $A,\ B \in \Xi$, and $b_e(A,\ B) = 0$ if and only if $A = B$. Also $b_e(A,\ B) = b_e(B,\ A)$. Hence it is clear that $(\Xi,\ b_e)$ is an EBbDS, but it is neither a BDS nor a metric space.

Definition 1.5. [1] *Let $\Xi \neq \emptyset$ be a set endowed with an extended Branciari b-distance b_e and and $\alpha \colon \Xi \times \Xi \to \mathbb{R}_+$.*

(a) *A sequence $\{\varkappa_n\}$ in Ξ converges to $\varkappa$ if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $b_e(\varkappa_n, \varkappa) < \epsilon$ for all $n \geq N$. For this particular case, we write $\lim_{n\to\infty} \varkappa_n = \varkappa$.*
(b) *A sequence $\{\varkappa_n\}$ in Ξ is called Cauchy if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $b_e(\varkappa_m, \varkappa_n) < \epsilon$ for all $m, n \geq N$.*
(c) *A b_e-metric space (Ξ, b_e) is complete if every Cauchy sequence in Ξ is convergent.*
(d) *A b_e-metric space (Ξ, b_e) is α-complete if every Cauchy sequence $\{\varkappa_n\}$ in Ξ with $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ is convergent in Ξ.*

For a self mapping $\Im_1$ on a nonempty set Ξ and a point $\varkappa \in \Xi$, we use the following notation: $\Im_1^{-1}(\varkappa) = \{\varpi \in \Xi : \Im_1\varpi = \varkappa\}$.

We apply the concepts discussed in [13] to an EBbDS.

Definition 1.6. *Let (Ξ, b_e) be an EBbDS, $\alpha \colon \Xi \times \Xi \to \mathbb{R}_+$ and $\Im_1, \Im_2, \Im_4 \colon \Xi \to \Xi$ be four mappings such that $\Im_1(\Xi) \subseteq \Im_4(\Xi)$ and $\Im_2(\Xi) \subseteq \Im_4(\Xi)$. The ordered pair $(\Im_1, \Im_2)$ is said to be:*

(a) α-weakly increasing with respect to $\Im_4$ if, for all $\varkappa \in \Xi$, we have $\alpha(\Im_1\varkappa, \Im_2\varpi) \geq 1$ for all $\varpi \in \Im_4^{-1}(\Im_1\varkappa)$ and $\alpha(\Im_2\varkappa, \Im_1\varpi) \geq 1$ for all $\varpi \in \Im_4^{-1}(\Im_2\varkappa)$.

(b) partially α-weakly increasing with respect to $\Im_4$ if $\alpha(\Im_1\varkappa, \Im_2\varpi) \geq 1$ for all $\varpi \in \Im_4^{-1}(\Im_1\varkappa)$.

Definition 1.7. *Let (Ξ, b_e) be an EBbDS, $\alpha \colon \Xi \times \Xi \to \mathbb{R}_+$ and $\Im_1, \Im_2 \colon \Xi \to \Xi$ be three mappings. The pair $(\Im_1, \Im_2)$ is said to be an α-compatible if*

$$\lim_{n\to\infty} b_e(\Im_1\Im_2\varkappa_n, \Im_2\Im_1\varkappa_n) = 0$$

whenever $\{\varkappa_n\}$ *is a sequence in* Ξ *such that* $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \Im_1 \varkappa_n = \lim_{n\to\infty} \Im_2 \varkappa_n = \upsilon$$

for some $\upsilon \in \Xi$.

Definition 1.8. *Let* (Ξ, b_e) *be an EBbDS,* $\alpha\colon \Xi \times \Xi \to \mathbb{R}_+$ *and* $\Im_1\colon \Xi \to \Xi$ *be two mappings. We say that* $\Im_1$ *is an* α*-continuous at a point* $\varkappa \in \Xi$ *if, for each sequence* $\{\varkappa_n\}$ *in* Ξ *with* $\varkappa_n \to \varkappa$ *as* $n \to \infty$ *and* $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$*, we have*

$$\lim_{n\to\infty} \Im_1 \varkappa_n = \Im_1 \varkappa.$$

Definition 1.9. *Let* (Ξ, b_e) *be an EBbDS, and* $\alpha\colon \Xi\times\Xi \to \mathbb{R}_+$*. A mapping* $\Im_1\colon \Xi \to \Xi$ *is said to be an* α*-dominating on* Ξ *if* $\alpha(\varkappa, \Im_1\varkappa) \geq 1$ *for each* $\varkappa$ *in* Ξ.

In [18], Jleli and Samet introduce a new type of control functions and generalized the Banach contraction theorem. Nashine and Kadelburg [22] used this concept to generalized the earlier work in b-metric spaces for two pairs of mappings. In this paper, we extend the work of Nashine and Kadelburg to an α-complete extended Branciari b-distance spaces.

2. Main results

In this section, the concept of an $\alpha - \lambda-$ rational contraction in an EBbDS is introduced.

We begin with the following concepts.

The set of all functions $\theta : (0, \infty) \to [1, \infty)$ satisfying the conditions listed below is denoted by Θ after [18].

- (θ_1) θ is strictly increasing;
- (θ_2) θ is continuous;
- (θ_3) for each sequence $\{\tau_n\} \subseteq (0,\ \infty)$, $\lim_{n\to\infty} \theta(\tau_n) = 1$ if and only if $\lim_{n\to\infty} \tau_n = 0$.

Now we are in a position to define new contractive concept.

Definition 2.1. *Let* (Ξ, b_e) *be an EBbDS and an* $\alpha\colon \Xi^2 \to \mathbb{R}_+$ *and* $\lambda_i : \Xi \to [0, 1)$ $(i = 1, 2, 3, 4, 5)$ *with* $\lambda = \sum_{i=1}^{5} \lambda_i < 1$*. The mappings* $\Im_1, \Im_2, \Im_3, \Im_4\colon \Xi \to \Xi$ *are said to be rational-*$\alpha - \lambda - JS$*-contractive, if there exist* $\gamma \in [0, 1)$*,* $\theta \in \Theta$ *such that for* $\varkappa, \varpi \in \Xi$

$$\begin{aligned} &\alpha(\Im_3\varkappa, \Im_4\varpi) \geq 1 \text{ with } b_e(\Im_3\varkappa, \Im_4\varpi) > 0, b_e(\Im_1\varkappa, \Im_2\varpi) > 0 \text{ implies} \\ &\theta(w(\varkappa, \varpi) b_e(\Im_1\varkappa, \Im_2\varpi)) \leq [\theta(\Delta_b(\varkappa, \varpi))]^\gamma \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Delta_b(\varkappa, \varpi) \;=\; & \lambda_1(\varkappa)\ b_e(\Im_3\varkappa, \Im_4\varpi) + \lambda_2(\varkappa)\ b_e(\Im_3\varkappa, \Im_1\varkappa) + \lambda_3(\varkappa)\ b_e(\Im_4\varpi, \Im_2\varpi) \\ & +\lambda_4(\varkappa)\ \frac{b_e(\Im_4\varpi, \Im_2\varpi)[1 + b_e(\Im_3\varkappa, \Im_1\varkappa)]}{w(\varkappa, \varpi)[1 + b_e(\Im_3\varkappa, \Im_4\varpi)]} \\ & +\lambda_5(\varkappa)\ \frac{b_e(\Im_3\varkappa, \Im_1\varkappa).b_e(\Im_4\varpi, \Im_2\varpi)}{w(\varkappa, \varpi).b_e(\Im_3\varkappa, \Im_4\varpi)}. \end{aligned}$$

We denote by $\mathfrak{J}(\Xi, \alpha, \lambda)$ *the collection of all rational-*$\alpha-\lambda-JS$*-contractive mappings on* (Ξ, b_e).

Let Ξ be a nonempty set and $\Im_1, \Im_2 : \Xi \to \Xi$. Then denote

$$Fix(\Im_1) := \{\varkappa \in \Xi : \Im_1 \varkappa = \varkappa\},$$
$$C(\Im_1, \Im_2) := \{\varkappa \in \Xi : \Im_1 \varkappa = \Im_2 \varkappa\}$$
$$CF(\Im_1, \Im_2) := \{\varkappa \in \Xi : \varkappa = \Im_1 \varkappa = \Im_2 \varkappa\}.$$

We can now state and demonstrate the outcome.

Theorem 2.2. *Let (Ξ, b_e) be an α-complete EBbDS and an $\alpha\colon \Xi \times \Xi \to [0, \infty)$. Let $\Im_1, \Im_3, \Im_2, \Im_4 \colon \Xi \to \Xi$ be given mappings satisfying*

- $(H1)$ $(\Im_1, \Im_3, \Im_2, \Im_4) \in \mathfrak{J}(\Xi, \alpha, \lambda)$;
- $(H2)$ $\Im_1(\Xi) \subseteq \Im_4(\Xi)$ *and* $\Im_2(\Xi) \subseteq \Im_3(\Xi)$;
- $(H3)$ *the pairs* $(\Im_1, \Im_2)$ *and* $(\Im_2, \Im_1)$ *are partially α-weakly increasing with respect to* $\Im_4$ *and* $\Im_3$, *respectively;*
- $(H4)$ α *is a transitive mapping, that is, for* $\varkappa, \varpi, \vartheta \in \Xi$,
$$\alpha(\varkappa, \varpi) \geq 1 \text{ and } \alpha(\varpi, \vartheta) \geq 1 \Rightarrow \alpha(\varkappa, \vartheta) \geq 1;$$
- $(H5)$ $\Im_1, \Im_2, \Im_3$ *and* $\Im_4$ *are α-continuous;*
- $(H6)$ *the pairs* $(\Im_1, \Im_3)$ *and* $(\Im_2, \Im_4)$ *are α-compatible.*

Then there exists $\zeta^* \in \Xi$ such that $\zeta^* \in C(\Im_1, \Im_3) \cap C(\Im_2, \Im_4)$. Moreover, if $\alpha(\Im_3 \zeta^*, \Im_4 \zeta^*) \geq 1$ or $\alpha(\Im_4 \zeta^*, \Im_3 \zeta^*) \geq 1$, then $\zeta^* \in C(\Im_1, \Im_2, \Im_3, \Im_4)$.

Proof. By using the condition (H2) and any starting point in Ξ, we can consider the sequences $\{\varkappa_n\}$ and $\{\varpi_n\}$ in Ξ that are defined by

$$\varpi_{2n+1} = \Im_4 \varkappa_{2n+1} = \Im_1 \varkappa_{2n}, \quad \varpi_{2n+2} = \Im_3 \varkappa_{2n+2} = \Im_2 \varkappa_{2n+1}$$

for $n \in \mathbb{N}^*$. Since $\varkappa_1 \in \Im_4^{-1}(\Im_1 \varkappa_0)$, $\varkappa_2 \in \Im_3^{-1}(\Im_2 \varkappa_1)$ and the pairs $(\Im_1, \Im_2)$ and $(\Im_2, \Im_1)$ satisfy (H3), we have

$$\alpha(\varpi_1, \varpi_2) = \alpha(\Im_1 \varkappa_0, \Im_2 \varkappa_1) \geq 1, \quad \alpha(\varpi_2, \varpi_3) = \alpha(\Im_2 \varkappa_1, \Im_1 \varkappa_2) \geq 1.$$

We obtain by repeating this process

$$\alpha(\varpi_n, \varpi_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}^*. \tag{2.2}$$

Step 1: First, we demonstrate that

$$\lim_{n \to \infty} b_e(\varpi_n, \varpi_{n+1}) = 0. \tag{2.3}$$

We define $\varrho_k = b_e(\varpi_k, \varpi_{k+1})$ for all $k \in \mathbb{N}^*$. If we suppose that $\varrho_{k_0} = 0$ for some $k_0 \in \mathbb{N}^*$, then $\varpi_{k_0} = \varpi_{k_0+1}$, and the proof is complete. Assume that $\varpi_n \neq \varpi_{n+1}$ for all $n \geq 0$. Then, for all $n \in \mathbb{N}^*$, $\varrho_n > 0$.

Assume n is an odd number. Since $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$, we can deduce from $(\Im_1, \Im_2, \Im_3, \Im_4) \in \mathfrak{J}(\Xi, \alpha, \lambda)$ that the condition (2.1) implies that

$$
\begin{aligned}
&\theta(w(\varkappa_{n-1},\varkappa_n)\varrho_n) = \theta(w(\varkappa_{n-1},\varkappa_n)b_e(\Im_1\varkappa_{n-1},\Im_2\varkappa_n)) \\
&\leq \left[\theta\begin{pmatrix} \lambda_1(\varkappa_{n-1})b_e(\Im_3\varkappa_{n-1},\Im_4\varkappa_n)+\lambda_2(\varkappa_{n-1})b_e(\Im_3\varkappa_{n-1},\Im_1\varkappa_{n-1}) \\ +\lambda_3(\varkappa_{n-1})b_e(\Im_4\varkappa_n,\Im_2\varkappa_n) \\ +\lambda_4(\varkappa_{n-1})\frac{b_e(\Im_4\varkappa_n,\Im_2\varkappa_n)[1+b_e(\Im_3\varkappa_{n-1},\Im_1\varkappa_{n-1})]}{w(\varkappa_{n-1},\varkappa_n)[1+b_e(\Im_3\varkappa_{n-1},\Im_4\varkappa_n)]} \\ +\lambda_5(\varkappa_{n-1})\frac{b_e(\Im_3\varkappa_{n-1},\Im_1\varkappa_{n-1}).b_e(\Im_4\varkappa_n,\Im_2\varkappa_n)}{w(\varkappa_{n-1},\varkappa_n).b_e(\Im_3\varkappa_{n-1},\Im_4\varkappa_n)} \end{pmatrix}\right]^{\gamma} \\
&= \left[\theta\begin{pmatrix} \lambda_1(\varpi_{n-1})b_e(\varpi_{n-1},\varpi_n)+\lambda_2(\varpi_{n-1})b_e(\varpi_{n-1},\varpi_n) \\ +\lambda_3(\varpi_{n-1})b_e(\varpi_n,\varpi_{n+1})+\lambda_4(\varpi_{n-1})\frac{b_e(\varpi_n,\varpi_{n+1})[1+b_e(\varpi_{n-1},\varpi_n)]}{w(\varpi_{n-1},\varpi_n)[1+b_e(\varpi_{n-1},\varpi_n)]} \\ +\lambda_5(\varpi_{n-1})\frac{b_e(\varpi_{n-1},\varpi_n).b_e(\varpi_n,\varpi_{n+1})}{w(\varpi_{n-1},\varpi_n).b_e(\varpi_{n-1},\varpi_n)} \end{pmatrix}\right]^{\gamma} \\
&\leq \left[\theta\begin{pmatrix} \lambda_1(\varpi_{n-1})b_e(\varpi_{n-1},\varpi_n)+\lambda_2(\varpi_{n-1})b_e(\varpi_{n-1},\varpi_n) \\ +\lambda_3(\varpi_{n-1})b_e(\varpi_n,\varpi_{n+1})+\lambda_4(\varpi_{n-1})b_e(\varpi_n,\varpi_{n+1}) \\ +\lambda_5(\varpi_{n-1})b_e(\varpi_n,\varpi_{n+1}) \end{pmatrix}\right]^{\gamma} \\
&\leq \left[\theta\begin{pmatrix} \lambda_1(\varpi_{n-1})\varrho_{n-1}+\lambda_2(\varpi_{n-1})\varrho_{n-1} \\ +\lambda_3(\varpi_{n-1})\varrho_n+\lambda_4(\varpi_{n-1})\varrho_n \\ +\lambda_5(\varpi_{n-1})\varrho_n \end{pmatrix}\right]^{\gamma}. \qquad (2.4)
\end{aligned}
$$

Since θ is strictly rising and $\gamma < 1$, we conclude that

$$
\begin{aligned}
w(\varpi_{n-1},\varpi_n)\varrho_n \leq{}& \lambda_1(\varpi_{n-1})\varrho_{n-1}+\lambda_2(\varpi_{n-1})\varrho_{n-1} \\
&+\lambda_3(\varpi_{n-1})\varrho_n+\lambda_4(\varpi_{n-1})\varrho_n+\lambda_5(\varpi_{n-1})\varrho_n. \qquad (2.5)
\end{aligned}
$$

If $\varrho_{n-1} \leq \varrho_n$ for some $n \in \mathbb{N}$, then from (2.5), we have $w(\varpi_n,\varpi_{n+1})\varrho_n \leq \lambda(\varpi_{n-1})\varrho_n$, which is a contradiction since $w \geq 1$ and $\lambda < 1$. Thus $\varrho_n \leq \varrho_{n-1}$ for all $n \in \mathbb{N}$ and the sequence $\{\varrho_n\}$ is a decreasing sequence of real numbers. As a result, there exists ζ such that

$$\lim_{n\to\infty}\varrho_n = \zeta.$$

From (2.4), we have

$$\theta(w(\varpi_{n-1},\varpi_n)\varrho_n) \leq (\theta(\varrho_{n-1}))^{\gamma} \leq (\theta(\varrho_{n-2}))^{\gamma^2}.$$

Thus,

$$1 \leq (\theta(\varrho_{n-1}))^{\gamma} \leq (\theta(\varrho_{n-2}))^{\gamma^2} \leq .. \leq (\theta(\varrho_0))^{\gamma^n}.$$

Using the limit in the preceding relation, we get

$$\lim_{n\to\infty}\theta(\varrho_{n-1}) = 1$$

and by the property (θ_3) of θ,

$$\lim_{n\to\infty}\varrho_{n-1} = 0.$$

Step 2: In this step, we will demonstrate that $\{\varpi_n\}$ is a Cauchy sequence, that is, for $m > n$, we prove

$$\lim_{n,m\to\infty} b_e(\varpi_n,\varpi_m) = 0.$$

Using (ebb3), we have

$$
\begin{aligned}
& b_e(\varpi_n,\ \varpi_m) \\
\leq\ & w(\varpi_n, \varpi_m)[b_e(\varpi_n, \varpi_{n+1}) + b_e(\varpi_{n+1}, \varpi_{n+2}) + b_e(\varpi_{n+2}, \varpi_m)] \\
\leq\ & w(\varpi_n, \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \\
& w(\varpi_n,\ \varpi_m) b_e(\varpi_{n+2},\ \varpi_m) \\
\leq\ & w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+2},\ \varpi_m)[b_e(\varpi_{n+2},\ \varpi_{n+3}) + b_e(\varpi_{n+3},\ \varpi_{n+4}) \\
& + b_e(\varpi_{n+4},\ \varpi_m)] \\
\leq\ & w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) \\
& w(\varpi_{n+2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) b_e(\varpi_{n+4},\ \varpi_m) \\
\vdots\ & \\
\leq\ & w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) \\
& w(\varpi_{n+2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \ldots + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) \ldots w(\varpi_{m-2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) \ldots w(\varpi_{m-2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) \\
\leq\ & w(\varpi_n,\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) w(\varpi_{n+1},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+1},\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + w(\varpi_n,\ \varpi_m) \\
& w(\varpi_{n+1},\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) w(\varpi_{n+3},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \ldots + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+1},\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) \ldots w(\varpi_{m-2},\ \varpi_m) b_e(\varpi_n, \varpi_{n+1}) + \\
& w(\varpi_n,\ \varpi_m) w(\varpi_{n+1},\ \varpi_m) w(\varpi_{n+2},\ \varpi_m) \ldots w(\varpi_{m-2},\ \varpi_m) w(\varpi_{m-1},\ \varpi_m) \\
& b_e(\varpi_n, \varpi_{n+1}).
\end{aligned}
$$

Applying $n, m \to \infty$ and using (2.3), we get

$$\lim_{n,m\to\infty} b_e(\varpi_n, \varpi_m) = 0.$$

Hence $\{\varpi_n\}$ is a Cauchy sequence.

Due to the validity of the inequality (2.2) and the α-completeness of EBbDS (Ξ, b_e), there exists a $\zeta^* \in \Xi$ such that

$$\lim_{n\to\infty} b_e(\varpi_n, \zeta^*) = 0,$$

and so

$$\lim_{n\to\infty} b_e(\varpi_{2n+1}, \zeta^*) = \lim_{n\to\infty} b_e(\Im_4 \varkappa_{2n+1}, \zeta^*) = \lim_{n\to\infty} b_e(\Im_1 \varkappa_{2n}, \zeta^*) = 0 \tag{2.6}$$

and

$$\lim_{n\to\infty} b_e(\varpi_{2n+2}, \zeta^*) = \lim_{n\to\infty} b_e(\Im_3 \varkappa_{2n+2}, \zeta^*) = \lim_{n\to\infty} b_e(\Im_2 \varkappa_{2n+1}, \zeta^*) = 0. \tag{2.7}$$

From (2.6) and (2.7), we have $\Im_1 \varkappa_{2n} \to \zeta^*$ and $\Im_3 \varkappa_{2n} \to \zeta^*$ as $n \to \infty$. Since $(\Im_1, \Im_3)$ is an α-compatible, by (2.2), we have

$$\lim_{n\to\infty} b_e(\Im_3 \Im_1 \varkappa_{2n}, \Im_1 \Im_3 \varkappa_{2n}) = 0. \tag{2.8}$$

By (2.2), the α-continuity of $\Im_3$, $\Im_1$, we obtain

$$\lim_{n\to\infty} b_e(\Im_3\Im_1\varkappa_{2n}, \Im_3\zeta^*) = 0 = \lim_{n\to\infty} b_e(\Im_1\Im_3\varkappa_{2n}, \Im_1\zeta^*). \tag{2.9}$$

By (B3) property, we have

$$\begin{aligned}&b_e(\Im_3\zeta^*, \Im_1\zeta^*)\\&\leq w(\Im_3\zeta^*, \Im_1\zeta^*)[b_e(\Im_3\zeta^*, \mathcal{S}\mathcal{J}\varkappa_{2n}) + b_e(\mathcal{S}\mathcal{J}\varkappa_{2n}, \mathcal{J}\mathcal{S}\varkappa_{2n}) + b_e(\mathcal{J}\mathcal{S}\varkappa_{2n}, \Im_1\zeta^*)]\end{aligned}$$

for all $n \in \mathbb{N}^*$. Passing to the limit as $n \to \infty$ in the above inequality and using (2.8)–(2.9), we obtain $b_e(\Im_3\zeta^*, \Im_1\zeta^*) \leq 0$.

As a result, it can be inferred that $b_e(\Im_3\zeta^*, \Im_1\zeta^*) = 0$ meaning that ζ^* is a coincidence point between $\Im_1$ and $\Im_3$. In a similar manner, we can demonstrate that ζ^* is a coincidence point for both $\Im_2$ and $\Im_4$.

This implies that $b_e(\Im_3\zeta^*, \Im_1\zeta^*) = 0$ and so $\zeta^* \in C(\Im_1, \Im_3)$, that is, ζ^* is a coincidence point of $\Im_1$ and $\Im_3$. Similarly, we can prove that ζ^* is also a coincidence point of $\Im_2$ and $\Im_4$.

Finally, we show that ζ^* is a coincidence point of $\Im_1, \Im_3, \Im_2$ and $\Im_4$ if and only if

$$\alpha(\Im_4\zeta^*, \Im_3\zeta^*) \geq 1 \text{ or } \alpha(\Im_3\zeta^*, \Im_4\zeta^*) \geq 1.$$

On the contrary, suppose that $\Im_1\zeta^* \neq \Im_2\zeta^*$. Then, from (2.1), we have

$$\begin{aligned}&\theta(w(\zeta^*, \zeta^*)b_e(\Im_1\zeta^*, \Im_2\zeta^*))\\&\leq \left[\theta\left(\begin{array}{l}\lambda_1(\zeta^*)\ b_e(\Im_3\zeta^*, \Im_4\zeta^*) + \lambda_2(\zeta^*)\ b_e(\Im_3\zeta^*, \Im_1\zeta^*)\\ +\lambda_3(\zeta^*)\ b_e(\Im_4\zeta^*, \Im_2\zeta^*) + \lambda_4(\zeta^*)\ \frac{b_e(\Im_4\zeta^*, \Im_2\zeta^*)[1+b_e(\Im_3\zeta^*, \Im_1\zeta^*)]}{w(\zeta^*, \zeta^*)[1+b_e(\Im_3\zeta^*, \Im_4\zeta^*)]}\\ +\lambda_5(\zeta^*)\ \frac{b_e(\Im_3\zeta^*, \Im_1\zeta^*).b_e(\Im_4\zeta^*, \Im_2\zeta^*)}{w(\zeta^*, \zeta^*).b_e(\Im_3\zeta^*, \Im_4\zeta^*)}\end{array}\right)\right]^\gamma\\&\leq [\theta(\lambda_1(\zeta^*)\ b_e(\Im_1\zeta^*, \Im_2\zeta^*))]^\gamma.\end{aligned} \tag{2.10}$$

Since θ is strictly rising and $\gamma < 1$, we conclude that

$$w(\zeta^*, \zeta^*)b_e(\Im_1\zeta^*, \Im_2\zeta^*) \leq \lambda_1(\zeta^*)\ b_e(\Im_1\zeta^*, \Im_2\zeta^*))$$

which is a contradiction, since $w \geq 1$ and $\lambda < 1$. Thus $\Im_1\zeta^* = \Im_2\zeta^*$, and hence $\zeta^* \in C(\Im_1, \Im_3, \Im_2, \Im_4)$. □

Under certain additional assumptions, the previous result may still be valid for $\Im_1, \Im_3, \Im_2, \Im_4$ that are not necessarily α-continuous. The following is the result.

Theorem 2.3. *Let (Ξ, b_e) be an α-complete EBbDS with coefficient $b \geq 1$, let $\alpha\colon \Xi \times \Xi \to \mathbb{R}_+$ and $\Im_1, \Im_3, \Im_2, \Im_4\colon \Xi \to \Xi$ be given mappings. Suppose that the assumptions $(H1)$–$(H4)$ of Theorem 2.2 hold, as well as:*

- $(\widehat{H5})$ *$\Im_4(\Xi)$ and $\Im_3(\Xi)$ are b-closed subsets of Ξ;*
- $(\widehat{H6})$ *the pairs $(\Im_1, \Im_3)$ and $(\Im_2, \Im_4)$ are weakly compatible;*
- $(H7)$ *Ξ is an α-regular, i.e., if $\{\varpi_n\}$ is a sequence in Ξ with $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for $n \in \mathbb{N}$ and $\varpi_n \to \varpi^*$ as $n \to \infty$, then $\alpha(\varpi_n, \varpi^*) \geq 1$ for $n \in \mathbb{N}$.*

Then there exists $\zeta^* \in \Xi$ such that $\zeta^* \in C(\Im_1, \Im_3) \cap C(\Im_2, \Im_4)$. Moreover, if $\alpha(\Im_3\zeta^*, \Im_4\zeta^*) \geq 1$ or $\alpha(\Im_4\zeta^*, \Im_3\zeta^*) \geq 1$, then $\zeta^* \in C(\Im_1, \Im_3, \Im_2, \Im_4)$.

Proof. In the line of proof of Theorem 2.2, we obtain a b-Cauchy sequence $\{\varpi_n\}$ in an α-complete EBbDS (Ξ, b_e). Hence, there exists $\zeta^* \in \Xi$ such that

$$\lim_{n\to\infty} b_e(\varpi_n, \zeta^*) = 0.$$

Combining hypothesis $(\widehat{H}5)$ for $\Im_4(\Xi)$ and $\{\varpi_{2n+1}\} \subseteq \Im_4(\Xi)$, we have $\zeta^* \in \Im_4(\Xi)$. Hence there exists $\xi \in \Xi$ such that $\zeta^* = \Im_4\xi$ and

$$\lim_{n\to\infty} b_e(\varpi_{2n+1}, \Im_4\xi) = \lim_{n\to\infty} b_e(\Im_4\varkappa_{2n+1}, \Im_4\xi) = 0.$$

Similarly, using hypothesis $(\widehat{H}5)$ for $\Im_3(\Xi)$ and $\{\varpi_{2n}\} \subseteq \Im_3(\Xi)$, we have $\zeta^* \in \Im_3(\Xi)$. Hence there exists $\zeta \in \Xi$ such that $\zeta^* = \Im_4\xi = \Im_3\zeta$ and

$$\lim_{n\to\infty} b_e(\varpi_{2n}, \Im_3\zeta) = \lim_{n\to\infty} b_e(\Im_3\varkappa_{2n}, \Im_4\zeta) = 0.$$

Additionally, we demonstrate that ζ^* is a coincidence point of $\Im_1$ and $\Im_3$. Since $\Im_4\varkappa_{2n+1} \to \zeta^* = \Im_3\zeta$ as $n \to \infty$, it follows from the hypothesis (H7), that is, α-regularity of Ξ that $\alpha(\Im_4\varkappa_{2n+1}, \Im_3\zeta) \geq 1$.

Contrarily, suppose that $\Im_1\zeta \neq \zeta^*$. Then, derived from (2.1), we have

$$\begin{aligned}
&\theta(w(\zeta,\varkappa_{2n+1})b_e(\Im_1\zeta,\Im_2\varkappa_{2n+1}))\\
&\leq \left[\theta\left(\begin{array}{l}
\lambda_1(\zeta)\ b_e(\Im_3\zeta,\Im_4\varkappa_{2n+1}) + \lambda_2(\zeta)\ b_e(\Im_3\zeta,\Im_1\zeta)\\
+\lambda_3(\zeta)\ b_e(\Im_4\varkappa_{2n+1},\Im_2\varkappa_{2n+1})\\
+\lambda_4(\zeta)\ \frac{b_e(\Im_4\varkappa_{2n+1},\Im_2\varkappa_{2n+1})[1+b_e(\Im_3\zeta,\Im_1\zeta)]}{w(\zeta,\varkappa_{2n+1})[1+b_e(\Im_3\zeta,\Im_4\varkappa_{2n+1})]}\\
+\lambda_5(\zeta)\ \frac{b_e(\Im_3\zeta,\Im_1\zeta).b_e(\Im_4\varkappa_{2n+1},\Im_2\varkappa_{2n+1})}{w(\zeta,\varkappa_{2n+1}).b_e(\Im_3\zeta,\Im_4\varkappa_{2n+1})}
\end{array}\right)\right]^{\gamma}.
\end{aligned} \tag{2.11}$$

Since $\lim_{n\to\infty} b_e(\Im_4\varkappa_{2n+1}, \Im_2\varkappa_{2n+1}) = 0$,

$$\theta(w(\zeta,\varkappa_{2n+1})b_e(\Im_1\zeta,\zeta^*)) \leq [\theta(b_e(\zeta^*,\Im_1\zeta))]^{\gamma}. \tag{2.12}$$

Since θ is strictly rising and $\gamma < 1$, we conclude that

$$w(\zeta,\varkappa_{2n+1})b_e(\Im_1\zeta,\zeta^*)) \leq \theta(b_e(\zeta^*,\Im_1\zeta))$$

a contradiction, except when $b_e(\Im_1\zeta,\zeta^*) = 0$. Hence $\zeta^* = \Im_1\zeta$ and so $\Im_3\zeta = \zeta^* = \Im_1\zeta$. Owing $(\widehat{H}6)$ for the pair $(\Im_1, \Im_3)$, we have

$$\Im_1\zeta^* = \Im_1\Im_3\zeta = \Im_3\Im_1\zeta = \Im_3\zeta^*.$$

Consequently, ζ^* is a point of coincidence for $\Im_1$ and $\Im_3$. Similarly, we can determine that ζ^* is a point of coincidence for the pair $(\Im_2, \Im_4)$. Using arguments, similar to those in the previous theorem, we can demonstrate that $\zeta^* \in C(\Im_1, \Im_3, \Im_2, \Im_4)$. □

2.1. **Results on common fixed point.**

Theorem 2.4. *According to the hypotheses of Theorem 2.2 (or Theorem 2.3), $\Im_1, \Im_3, \Im_2, \Im_4$ have a common fixed point in Ξ if the following condition is met:*

$(H8)$ *$\Im_3$ or $\Im_4$ is an α-dominating map.*

Proof. From Theorem 2.2 (or Theorem 2.3), there exists a $\zeta^* \in \Xi$ such that $\zeta^* \in C(\Im_1, \Im_3, \Im_2, \Im_4)$. Since the pair $(\Im_1, \Im_3)$ is weakly compatible, we have $\Im_1\Im_3\zeta^* = \Im_3\Im_1\zeta^*$. Let $\mu^* = \Im_1\zeta^* = \Im_3\zeta^*$. Therefore, we have $\Im_1\mu^* = \Im_3\mu^*$. Similarly, since the pair $(\Im_2, \Im_4)$ is weakly compatible, we have $\Im_2\Im_4\zeta^* = \Im_4\Im_2\zeta^*$. Let $\mu^* = \Im_2\zeta^* = \Im_4\zeta^*$. Therefore, we have $\Im_2\mu^* = \Im_4\mu^*$.
An α-dominating of mapping $\Im_3$ (or $\Im_4$),

$$\alpha(\mu^*, \Im_3\mu^*) = \alpha(\Im_4\zeta^*, \Im_3\mu^*) \geq 1.$$

If $\mu^* = \zeta^*$, then ζ^* is a common fixed point of $\Im_1, \Im_3, \Im_2$ and $\Im_4$. If $\mu^* \neq \zeta^*$, then, using $\alpha(\Im_4\zeta^*, \Im_3\mu^*) \geq 1$, from (2.1), we have

$$\theta(w(\zeta^*,\mu^*)b_e(\Im_1\zeta^*,\Im_2\mu^*)) \leq \left[\theta\begin{pmatrix} \lambda_1(\zeta^*)\ b_e(\Im_3\zeta^*,\Im_4\mu^*)+\lambda_2(\zeta^*)\ b_e(\Im_3\zeta^*,\Im_1\zeta^*) \\ +\lambda_3(\zeta^*)\ b_e(\Im_4\mu^*,\Im_2\mu^*) \\ +\lambda_4(\zeta^*)\ \frac{b_e(\Im_4\mu^*,\Im_2\mu^*)[1+b_e(\Im_3\zeta^*,\Im_1\zeta^*)]}{w(\zeta^*,\mu^*)[1+b_e(\Im_3\zeta^*,\Im_4\mu^*)]} \\ +\lambda_5(\zeta^*)\ \frac{b_e(\Im_3\zeta^*,\Im_1\zeta^*).b_e(\Im_4\mu^*,\Im_2\mu^*)}{w(\zeta^*,\mu^*).b_e(\Im_3\zeta^*,\Im_4\mu^*)}\end{pmatrix}\right]^{\gamma} \tag{2.13}$$

implies that

$$\theta(w(\zeta^*,\mu^*)b_e(\mu^*,\Im_2\mu^*)) \leq [\theta(\lambda_1(\zeta^*)\ b_e(\mu^*,\Im_2\mu^*))]^{\gamma}.$$

Since θ is strictly rising and $\gamma<1$, we conclude that

$$w(\zeta^*,\mu^*)b_e(\mu^*,\Im_2\mu^*)) \leq \lambda_1(\zeta^*)\ b_e(\mu^*,\Im_2\mu^*))$$

a contradiction. Hence $\mu^*=\Im_2\mu^*$ which implies that μ^* is a common fixed point of $\Im_1,\Im_3,\Im_2$ and $\Im_4$. □

2.2. **Uniqueness of common fixed point.** We will consider the following hypothesis to ensure the uniqueness of the common fixed point for the pair $(\Im_1,\Im_3)$ of mappings.

$$(H9): \qquad \text{for all } \varkappa,\varpi\in CF(\Im_3,\Im_4),\ \ \alpha(\varkappa,\varpi)\geq 1 \text{ or } \alpha(\varpi,\varkappa)\geq 1.$$

Theorem 2.5. *If the condition $(H9)$ is true for the pair $(\Im_4,\Im_3)$ and add to the hypotheses of Theorem 2.4, the uniqueness is attained.*

Proof. Let's pretend that ξ^* is another fixed point shared by $\Im_1,\Im_3,\Im_2$ and $\Im_4$ and that, in contrast to what will be proved, $b_e(\Im_1\zeta^*,\Im_2\xi^*)=b_e(\zeta^*,\xi^*)>0$. In order to prove that $\zeta^*\neq\xi^*\in CF(\Im_3,\Im_4)$, we use $(H9)$.

$$\alpha(\Im_4\zeta^*,\Im_3\xi^*)=\alpha(\zeta^*,\xi^*)\geq 1. \tag{2.14}$$

Now we can replace $\varkappa$ by ζ^* and ϖ by ξ^* in the condition (2.1), and we get easily with (2.14)

$$\begin{aligned}&\theta(w(\zeta^*,\xi^*)b_e(\Im_1\zeta^*,\Im_2\xi^*)) \\ &\leq \left[\theta\begin{pmatrix} \lambda_1(\zeta^*)b_e(\Im_3\zeta^*,\Im_4\xi^*)+\lambda_2(\zeta^*)b_e(\Im_3\zeta^*,\Im_1\zeta^*) \\ +\lambda_3(\zeta^*)b_e(\Im_4\xi^*,\Im_2\xi^*)+\lambda_4(\zeta^*)\frac{b_e(\Im_4\xi^*,\Im_2\xi^*)[1+b_e(\Im_3\zeta^*,\Im_2\xi^*)}{w(\zeta^*,\xi^*)[1+b_e(\Im_3\zeta^*,\Im_4\xi^*)]} \\ +\lambda_5(\zeta^*)\frac{b_e(\Im_3\zeta^*,\Im_1\zeta^*).b_e(\Im_4\xi^*,\Im_2\xi^*)}{w(\zeta^*,\xi^*)).b_e(\Im_3\zeta^*,\Im_4\xi^*)}\end{pmatrix}\right]^{\gamma} \\ &\leq [\theta(\lambda_1(\zeta^*)b_e(\zeta^*,\xi^*))]^{\gamma}.\end{aligned}$$

As θ is strictly increasing and $\gamma<1$, we get

$$w(\zeta^*,\xi^*)b_e(\zeta^*,\xi^*) \leq \lambda_1(\zeta^*)b_e(\zeta^*,\xi^*),$$

a contradiction and hence $\zeta^*=\xi^*$. □

3. Illustration

Example 3.1. *Consider $\Xi = [0,1]$ and define $b_e : \Xi^2 \to \mathbb{R}_+$ by $b_e(\varkappa, \varpi) = |\varkappa - \varpi|^2$ and let $\alpha\colon \Xi \times \Xi \to \mathbb{R}_+$ be given as*

$$\alpha(\varkappa, \varpi) = \begin{cases} \varkappa + \varpi, & \text{if } \varkappa \geq \varpi \\ 0, & \text{otherwise.} \end{cases}$$

Then (Ξ, b_e) is an α-complete EBbDS with $w(\varkappa, \varpi) = \varkappa + \varpi + 2.5$ but neither a BDS (Ξ, b) nor a metric space (Ξ, d). For instances

$$b_e(0,1) = 1 \nleq 0.5 = b_e(0, 0.5) + b_e(0.5, 1)$$

and

$$b_e(0,1) = 1 \nleq 0.4902 = b_e(0, 0.5) + b_e(0.5, 0.99) + b_e(0.99, 1)$$

but

$$\begin{aligned} b_e(\varkappa, \varpi) &= |\varkappa - \varpi|^2 \\ &= |\varkappa - \mu + \mu - \upsilon + \upsilon - \varpi|^2 \\ &\leq |\varkappa - \mu|^2 + |\mu - \upsilon|^2 + |\upsilon - \varpi|^2 + 2|\varkappa - \mu||\mu - \upsilon| \\ &\quad + 2|\mu - \upsilon||\upsilon - \varpi| + 2|\upsilon - \varpi||\varkappa - \mu| \\ &\leq \left(\varkappa + \varpi + \frac{5}{2}\right)[|\varkappa - \mu|^2 + |\mu - \upsilon|^2 + |\upsilon - \varpi|^2] \\ &= w(\varkappa, \varpi)[b_e(\varkappa, \mu) + b_e(\mu,\ \upsilon) + b_e(\upsilon, \varpi)] \end{aligned}$$

for all $\varkappa, \varpi, \mu, \upsilon \in \Xi$.
Consider the mappings $\Im_1, \Im_3, \Im_2, \Im_4 \colon \Xi \to \Xi$ defined by:

$$\Im_1\varkappa = \begin{cases} 0, & \text{if } 0 \leq \varkappa \leq 1/4 \\ 1/16, & \text{if } 1/4 < x \leq 1; \end{cases} \qquad \Im_2\varkappa = 0 \text{ for } 0 \leq \varkappa \leq 1;$$

$$\Im_4\varkappa = \begin{cases} \varkappa, & \text{if } 0 \leq \varkappa \leq 1/4 \\ 1, & \text{if } 1/4 < \varkappa \leq 1; \end{cases} \qquad \Im_3\varkappa = \begin{cases} 0, & \text{if } x = 0 \\ 1/4, & \text{if } 0 < \varkappa \leq 1/4 \\ 1, & \text{if } 1/4 < \varkappa \leq 1. \end{cases}$$

All but one of the necessary conditions for Theorem 2.5 to hold are immediately apparent is (H1).

Take $\theta \in \Theta$ defined by $\theta(\tau) = exp(\tau exp(\tau))$ $(\tau > 0)$ and $\gamma = \frac{9}{10}$, $\lambda_i : \Xi \to [0,1)$ by $\lambda_i(\varkappa) = \frac{8+\varkappa}{50}$ for all $\varkappa \in \Xi$, $i \in \{1,2,3,4,5\}$ so that $\lambda = \sum_{i=1}^{5} \lambda_i = [\frac{8}{50}, \frac{45}{50}] < 1$. We will examine the contractual condition specified by (2.1)*, that is, $b_e(\Im_1\varkappa, \Im_1\varpi) > 0$, $b_e(\Im_3\varkappa, \Im_4\varpi) > 0$. Consider the cases below:*

(1) $1/4 < \varkappa \leq 1$, $\varpi = 0$. *Then $\alpha(\Im_3\varkappa, \Im_4\varpi) = 1$, $b_e(\Im_1\varkappa, \Im_2\varpi) = (1/16)^2 \neq 0$, $b_e(\Im_3\varkappa, \Im_4\varpi) = 1 \neq 0$, $\Delta_b(\varkappa, \varpi) \geq 2.9532$, and then* (2.1) *holds true as*

$$\theta\left(\frac{8+\varkappa}{50} b_e(\Im_1\varkappa, \Im_2\varpi)\right) = 1.01395462 \leq 2.9279 = (\theta(\Delta_b(\varkappa, \varpi)))^{\frac{9}{10}}$$

(2) $1/4 < \varkappa \leq 1$, $0 < \varpi \leq 1/4$. *Then $\alpha(\Im_3\varkappa, \Im_4\varpi) = (1, 5/4] \geq 1$, $b_e(\Im_1\varkappa, \Im_2\varpi) = (1/16)^2 \neq 0$, $b_e(\Im_3\varkappa, \Im_4\varpi) = 3/4 \neq 0$, $\Delta_b(\varkappa, \varpi) \geq 1.289$. In this case* (2.1) *reduces* *to*

$$\theta\left(\frac{8+\varkappa}{50}b_e(\Im_1\varkappa,\Im_2\varpi)\right)=1.01395462\leq 1.3984=(\theta(\Delta_b(\varkappa,\varpi)))^{\frac{9}{10}}$$

and holds true for the chosen value of γ*. Other cases are not true as* $\alpha(\Im_3\varkappa,\Im_4\varpi)\ngeq 1$ *with* $b_e(\Im_3\varkappa,\Im_4\varpi)=0$*.*

4. System of Urysohn integral equations

In this section, we discuss the existence and uniqueness of common solution of following system of Urysohn integral equations:

$$\begin{cases}\varkappa(\tau)=\wp_1(\tau)+\int_0^T\Phi_1(\tau,s,\varkappa(s))\,ds, & \tau\in[0,T],\\ \varkappa(\tau)=\wp_2(\tau)+\int_0^T\Phi_2(\tau,s,\varkappa(s))\,ds, & \tau\in[0,T],\\ \varkappa(\tau)=\wp_3(\tau)+\int_0^T\Phi_3(\tau,s,\varkappa(s))\,ds, & \tau\in[0,T],\\ \varkappa(\tau)=\wp_4(\tau)+\int_0^T\Phi_4(\tau,s,\varkappa(s))\,ds, & \tau\in[0,T],\end{cases}\tag{4.1}$$

where $T>0$, $\tau\in[0,T]$, and $\wp_i\colon[0,T]\to\mathbb{R}$ and $\Phi_i\colon[0,T]^2\times\mathbb{R}\to\mathbb{R}$ ($i\in\{1,2,3,4\}$) are given mappings.

Let $I=[0,T]$ and $\Xi:=C(I,\mathbb{R})$ be equipped with the usual maximum norm, i.e., $\|\varkappa\|_\Xi=\max_{\tau\in I}|\varkappa(\tau)|$, for $\varkappa\in C(I,\mathbb{R})$. Then $(\Xi,\|\cdot\|_\Xi)$ is a complete metric space. The distance in Ξ is given by

$$d_\infty(\varkappa,\varpi)=\max_{\tau\in I}|\varkappa(\tau)-\varpi(\tau)|\quad\text{for all }\varkappa,\varpi\in\Xi.$$

Moreover, we can define a EBbDS b_e on Ξ by $b_e(\varkappa,\varpi)=[d_\infty(\varkappa,\varpi)]^p$ for some $p>1$ and all $\varkappa,\varpi\in\Xi$. Since (Ξ,d_∞) is complete, we deduce that (Ξ,b_e) is a complete EBbDS with $w(\varkappa,\varpi)=\varkappa+\varpi+2^{p-1}$. Throughout this section, for each $i\in\{1,2,3,4\}$ and Φ_i in (4.1), we will denote by $\Omega_i\colon\Xi\to\Xi$ the operator defined by:

$$\Omega_i\varkappa(\tau):=\int_0^T\Phi_i(\tau,s,\varkappa(s))\,ds,\quad\tau\in\Xi,\quad\tau\in I.$$

We will also use the following partial order on Ξ:

$$\varkappa\preceq\varpi\iff\varkappa(\tau)\leq\varpi(\tau)\text{ for all }\tau\in[0,T].$$

Theorem 4.1. *Assume the following hypotheses are correct:*

(U1): *There exist* $\lambda_i:\Xi\to[0,1)$ *(*$i=\{1,2,3,4,5\}$*),* $\gamma\in[0,1)$ *and* $p>1$ *such that for all* $\varkappa,\varpi\in\Xi$*,*

$$\begin{cases}2\varkappa-\Omega_3\varkappa-\wp_3\preceq 2\varpi-\Omega_4\varpi-\wp_4\\ \quad or\\ 2\varkappa-\Omega_4\varkappa-\wp_4\preceq 2\varpi-\Omega_3\varpi-\wp_3\end{cases}\tag{4.2}$$

implies that

$$\begin{aligned}(\varkappa+\varpi+2^{p-1})\max_{\tau\in I}\mathcal{W}_1(\varkappa,\varpi)(\tau)\,\exp\{(\varkappa+\varpi+2^{p-1})\max_{\tau\in I}\mathcal{W}_1(\varkappa,\varpi)(\tau)\}&\\ \leq\gamma^2\max_{\tau\in I}\Delta_b(\varkappa,\varpi)(\tau)\,\exp\{\max_{\tau\in I}\Delta_b(\varkappa,\varpi)(\tau)\},&\end{aligned}\tag{4.3}$$

where

$$\Delta_b(\varkappa,\varpi)(\tau) = \lambda_1(\varkappa)\mathcal{W}_2(\varkappa,\varpi)(\tau) + \lambda_2(\varkappa)\mathcal{W}_3(\varkappa,\varpi)(\tau) + \lambda_3(\varkappa)\mathcal{W}_4(\varkappa,\varpi)(\tau) + \lambda_4(\varkappa)\frac{\mathcal{W}_4(\varkappa,\varpi)(\tau)[1+\mathcal{W}_5(\varkappa,\varpi)(\tau)]}{w(\varkappa,\varpi)[1+\mathcal{W}_2(\varkappa,\varpi)(\tau)]} + \lambda_5(\varkappa)\frac{\mathcal{W}_5(\varkappa,\varpi)(\tau).\mathcal{W}_4(\varkappa,\varpi)(\tau)}{w(\varkappa,\varpi)\mathcal{W}_2(\varkappa,\varpi)(\tau)}$$

and

$$\begin{aligned}
\mathcal{W}_1(\varkappa,\varpi)(\tau) &= |\Omega_1\varkappa(\tau) + \wp_1(\tau) - \Omega_2\varpi(\tau) - \wp_2(\tau)|^p \\
\mathcal{W}_2(\varkappa,\varpi)(\tau) &= |2\varkappa(\tau) - \Omega_3\varkappa(\tau) - \wp_3(\tau) - 2\varpi(\tau) + \Omega_4\varpi(\tau) + \wp_4(\tau)|^p \\
\mathcal{W}_3(\varkappa,\varpi)(\tau) &= |\Omega_1\varkappa(\tau) + \wp_1(\tau) - 2\varkappa(\tau) + \Omega_3\varkappa(\tau) + \wp_3(\tau)|^p \\
\mathcal{W}_4(\varkappa,\varpi)(\tau) &= |\Omega_2\varpi(\tau) + \wp_2(\tau) - 2\varpi(\tau) + \Omega_4\varpi(\tau) + \wp_4(\tau)|^p \\
\mathcal{W}_5(\varkappa,\varpi)(\tau) &= |2\varkappa(\tau) - \Omega_3\varkappa(\tau) - \wp_3(\tau) - \Omega_1\varkappa(\tau) - \wp_1(\tau)|^p.
\end{aligned}$$

(U2): *For each $\varkappa \in \Xi$, there is some $\varpi \in \Xi$ such that*

$$\Omega_1\varkappa + \wp_1 = 2\varpi - \Omega_4\varpi - \wp_4$$

and for each $\varkappa \in \Xi$ there is some $\varpi \in \Xi$ such that

$$\Omega_2\varkappa + \wp_2 = 2\varpi - \Omega_3\varpi - \wp_3.$$

(U3): *For all $\varkappa, \varpi \in \Xi$,*

$$2\varpi - \Omega_4\varpi - \wp_4 = \Omega_1\varkappa + \wp_1 \implies \Omega_1\varkappa + \wp_1 \preceq \Omega_2\varpi + \wp_2,$$

and for all $\varkappa, \varpi \in \Xi$,

$$2\varpi - \Omega_3\varpi - \wp_3 = \Omega_2\varkappa + \wp_2 \implies \Omega_2\varkappa + \wp_2 \preceq \Omega_1\varpi + \wp_1.$$

(U4): *The mappings $\wp_i\colon I \to \mathbb{R}$ and $\Phi_i\colon [0,T]^2 \times \mathbb{R} \to \mathbb{R}$ ($i \in \{1,2,3,4\}$) are continuous.*

(U5$_1$): *If $\{\varkappa_n\}$ is a sequence in Ξ such that $\varkappa_n \preceq \varkappa_{n+1}$ for all $n \in \mathbb{N}$ and $\varpi \in \Xi$ is such that*

$$\max_{t\in I} |\Omega_1\varkappa_n(\tau) + \wp_1(\tau) - \varpi(\tau)|^p \to 0 \text{ as } n \to \infty,$$
$$\max_{\tau\in I} |2\varkappa_n(\tau) - \Omega_3\varkappa_n(\tau) - \wp_3(\tau) - \varpi(\tau)|^p \to 0 \text{ as } n \to \infty,$$

then

$$\max_{\tau\in I} |[\wp_1(\tau) + \Omega_1(2\varkappa_n(\tau) - \Omega_3\varkappa_n(\tau) - \wp_3(\tau))] - [2(\Omega_1\varkappa_n(\tau) + \wp_1(\tau)) - \Omega_3(\Omega_1\varkappa_n(\tau) + \wp_1(\tau)) - \wp_3(\tau)]|^p \to 0 \text{ as } n \to \infty.$$

(U5$_2$): *If $\{\varkappa_n\}$ is a sequence in Ξ such that $\varkappa_n \preceq \varkappa_{n+1}$ for all $n \in \mathbb{N}$ and $\varpi \in \Xi$ is such that*

$$\max_{\tau\in I} |\Omega_2\varkappa_n(\tau) + \wp_2(\tau) - \varpi(\tau)|^p \to 0 \text{ as } n \to \infty,$$
$$\max_{\tau\in I} |2\varkappa_n(\tau) - \Omega_4\varkappa_n(\tau) - \wp_4(\tau) - \varpi(\tau)|^p \to 0 \text{ as } n \to \infty,$$

then

$$\max_{\tau\in I} |[\wp_2(\tau) + \Omega_2(2\varkappa_n(\tau) - \Omega_4\varkappa_n(\tau) - \wp_4(\tau))] - [2(\Omega_2\varkappa_n(\tau) + \wp_2(\tau)) - \Omega_4(\Omega_2\varkappa_n(\tau) + \wp_2(\tau)) - \wp_4(\tau)]|^p \to 0 \text{ as } n \to \infty.$$

(U6): *$\varkappa \preceq 2\varkappa - \Omega_3\varkappa - \wp_3$ for all $\varkappa \in \Xi$ or $\varkappa \preceq 2\varkappa - \Omega_4\varkappa - \wp_4$ for all $\varkappa \in \Xi$.*

Then the system (4.1) *has a solution. Moreover, if*

(U7): *for any two solutions $\varkappa^*, \varpi^*$ of the system* (4.1)*,* $\varkappa^* \preceq \varpi^*$ *or* $\varpi^* \preceq \varkappa^*$ *holds,*

then the solution of (4.1) *is unique.*

Proof. Consider four mappings $\Im_1, \Im_2, \Im_3, \Im_4 \colon \Xi \to \Xi$ by

$$\begin{aligned}
\Im_1\varkappa(\tau) &= \Omega_1\varkappa(\tau) + \wp_1(\tau) = \int_0^T \Phi_1(\tau, s, \varkappa(s))\, ds + \wp_1(\tau)\\
\Im_2\varkappa(\tau) &= \Omega_2\varkappa(\tau) + \wp_2(\tau) = \int_0^T \Phi_2(\tau, s, \varkappa(s))\, ds + \wp_2(\tau)\\
\Im_3\varkappa(\tau) &= 2\varkappa(\tau) - \Omega_3\varkappa(\tau) - \wp_3(\tau) = 2\varkappa(\tau) - \int_0^T \Phi_3(\tau, s, \varkappa(s))\, ds - \wp_3(\tau)\\
\Im_4\varkappa(\tau) &= 2\varkappa(\tau) - \Omega_4\varkappa(\tau) - \wp_4(\tau) = 2\varkappa(\tau) - \int_0^T \Phi_4(\tau, s, \varkappa(s))\, ds - \wp_4(\tau).
\end{aligned} \tag{4.4}$$

Define also a function $\alpha \colon \Xi^2 \to \mathbb{R}_+$ by

$$\alpha(\varkappa, \varpi) = \begin{cases} 1, & \text{if } \varkappa(\tau) \leq \varpi(\tau) \text{ for all } \tau \in I\\ 0, & \text{otherwise.} \end{cases}$$

We will check the validity of conditions (H1)–(H6) of Theorem 2.2 and (H8) of Theorem 2.4, as well as (under assumption (U7)), (H9) of Theorem 2.5.

(H1). By the definition (4.4) of mappings $\Im_1, \Im_2, \Im_3, \Im_4$ and the definition of an EBbDS b_e, we have that, for all $\varkappa, \varpi \in \Xi$,

$$\begin{cases}
b_e(\Im_1\varkappa, \Im_2\varpi) = \max\limits_{\tau\in[0,T]} |\Omega_1\varkappa(\tau) + \wp_1(\tau) - \Omega_2 v(\tau) - \wp_2(\tau)|^p\\
b_e(\Im_3\varkappa, \Im_4\varpi) = \max\limits_{\tau\in[0,T]} |2\varkappa(\tau) - \Omega_3\varkappa(\tau) - \wp_3(\tau) - 2\varpi(\tau) + \Omega_4\varpi(\tau) + \wp_4(\tau)|^p\\
b_e(\Im_1\varpi, \Im_3\varkappa) = \max\limits_{\tau\in[0,T]} |\Omega_1\varkappa(\tau) + \wp_1(\tau) - 2\varkappa(\tau) + \Omega_3\varkappa(\tau) + \wp_3(\tau)|^p\\
b_e(\Im_2\varpi, \Im_4\varpi) = \max\limits_{\tau\in[0,T]} |\Omega_2\varpi(\tau) + \wp_2(\tau) - 2\varpi(\tau) + \Omega_4\varpi(\tau) + \wp_4(\tau)|^p\\
b_e(\Im_3\varkappa, \Im_1\varkappa) = \max\limits_{\tau\in[0,T]} |2\varkappa(\tau) - \Omega_3\varkappa(\tau) - \wp_3(\tau) - \Omega_1\varkappa(\tau) - \wp_1(\tau)|^p.
\end{cases}$$

Suppose that $\alpha(\Im_3\varkappa, \Im_4\varpi) \geq 1$. Then, $\Im_3\varkappa \preceq \Im_4\varpi$, i.e., the assumption (4.2) of (U1) holds, and consequently, so does its conclusion (4.3). This, however, indicates that the implication (2.1) holds true for the function that the implication (2.1) is valid for the function $\theta \in \Theta$ given as $\theta(\tau) = \exp\{\sqrt{\tau \exp(\tau)}\}$. Thus, (H1) is demonstrated.

(H2) is a direct consequence of the assumption (U2).

(H3). Let $\varkappa \in \Xi$ and $\varpi \in \Im_4^{-1}(\Im_1\varkappa)$. Then $2\varpi - \Omega_4\varpi - \wp_4 = \Omega_1\varkappa + \wp_1$, and by the assumption (U3), $\Omega_1\varkappa + \wp_1 \preceq \Omega_2\varpi + \wp_2$ holds. That is $\Im_1\varkappa \preceq \Im_2\varpi$, and so $\alpha(\Im_1\varkappa, \Im_2\varpi) \geq 1$. As a result, the pair $(\Im_1, \Im_2)$ is a partially α-weakly increasing w.r.t. $\Im_4$. Similar to this, the pair $(\Im_2, \Im_1)$ is a partially α-weakly increasing w.r.t. $\Im_3$.

(H4) is easily derived from the definition of mapping α, and (H5) is derived from the assumption (U4).

(H6). Let $\{\varkappa_n\}$ be a sequence in Ξ such that $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$, i.e., $\varkappa_n \preceq \varkappa_{n+1}$ for $n \in \mathbb{N}$, and let $\lim_{n\to\infty} \Im_1 \varkappa_n = \lim_{n\to\infty} \Im_3 \varkappa_n = \varpi$ in (Ξ, b_e), i.e.,

$$\max_{\tau\in I} |\Omega_1 \varkappa_n(\tau) + \wp_1(\tau) - \varpi(\tau)|^p \to 0 \text{ as } n \to \infty,$$
$$\max_{\tau\in I} |2\varkappa_n(\tau) - \Omega_3 \varkappa_n(\tau) - \wp_3(\tau) - \varpi(\tau)|^p \to 0 \text{ as } n \to \infty.$$

By the assumption ($\mathrm{U5}_1$), it follows that

$$\begin{aligned}\max_{\tau\in I} &|[\wp_1(\tau) + \Omega_1(2\varkappa_n(\tau) - \Omega_3\varkappa_n(\tau) - \wp_3(\tau))] \\ &- [2(\Omega_1\varkappa_n(\tau) + \wp_1(\tau)) - \Omega_3(\Omega_1\varkappa_n(\tau) + \wp_1(\tau)) - \wp_3(\tau)]|^p \to 0 \text{ as } n \to \infty,\end{aligned}$$

i.e., $\lim_{n\to\infty} b_e(\Im_1\Im_3\varkappa_n, \Im_3\Im_1\varkappa_n) = 0$. Hence, the pair $(\Im_1, \Im_3)$ is an α-compatible. Similarly, it follows from ($\mathrm{U5}_2$) that the pair $(\Im_2, \Im_4)$ is an α-compatible.

The condition (H8) (that $\Im_3$ or $\Im_4$ is an α-dominating map) follows directly from the assumption (U6).

As a result, all of the conditions of Theorem 2.4 are met, and the mappings $\Im_1, \Im_2, \Im_3, \Im_4$ have a common fixed point $\varkappa^* \in \Xi$. It is obvious that $\varkappa^*$ is a solution of the system (4.1).

Finally, if the assumption (U7) is satisfied, the condition (H9) of Theorem 2.5 holds, and thus the solution of (4.1) is unique. □

Conclusion

We introduce a notion of an α-complete extended Branciari b-distance space and rational $\alpha - \lambda - JS$-contractive conditions, and derive coincidence point, common fixed points, and uniqueness of fixed points for two pairs of mappings. We use these findings to obtain the solution of a system of Urysohn integral equations. An example is given to illustrate the result.

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