Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 3(2024), Pages 28-44
https://doi.org/10.54671/BMAA-2024-3-3

ON GENERALIZED STATISTICAL CONVERGENCE IN QUATERNION VALUED GENERALIZED METRIC SPACES

ÖMER KIŞI, BURAK ÇAKAL, MEHMET GÜRDAL

Abstract. We present and explore the concept of a quaternion-valued g-metric space with specific foundational properties related to convergence in this study. The main goal of this study is to introduce λ-statistically convergent sequences and λ-statistically g-Cauchy sequences, examining their properties within quaternion-valued g-metric spaces. The research also includes discussions on natural inclusion theorems relevant to these concepts.

1. Introduction

The investigation of sequence convergence and summability theory has been a pivotal and dynamic area of research in pure mathematics for many years. Furthermore, its substantial contributions extend to topology, functional analysis, Fourier analysis, measure theory, applied mathematics, mathematical modeling, computer science, and other disciplines. The concept of statistical convergence of sequences has gained widespread application in mathematics in recent times. Statistical convergence, initially explored by Fast [10]. Since then, several mathematicians have explored the properties of convergence and statistical convergence, applying these concepts to various fields. For further reference, see [7, 12, 13, 14, 17, 18, 19, 28, 29, 31, 32, 33]. Mursaleen [26], on the other hand, introduced the concept of λ-statistical sonvergence as a novel approach, elucidating its connections to statistical convergence, strongly Cesáro summable and strongly (V, λ)-summable. Recently, the concepts of λ-statistical convergence, almost λ-statistical convergence and invariant statistical convergence were generalized by Braha [5], Esi et al. [9], Kişi and Nuray [23], Savaş [30] and Savaş and Nuray [34]. Also, the readers should refer to the monographs [4], and [27], and recent papers [21], [35], [36], [37], [38], [39], [40] and [41] for the background and related topics on the sequence spaces.

In mathematical analysis, the idea of physical distance is expanded upon by a distance function or metric. This notion can be generalized in various ways (see, for instance, [22]). One notable generalization is the G-metric space introduced by Mustafa and Sims [25]. Metrics in this domain indicate the separation of three

2000 *Mathematics Subject Classification.* 40A05, 40C05, 40D25.
Key words and phrases. λ-statistical convergence; quaternion valued g-metric space; Cauchy sequence; invariant convergence.

Submitted July 1, 2024. Accepted September 4, 2024. Published September 11, 2024.
Communicated by F. Basar.

locayions. Choi et al. [6] introduced a g-metric, an extended form of a distance function. The ordinary distance between two points and the G-metric between three points are generalized by the $n+1$ point distance known as the g-metric of degree n. By advancing the metric concept, Abazari established the idea of statistical g-convergence in [1] (see also [15, 16, 24]).

A number system that extends beyond complex numbers is known as quaternions. Irish mathematician Hamilton originally formulated it in 1843 to describe mechanics in three dimensions. One characteristic of quaternions is their noncommutative multiplication of two quaternions. Detailed discussions on quaternion analysis can be found in [3] and references therein. This paper proposes a type of convergence in quaternion-valued g-metric spaces to expand on the g-metric spaces introduced by [6], quanternion valued g-metric spaces by [20], and various well-known forms of statistical convergence in the literature. The motivation behind this proposal arises from the practical applications of quaternions and fixed point theorems.

The primary objective of this research is to define λ-statistically convergent sequence and λ-statistically g-Cauchy sequence and to investigate their features in a quaternion valued g-metric spaces. In addition to these definitions, natural inclusion theorems will also be discussed.

2. Preliminaries

In this section, we will examine specific definitions and results that form the basis of the current study. We begin with several key definitions.

We will introduce some fundamental notations for quaternionic spaces. The four-dimensional real algebra with unity is known as the space of quaternions, denoted by $\mathbf{Q}$. The null element of $\mathbf{Q}$ is denoted by $0_{\mathbf{Q}}$, and the multiplicative identity of $\mathbf{Q}$ is denoted by $1_{\mathbf{Q}}$. Within $\mathbf{Q}$, there exist three imaginary units represented by the symbols i, j, k. By definition, these units satisfy:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i \text{ and } ki = -ik = j.$$

For each $\rho = \varsigma_0 + \varsigma_1 i + \varsigma_2 j + \varsigma_3 k$; where $\varsigma_0, \varsigma_1, \varsigma_2$ and ς_3 belong to $\mathbb{R}$, the elements $1, i, j, k$ are assumed to constitute a real vector basis of $\mathbf{Q}$. Given $\rho = \varsigma_0 + \varsigma_1 i + \varsigma_2 j + \varsigma_3 k \in \mathbf{Q}$, we recall that:

(i) $\bar{\rho} = \varsigma_0 - \varsigma_1 i - \varsigma_2 j - \varsigma_3 k$ is the conjugate quaternion of ρ,

(ii) $|\rho| = \sqrt{\rho\bar{\rho}} = \sqrt{\varsigma_0^2 + \varsigma_1^2 + \varsigma_2^2 + \varsigma_3^2} \in \mathbb{R}$

(iii) $\mathrm{Re}(\rho) = \frac{1}{2}(\rho + \bar{\rho}) = \varsigma_0 \in \mathbb{R}$

(iv) $\mathrm{Im}(\rho) = \frac{1}{2}(\rho - \bar{\rho}) = \varsigma_1 i + \varsigma_2 j + \varsigma_3 k$ is the imaginary part of ρ.

When $\rho = \mathrm{Re}(\rho)$, the element $\rho \in \mathbf{Q}$ is said to be real. It is obvious that ρ is real only iff $\rho = \bar{\rho}$. If $\bar{\rho} = -\rho$ or $\rho = \mathrm{Im}(\rho)$, ρ is said to be imaginary.

The concept of a complex metric space was introduced by Azam et al. [3] as follows:

Definition 1. *([3]) Let X be a nonempty set, and suppose the mapping $d_{\mathbf{C}} : X \times X \to \mathbf{C}$ satisfies the following conditions:*
(i) $0 \prec d_{\mathbf{C}}(\tau_1, \tau_2)$, for all $\tau_1, \tau_2 \in X$ and $d_{\mathbf{C}}(\tau_1, \tau_2) = 0$ iff $\tau_1 = \tau_2$,
(ii) $d_{\mathbf{C}}(\tau_1, \tau_2) = d_{\mathbf{C}}(\tau_2, \tau_1)$ for all $\tau_1, \tau_2 \in X$,
(iii) $d_{\mathbf{C}}(\tau_1, \tau_2) \preceq d_{\mathbf{C}}(\tau_1, \tau_3) + d_{\mathbf{C}}(\tau_3, \tau_2)$ for all $\tau_1, \tau_2, \tau_3 \in X$.
Then $(X, d_{\mathbf{C}})$ is called a complex metric space.

Ahmed et al. [8] extended the above definition to Clifford analysis as follows:

Definition 2. *([8]) Let X be a nonempty set and suppose that the mapping $d_{\mathbf{Q}}$: $X \times X \rightarrow \mathbf{Q}$ satisfies the following.*
(i) $0 \prec d_{\mathbf{Q}}(\tau_1, \tau_2)$, for all $\tau_1, \tau_2 \in X$ and $d_{\mathbf{Q}}(\tau_1, \tau_2) = 0$ iff $\tau_1 = \tau_2$,
(ii) $d_{\mathbf{Q}}(\tau_1, \tau_2) = d_{\mathbf{Q}}(\tau_2, \tau_1)$ for all $\tau_1, \tau_2 \in X$,
(iii) $d_{\mathbf{Q}}(\tau_1, \tau_2) \preceq d_{\mathbf{Q}}(\tau_1, \tau_3) + d_{\mathbf{Q}}(\tau_3, \tau_2)$ for all $\tau_1, \tau_2, \tau_3 \in X$.
Then $(X, d_{\mathbf{Q}})$ is called a quaternion valued metric space.

Ahmed et al. [8] introduced a partial order $\preceq$ on $\mathbf{Q}$ (space of all quaternions).

Definition 3. *Let $\rho_1, \rho_2 \in \mathbf{Q}$. Then $\rho_1 \preceq \rho_2$ iff $\operatorname{Re}(\rho_1) \leq \operatorname{Re}(\rho_2)$ and $\operatorname{Im}_{\boldsymbol{s}}(\rho_1) \leq \operatorname{Im}_s(\rho_2)$, $\rho_1, \rho_2 \in \mathbf{Q}$, $s = i, j, k$ where $\operatorname{Im} m_i = b$, $\operatorname{Im} m_j = c$, $\operatorname{Im} m_k = d$. It was observed that $\rho_1 \preceq \rho_2$, if any one of the below circumstances is true:*
(a) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_1}(\rho_1) = \operatorname{Im}_{s_1}(\rho_2)$ where $s_1 = j, k$, $\operatorname{Im}_i(\rho_1) < \operatorname{Im}_i(\rho_2)$;
(b) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_2}(\rho_1) = \operatorname{Im}_{s_2}(\rho_2)$ where $s_2 = i, k$, $\operatorname{Im}_j(\rho_1) < \operatorname{Im}_j(\rho_2)$;
(c) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_3}(\rho_1) = \operatorname{Im}_{s_3}(\rho_2)$ where $s_3 = i, j$, $\operatorname{Im}_k(\rho_1) < \operatorname{Im}_k(\rho_2)$;
(d) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_1}(\rho_1) = \operatorname{Im}_{s_1}(\rho_2)$, $\operatorname{Im} m_i(\rho_1) = \operatorname{Im}_i(\rho_2)$;
(e) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_2}(\rho_1) = \operatorname{Im}_{s_2}(\rho_2)$, $\operatorname{Im} m_j(\rho_1) = \operatorname{Im} m_j(\rho_2)$;
(f) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_3}(\rho_1) < \operatorname{Im}_{s_3}(\rho_2)$, $\operatorname{Im} m_k(\rho_1) = \operatorname{Im} m_k(\rho_2)$;
(g) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2)$, $\operatorname{Im}_s(\rho_1) = \operatorname{Im}_s(\rho_2)$;
(h) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_1}(\rho_1) = \operatorname{Im}_{s_1}(\rho_2)$, $\operatorname{Im}_i(\rho_1) < \operatorname{Im}_i(\rho_2)$;
(i) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_2}(\rho_1) = \operatorname{Im}_{s_2}(\rho_2)$, $\operatorname{Im} m_j(\rho_1) < \operatorname{Im} m_j(\rho_2)$;
(j) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_I}(\rho_1) = \operatorname{Im}_{s_I}(\rho_2)$, $\operatorname{Im}_k(\rho_1) < \operatorname{Im} m_k(\rho_2)$;
(k) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_1}(\rho_1) < \operatorname{Im}_{s_1}(\rho_2)$, $\operatorname{Im}_i(\rho_1) = \operatorname{Im} m_i(\rho_2)$;
(l) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_2}(\rho_1) < \operatorname{Im}_{s_2}(\rho_2)$, $\operatorname{Im} m_j(\rho_1) = \operatorname{Im} m_j(\rho_2)$;
(m) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_{s_3}(\rho_1) = \operatorname{Im}_{s_3}(\rho_2)$, $\operatorname{Im}_k(\rho_1) = \operatorname{Im}(\rho_k)$;
(n) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2)$, $\operatorname{Im}_s(\rho_1) < \operatorname{Im}_s(\rho_2)$;
(o) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2)$, $\operatorname{Im}_s(\rho_1) = \operatorname{Im}(m_s)$.

To be more precise, we write $\rho_1 \prec \rho_2$ if only *(n)* is fulfilled and $\rho_1 \precnsim \rho_2$ if $\rho_1 \neq \rho_2$ and one from *(a)* to *(o)* is satisfied

Remark. *Noteworthy is the fact that $\rho_1 \preceq \rho_2 \Rightarrow |\rho_1| \leq |\rho_2|$.*

Inspired by the research of Ahmed et al. [8], Adewale et al. [2] offered the subsequent definition.

Definition 4. *([2]) Let $\mathbf{Q}$ be a collection of quaternions, X a nonempty set, and $G^{\mathbf{Q}} : X \times X \times X \rightarrow \mathbf{Q}$ a function that satisfies the following properties:*
(a) $G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3) = 0$ iff $\tau_1 = \tau_2 = \tau_3$,
(b) $0 \prec G^{\mathbf{Q}}(\tau_1, \tau_1, \tau_2)$, $\forall \tau_1, \tau_2 \in X$, with $\tau_1 \neq \tau_2$,
(c) $G^{\mathbf{Q}}(\tau_1, \tau_1, \tau_2) \preceq G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3)$, $\forall \tau_1, \tau_2, \tau_3 \in X$, with $\tau_3 \neq \tau_2$,
(d) $G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3) = G^{\mathbf{Q}}(\tau_2, \tau_3, \tau_1) = G^{\mathbf{Q}}(\tau_1, \tau_3, \tau_2) = \ldots$ (symmetry),
(e) There exists a real number $r \geq 1$ such that

$$G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3) \leq r\left[G^{\mathbf{Q}}(\tau_1, a, a) + G^{\mathbf{Q}}(a, \tau_2, \tau_3)\right],$$

$\forall a, \tau_1, \tau_2, \tau_3 \in X$ (rectangle inequality).
Then, the function $G^{\mathbf{Q}}$ is called a quaternion G-metric and $\left(X, G^{\mathbf{Q}}\right)$ is referred to as the $G^{\mathbf{Q}}$-metric space. A $G^{\mathbf{Q}}$-metric space is considered complete if every Cauchy sequence in it is $G^{\mathbf{Q}}$-convergent.

The following is an extension of G-metric space with degree l.

Definition 5. *([2]) Let X be a non-empty set. A function $g : X^{p+1} \to \mathbb{R}^+$ is called a g-metric space with order p on X if it satisfies the following conditions:*
(a) $g(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) = 0$ iff $\tau_0 = \tau_1 = \ldots = \tau_p$,
(b) $g(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) = g\left(\tau_{\sigma(0)}, \tau_{\sigma(1)}, \tau_{\sigma(2)}, \ldots, \tau_{\sigma(p)}\right)$ for permutation σ on $\{0, 1, 2, \ldots, p\}$,
(c) $g(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) \leq g(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p)$ for all $(\tau_0, \tau_1, \tau_2, \ldots, \tau_p), (\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p) \in X^{p+1}$ with $\{\tau_i : i = 0, 1, \ldots, p\} \subseteq \{\varsigma_i : i = 0, 1, \ldots, p\}$,
(d) For every $\tau_0, \tau_1, \ldots, \tau_{\mathfrak{s}}, \varsigma_0, \varsigma_1, \ldots \varsigma_{\mathfrak{t}}$, $w \in X$ with $\mathfrak{s} + \mathfrak{t} + 1 = l$,

$$g(\tau_0, \tau_1, \tau_2, \ldots, \tau_{\mathfrak{s}}, \varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_{\mathfrak{t}}) \leq$$
$$g(\tau_0, \tau_1, \tau_2, \ldots, \tau_{\mathfrak{s}}, w, w, \ldots w) + g(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_{\mathfrak{t}}, w, w \ldots w).$$

The pair (X, g) is called g-metric space with degree p. For $p = 1, 2$ respectively, it is respectively equivalent to metric and G-metric space.

The statistical convergence of real valued sequences is based on the concept of natural density of subsets of $\mathbb{N}$, the set of all positive integers, which is defined as follows: Let (X, d) be a metric space. A real number sequence (τ_k) is said to be statistically convergent to the number τ if for every $\varepsilon > 0$,

$$\lim_n n^{-1} \left|\{j \leq n : d(\tau_j, \tau) \geq \varepsilon\}\right| = 0,$$

where the number of elements in the contained set is indicated by the vertical bars.

The idea of λ-statistical convergence of sequences $\tau = (\tau_j)$ of real numbers has been studied by Mursaleen [26]. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. The generalized de la Vallèe-Poussin mean is defined by

$$t_n(\tau) = \frac{1}{\lambda_n} \sum_{j \in I_n} \tau_j,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$. If $\lambda_n = n$, then (V, λ)-summability reduces to $(C, 1)$-summability.

A real number sequence $\tau = (\tau_j)$ is said to be λ-statistically convergent to the number τ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left|\{j \in I_n : d(\tau_j, \tau) \geq \varepsilon\}\right| = 0.$$

In this case we write st-λ-$\lim \tau_j = \tau$ or $\tau_j \to \tau\,(st - \lambda)$.

Note that if $\lambda_j = j$ is taken here, λ-statistical convergence is reduced to statistical convergence.

The following definitions were given by Abazari [1].

Definition 6. *([1]) If $p \in \mathbb{N}$ and $K \in \mathbb{N}^p$, and*

$$K(j) = \{(i_1, i_2, \ldots, i_p) \leq j : (i_1, i_2, \ldots, i_p) \in K\},$$

then

$$\delta_p(K) = \lim_{j \to \infty} \frac{p!}{j^p} |K(j)|,$$

is called p-dimensional asymptotic (or natural density) of the set K.

Definition 7. *([1]) Let (τ_j) be a sequence in a g-metric space (Y, g). Then, the following statements hold:*

(i) (τ_j) *is statistically convergent to* τ *, provided for all* $\varepsilon > 0$,

$$\lim_{j\to\infty} \frac{p!}{j^p} \left| \left\{ i_1, i_2, ..., i_p \leq j : g\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right) \geq \varepsilon \right\} \right| = 0,$$

and is indicated by $gS\text{-}\lim_{j\to\infty} \tau_j = \tau$.

(ii) (τ_j) *is called to be statistical g-Cauchy, provided for all* $\varepsilon > 0$*, there exists* $i_\varepsilon \in \mathbb{N}$ *so that*

$$\lim_{j\to\infty} \frac{p!}{j^p} \left| \left\{ i_1, i_2, ..., i_p \leq j : g\left(\tau_{i_\varepsilon}, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right) \right\} \right| = 0.$$

The following definition was given by Jan and Jalal [20].

Definition 8. *([20]) Let* X *be a non-empty set. A function* $g_{\mathbf{Q}} : X^{p+1} \to \mathbf{Q}$ *(where* $\mathbf{Q}$ *is the space of quaternions) is called quaternion valued g-metric space with order* p *on* X *if it satisfies the following conditions:*
(a) $g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) = 0$ *iff* $\tau_0 = \tau_1 = \ldots = \tau_p$,
(b) $g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) = g_{\mathbf{Q}}\left(\tau_{\sigma(0)}, \tau_{\sigma(1)}, \tau_{\sigma(2)}, \ldots, \tau_{\sigma(p)}\right)$ *for permutation* σ *on* $\{0, 1, 2, \ldots p\}$,
(c) $g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) \preceq g_{\mathbf{Q}}(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p)$ *for all*

$$(\tau_0, \tau_1, \tau_2, \ldots, \tau_p), \ (\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p) \in X^{p+1},$$

with $\{\tau_i : i = 0, 1, \ldots p\} \subsetneq \{\varsigma_i : i = 0, 1, \ldots p\}$,
(d) For all $\tau_0, \tau_1, \ldots, \tau_{\mathfrak{s}}, \varsigma_0, \varsigma_1, \ldots \varsigma_{\mathfrak{t}}, v \in X$ *with* $\mathfrak{s} + \mathfrak{t} + 1 = p$,

$$\begin{aligned} & g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_{\mathfrak{s}}, \varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_{\mathfrak{t}}) \leq g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_{\mathfrak{s}}, v, v, \ldots v) \\ & + g_{\mathbf{Q}}(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_{\mathfrak{t}}, v, v \ldots v). \end{aligned}$$

The pair $(X, g_{\mathbf{Q}})$ *is called quaternion valued* $g_{\mathbf{Q}}$*-metric space with degree* p*. For* $p = 1, 2$ *respectively, it is equivalent to quaternion valued metric and quaternion valued G-metric space.*

Definition 9. *([20]) A* $g_{\mathbf{Q}}$*-metric on* X *is called multiplicity independent with degree* p *if the following holds*

$$g_{\mathbf{Q}}(\tau_0, \ldots, \tau_p) = g_{\mathbf{Q}}(\varsigma_0, \ldots, \varsigma_p),$$

for all $(\tau_0, \tau_1, \ldots, \tau_p), (\varsigma_0, \varsigma_1, \ldots, \varsigma_p) \in X^{p+1}$ *with*

$$\{\tau_i : i = 0, \ldots, p\} = \{\varsigma_i : i = 0, \ldots, p\}.$$

Note that for a given multiplicity independent $g_{\mathbf{Q}}$-metric with order 2, it holds that $g_{\mathbf{Q}}(\tau, \varsigma, \varsigma) = g_{\mathbf{Q}}(\tau, \tau, \varsigma)$. For a given multiplicity independent $g_{\mathbf{Q}}$-metric with order 3, it holds that $g_{\mathbf{Q}}(\tau, \varsigma, \varsigma, \varsigma) = g_{\mathbf{Q}}(\tau, \tau, \varsigma, \varsigma) = g_{\mathbf{Q}}(\tau, \tau, \tau, \varsigma)$ and $g_{\mathbf{Q}}(\tau, \tau, \varsigma, z) = g_{\mathbf{Q}}(\tau, \varsigma, \varsigma, z) = g_{\mathbf{Q}}(\tau, \varsigma, z, z)$.

Remark. *If we allow equality under the conditions of monotonicity in Definition 9 that is*

$$g_{\mathbf{Q}}(\tau_0, \ldots, \tau_p) \preceq g_{\mathbf{Q}}(\varsigma_0, \ldots, \varsigma_p)$$

for $(\tau_0, \ldots, \tau_p), (\varsigma_0, \ldots, \varsigma_p) \in X^{p+1}$ *with* $\{\tau_i : i = 0, \ldots, p\} \subseteq \{\varsigma_i : i = 0, \ldots, p\}$*, then every* $g_{\mathbf{Q}}$*-metric becomes multiplicity independent.*

3. Main Results

Following the definitions and findings presented above, we aim to introduce new concepts of λ-statistically convergent sequences with regard to the metrics on quaternion valued g-metric spaces in this section. Alongside these definitions, we will discuss natural inclusion theorems.

We are now ready to define λ-statistical convergence in a quaternion-valued g-metric space $(X, g_{\mathbf{Q}})$.

Definition 10. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) λ-statistically converges to τ if for every $q \in \mathbf{Q}$ with $0 \prec q$ such that*

$$\begin{aligned}\lim_{j\to\infty} \tfrac{p!}{(\lambda_j)^p} \Big| \big\{ (i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \le j\ (j \in \mathbb{N}) : \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \ge |q| \big\} \Big| = 0,\end{aligned}$$

where $I_j^p = [j - \lambda_j + 1, j] \times [j - \lambda_j + 1, j] \times ... \times [j - \lambda_j + 1, j]$ (p times) for $j = 1, 2, ...$ and denoted by $g_{\mathbf{Q}}(st - \lambda) - \lim_{j\to\infty} \left\{\tau_{i_1}, \tau_{i_2}, \ldots \tau_{i_p}\right\} = \tau$ or $(\tau_j) \overset{g_{\mathbf{Q}}(st-\lambda)}{\rightarrow} \tau$.

When $\lambda_j = j$, for all j, then the notion of $g_{\mathbf{Q}}(st - \lambda)$-statistically convergence of sequences is reduced to the quaternion valued g-statistical convergence for sequences in [20].

Definition 11. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. Then, (τ_j) is bounded if there exists a positive number $\mathfrak{B}$ such that $\left|\left\{\tau_{j_1}, \tau_{j_2}, \ldots, \tau_{j_p}\right\}\right| \le \mathfrak{B}$ for all $\{(i_1, i_2, \ldots, i_p)\}$.*

We denote the set of all quaternion valued bounded sequences by $\ell_\infty^{\mathbf{Q}}$.

Theorem 3.1. *If $g_{\mathbf{Q}} - \lim \tau_k = \tau$ then $g_{\mathbf{Q}}(st - \lambda) - \lim \tau_k = \tau$.*

Proof. Let $g_{\mathbf{Q}} - \lim \tau_k = \tau$. Then for all $0 \prec q \in \mathbf{Q}$ there exists $\Upsilon \in \mathbb{N}$ such that

$$\{k_1, k_{2,}, \ldots, k_p\} \ge \Upsilon \Rightarrow g_{\mathbf{Q}}\left(\tau, \tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_p}\right) \prec q.$$

The set

$$\begin{aligned}A(\varepsilon) = \big\{ (k_1, k_{2,}, \ldots, k_p) \in I_j^p,\ k_{1,} k_2, \ldots, k_p \le j\ (j \in \mathbb{N}) : \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{k_1}, \tau_{k_2}, ..., \tau_{k_p}\right)\right| \ge |q| = \varepsilon \big\} \subset \{1, 2, 3, \ldots\}^p,\end{aligned}$$

where $q = \frac{\varepsilon}{2} + i\frac{\varepsilon}{2} + j\frac{\varepsilon}{2} + k\frac{\varepsilon}{2}$, $\delta_\lambda(A(\varepsilon)) = 0$. Hence, $g_{\mathbf{Q}}(st - \lambda) - \lim \tau_k = \tau$ as desired. □

The following example shows that the converse of Theorem 3.1 need not be true, in general.

Example 1. *Let $X = \mathbb{R}$ and $G_{\mathbf{Q}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbf{Q}$ be a quaternion valued G-metric space defined by*

$$\begin{aligned}G_{\mathbf{Q}}(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3) = {} & \left|\mathfrak{z}_0^1 - \mathfrak{z}_0^2\right| + \left|\mathfrak{z}_0^1 - \mathfrak{z}_0^3\right| + \left|\mathfrak{z}_0^2 - \mathfrak{z}_0^3\right| \\ & + i\left(\left|\mathfrak{z}_1^1 - \mathfrak{z}_1^2\right| + \left|\mathfrak{z}_1^1 - \mathfrak{z}_1^3\right| + \left|\mathfrak{z}_1^2 - \mathfrak{z}_1^3\right|\right) \\ & + j\left(\left|\mathfrak{z}_2^1 - \mathfrak{z}_2^2\right| + \left|\mathfrak{z}_2^1 - \mathfrak{z}_2^3\right| + \left|\mathfrak{z}_2^2 - \mathfrak{z}_2^3\right|\right) \\ & + k\left(\left|\mathfrak{z}_3^1 - \mathfrak{z}_3^2\right| + \left|\mathfrak{z}_3^1 - \mathfrak{z}_3^3\right| + \left|\mathfrak{z}_3^2 - \mathfrak{z}_3^3\right|\right),\end{aligned}$$

where $\mathfrak{p}_r = \mathfrak{z}_0^r + \mathfrak{z}_1^r i + \mathfrak{z}_2^r j + \mathfrak{z}_3^r k$ *for* $r = 1, 2, 3$. *Let* $\tau_m = \left\{ x_{i_1}, x_{i_2}, \ldots x_{i_p} \right\}$ *be a sequence defined as*

$$\tau_k = \begin{cases} k, & \text{if } k \text{ is a square} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $g_{\mathbf{Q}}(st - \lambda) - \lim \tau_k = 1$, *since the cardinality of the set*

$$|\{k \in I_k^p,\ k \leq n\ (n \in \mathbb{N}) : |g_{\mathbf{Q}}(1, \tau_k)| > |q| = \varepsilon\}| \leq \sqrt{k},$$

for every $\varepsilon > 0$ *and*

$$q = \frac{\varepsilon}{2} + i\frac{\varepsilon}{2} + j\frac{\varepsilon}{2} + k\frac{\varepsilon}{2}.$$

But (τ_k) *is neither convergent nor bounded.*

Following theorem shows that the statistical limit in a quaternion valued g-metric space is unique.

Theorem 3.2. *If a sequence* (τ_j) *is* λ*-statistically convergent in* $(X, g_{\mathbf{Q}})$, *then* $g_{\mathbf{Q}}(st - \lambda) - \lim \tau_j$ *is unique.*

Proof. Suppose that (τ_j) λ-statistically converges in $(X, g_{\mathbf{Q}})$. Let $g_{\mathbf{Q}}(st - \lambda) - \lim \tau_j = \tau_1$ and $g_{\mathbf{Q}}(st - \lambda) - \lim \tau_j = \tau_2$. Given $\varepsilon > 0$ and $0 \prec q \in \mathbf{Q}$, let

$$q = \frac{\varepsilon}{4p} + i\frac{\varepsilon}{4p} + j\frac{\varepsilon}{4p} + k\frac{\varepsilon}{4p}.$$

Define the following sets as:

$$K_1(\varepsilon) = \Big\{(k_1, k_2, \ldots, k_p) \in I_j^p,\ k_1, k_2, \ldots, k_p \leq j\ (j \in \mathbb{N}) : \\ \left|g_{\mathbf{Q}}\left(\tau_1, \tau_{k_1}, \tau_{k_2}, \ldots . \tau_{k_p}\right)\right| \geq |q| = \frac{\varepsilon}{2p}\Big\},$$

$$K_2(\varepsilon) = \Big\{(k_1, k_2, \ldots, k_p) \in I_j^p,\ k_1, k_2, \ldots, k_p \leq j\ (j \in \mathbb{N}) : \\ \left|g_{\mathbf{Q}}\left(\tau_2, \tau_{k_1}, \tau_{k_2}, \ldots . \tau_{k_p}\right)\right| \geq |q| = \frac{\varepsilon}{2p}\Big\}.$$

Since $g_{\mathbf{Q}}(st - \lambda) - \text{Iim}\left(\tau_{k_1}, \tau_{k_2}, \ldots \tau_{k_p}\right) = \tau_1$, we have $\delta_\lambda(K_1(\varepsilon)) = 0$. Similarly $g_{\mathbf{Q}}(st - \lambda) - \text{Iim}\left(\tau_{k_1}, \tau_{k_2}, \ldots \tau_{k_p}\right) = \tau_2$, implies $\delta_\lambda(K_2(\varepsilon)) = 0$.

Let $K(\varepsilon) = K_1(\varepsilon) \cup K_2(\varepsilon)$. Then $\delta_\lambda(K(\varepsilon)) = 0$ and we have $K^c(\varepsilon)$ is non-empty and $\delta_\lambda(K^c(\varepsilon)) = 1$. Suppose $\{k_1, k_2, \ldots, k_p\} \in K^c(\varepsilon)$, then by Definition 8, we have:

$$\begin{aligned} |g_{\mathbf{Q}}(\tau_1, \tau_2, \tau_2, ..., \tau_2)| &\leq |g_{\mathbf{Q}}(\tau_1, \tau_m, \tau_m, ..., \tau_m)| + |g_{\mathbf{Q}}(\tau_m, \tau_2, \tau_2, ..., \tau_2)| \\ &\leq |g_{\mathbf{Q}}(\tau_1, \tau_m, \tau_m, ..., \tau_m)| + p\,|g_{\mathbf{Q}}(\tau_2, \tau_m, ..., \tau_m)| \\ &\leq \left|g_{\mathbf{Q}}\left(\tau_1, \tau_{k_1}, \tau_{k_2}, ..., \tau_{k_p}\right)\right| + p\left|g_{\mathbf{Q}}\left(\tau_2, \tau_{k_1}, \tau_{k_2}, ..., \tau_{k_p}\right)\right| \\ &\leq p\left|g_{\mathbf{Q}}\left(\tau_1, \tau_{k_1}, \tau_{k_2}, ..., \tau_{k_p}\right)\right| + p\left|g_{\mathbf{Q}}\left(\tau_2, \tau_{k_1}, \tau_{k_2}, ..., \tau_{k_p}\right)\right| \\ &\leq p\left(\frac{\varepsilon}{2p} + \frac{\varepsilon}{2p}\right) = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $g_{\mathbf{Q}}(\tau_1, \tau_2, \tau_2 \ldots, \tau_2) = 0$, therefore $\tau_1 = \tau_2$. $\square$

Definition 12. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) is said to be λ-statistically $g_{\mathbf{Q}}$-Cauchy if for every $q \in \mathbf{Q}$ with $0 \prec q$, there exists $i_r \prec \mathbf{Q}$ such that*

$$\begin{gathered}\lim_{j\to\infty} \tfrac{p!}{(\lambda_j)^p} \left|\left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) :\right.\right.\\ \left.\left.\left|g_{\mathbf{Q}}\left(\tau_{i_r}, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \geq |q|\right\}\right| = 0,\end{gathered}$$

$(X, g_{\mathbf{Q}})$ is called a complete quaternion valued g-metric space.

If $\lambda_j = j$, for all j, then the notion of λ-statistically $g_{\mathbf{Q}}$-Cauchy sequence is reduced to the concept of statistical $g_{\mathbf{Q}}$-Cauchy sequence in [20].

Theorem 3.3. *Let $(X, g_{\mathbf{Q}})$ be a complete quaternion valued g-metric space. Then, a sequence (τ_j) of points in $(X, g_{\mathbf{Q}})$ is λ-statistically g-convergent if and only if it is λ-statistically $g_{\mathbf{Q}}$-Cauchy.*

Proof. Suppose that $g_{\mathbf{Q}}(st-\lambda) - \lim \tau_j = \tau$. Then, we get $\delta_\lambda(A(\varepsilon)) = 0$, where

$$\begin{gathered}A(\varepsilon) = \left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) :\right.\\ \left.\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \geq |q| = \frac{\varepsilon}{2}\right\},\end{gathered}$$

where $q = \frac{\varepsilon}{4} + i\frac{\varepsilon}{4} + j\frac{\varepsilon}{4} + k\frac{\varepsilon}{4}$.

This implies that

$$\begin{gathered}\delta_\lambda\left(A^c(\varepsilon)\right) = \left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) :\right.\\ \left.\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| < |q| = \frac{\varepsilon}{2}\right\} = 1,\end{gathered}$$

Let $(m_1, m_2, m_3, \ldots, m_s) \in A^c(\varepsilon)$. Then

$$\left|g_{\mathbf{Q}}\left(\tau, \tau_{m_1}, \tau_{m_2}, ..., \tau_{m_s}\right)\right| < |q| = \frac{\varepsilon}{2}.$$

Let

$$\begin{gathered}B(\varepsilon) = \left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) :\right.\\ \left.\left|g_{\mathbf{Q}}\left(\tau_{m_1}, \tau_{m_2}, \ldots, \tau_{m_s}, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| \geq \varepsilon\right\},\end{gathered}$$

we need to demonstrate that $B(\varepsilon) \subset A(\varepsilon)$. Let $j \in B(\varepsilon)$. That is $(i_1, i_2, \ldots, i_p) \leq j$ $(j \in \mathbb{N}) \in A(\varepsilon)$. Otherwise, if $\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| \leq \varepsilon$ then

$$\begin{aligned}\varepsilon \leq \left|g_{\mathbf{Q}}\left(\tau_{m_1}, \tau_{m_2}, \ldots, \tau_{m_s}, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| &\leq g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\\ &+ g_{\mathbf{Q}}\left(\tau, \tau_{m_1}, \tau_{m_2}, \ldots, \tau_{m_s}\right)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which is not possible. Hence $B(\varepsilon) \subset A(\varepsilon)$, which implies that $\left\{\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right\}$ is λ-statistically $g_{\mathbf{Q}}$-Cauchy.

Conversely, suppose that (τ_k) is $g_{\mathbf{Q}}(st-\lambda)$-Cauchy but not $g_{\mathbf{Q}}(st-\lambda)$-convergent. Then, there exist $\tau_{k_1}, \tau_{k_2}, \ldots \tau_{k_m} \in I_j^p$ $(j \in \mathbb{N})$ such that $\delta_\lambda(G(\varepsilon)) = 0$ where

$$\begin{gathered}G(\varepsilon) = \left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) :\right.\\ \left.\left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}, \tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_m}\right)\right| \geq \frac{\varepsilon}{2}\right\},\end{gathered}$$

and $\delta_\lambda(D(\varepsilon)) = 0$ where

$$D(\varepsilon) = \left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \le j\ (j \in \mathbb{N}) : \right.$$
$$\left. \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| < \frac{\varepsilon}{2}\right\},$$

that is $\delta_\lambda\left(D^c(\varepsilon)\right) = 1$.
Since

$$\left|g_{\mathbf{Q}}\left(\tau_{m_1}, \tau_{m_2}, \ldots, \tau_{m_s}, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| \le 2\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| < \varepsilon$$

if $\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| < \frac{\varepsilon}{2}$. Therefore $\delta_\lambda\left(G^c(\varepsilon)\right) = 0$ that is $\delta_\lambda(G(\varepsilon)) = 1$, which leads the contradiction, since $\left\{\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right\}$ was $g_{\mathbf{Q}}(st-\lambda)$-Cauchy. Hence, (τ_k) is a $g_{\mathbf{Q}}(st-\lambda)$-convergent sequence. □

Now, we establish the relation between strongly Cesáro summability, strong summability and $g_{\mathbf{Q}}$-statistical convergence in quaternion valued g-metric space.

Definition 13. *A sequence (τ_j) is said to be strongly Cesáro summable to limit τ in $(X, g_{\mathbf{Q}})$ if*

$$\lim_{j\to\infty} \frac{p!}{j^p} \sum_{(i_1,i_2,\ldots,i_p)=1}^{j} g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right) = 0,$$

and write it as $\tau_j \to \tau[C, 1, g_{\mathbf{Q}}]$. In this case τ is the $[C, 1, g_{\mathbf{Q}}]$-limit of (τ_j).

Definition 14. *A sequence (τ_j) is said to be strongly λ-summable to limit τ in $(X, g_{\mathbf{Q}})$ if*

$$\lim_{j\to\infty} \frac{p!}{(\lambda_j)^p} \sum_{(i_1,i_2,\ldots,i_p)\in I_j^p} g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right) = 0,$$

and write it as $\tau_j \to \tau[V, \lambda, g_{\mathbf{Q}}]$. In this case τ is the $[V, \lambda, g_{\mathbf{Q}}]$-limit of (τ_j).

Proofs of the following result are routine works, so ommitted.

Theorem 3.4. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space. Then, following statements hold:*
(i) If $\tau_j \to \tau[V, \lambda, g_{\mathbf{Q}}]$ then $\tau_j \to \tau\left(g_{\mathbf{Q}}(st-\lambda)\right)$, and the inclusion $[V, \lambda, g_{\mathbf{Q}}] \subseteq g_{\mathbf{Q}}(st-\lambda)$.
(ii) If $\tau_j \in \ell_\infty^{\mathbf{Q}}$ and $\tau_j \to \tau\left(g_{\mathbf{Q}}(st-\lambda)\right)$, then $\tau_j \to \tau[V, \lambda, g_{\mathbf{Q}}]$.
(iii) $g_{\mathbf{Q}}(st-\lambda) \cap \ell_\infty^{\mathbf{Q}} = [V, \lambda, g_{\mathbf{Q}}] \cap \ell_\infty^{\mathbf{Q}}$.

Proof. (i) Let

$$K_\varepsilon(p) = \left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \le j\ (j \in \mathbb{N}) : \right.$$
$$\left. \left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right)\right| \ge \varepsilon\right\},$$

Now, since $\tau_j \to \tau[V, \lambda, g_{\mathbf{Q}}]$ for all $0 \prec q \in \mathbf{Q}$ we have

$$\begin{aligned}
0 \longleftarrow & \frac{p!}{(\lambda_j)^p} \sum_{(i_1,i_2,\ldots,i_p)\in I_j^p}^{j} g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right) \\
\geq & \frac{p!}{(\lambda_j)^p} \left\{ \sum_{\substack{i_1,i_2,\ldots,i_p=1 \\ (i_1,i_2,\ldots,i_p)\notin K_\varepsilon(p)}}^{j} g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right) \right. \\
& \left. + \sum_{\substack{i_1,i_2,\ldots,i_p=1 \\ (i_1,i_2,\ldots,i_p)\in K_\varepsilon(p)}}^{j} g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right) \right\} \\
\geq & \frac{p!}{(\lambda_j)^p} \varepsilon \left|K_\varepsilon(p)\right|, \text{ as } j \to \infty.
\end{aligned}$$

That is, $\lim_{j\to\infty} \frac{p!}{(\lambda_j)^p} |K_\varepsilon(p)| = 0$ and $\delta_\lambda\left(K_\varepsilon(p)\right) = 0$. Hence, we have $\tau_j \to \tau\left(g_{\mathbf{Q}}(st-\lambda)\right)$.

It is easy to see that the inclusion $[V, \lambda, g_{\mathbf{Q}}] \subseteq g_{\mathbf{Q}}(st-\lambda)$ is proper.

(ii) Suppose that $\left\{\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right\}$ is bounded and λ-statistically $g_{\mathbf{Q}}$-convergent to τ in $(X, g_{\mathbf{Q}})$. Then, for $\varepsilon > 0$, we have $\delta_\lambda\left(K_\varepsilon(p)\right) = 0$. Since $\left\{\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right\} \in \ell_\infty^{\mathbf{Q}}$, there exist a $M > 0$ such that $\left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, l\right)\right|^P \leq M$. We have

$$\begin{aligned}
& \frac{p!}{(\lambda_j)^p} \sum_{i_1,i_2,\ldots,i_p\in I_j^p} \left|g\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right)\right| \\
= & \frac{p!}{(\lambda_j)^p} \sum_{\substack{i_1,i_2,\ldots,i_p=1 \\ i_1,i_2,\ldots,i_p\notin K_\varepsilon(p)}}^{j} \left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right)\right| \\
+ & \frac{p!}{(\lambda_j)^p} \sum_{\substack{i_1,i_2,\ldots,i_p=1 \\ (i_1,i_2,\ldots,i_p)\in K_\varepsilon(p)}}^{j} \left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right)\right|. \\
= & S_1(j) + S_2(j),
\end{aligned}$$

where

$$S_1(j) := \frac{p!}{(\lambda_j)^p} \sum_{\substack{i_1,i_2,\ldots,i_p=1 \\ (i_1,i_2,\ldots,i_p)\notin K_\varepsilon(p)}}^{j} \left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right)\right|$$

and

$$S_2(j) := \frac{p!}{(\lambda_j)^p} \sum_{\substack{i_1,i_2,\ldots,i_p=1 \\ (i_1,i_2,\ldots,i_p)\in K_\varepsilon(p)}}^{j} \left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right)\right|.$$

Now if $\{i_1, i_2, \ldots, i_p\} \notin K_\varepsilon(p)$ then $S_1(j) < \varepsilon$. For $\{i_1, i_2, \ldots, i_p\} \in K_\varepsilon(p)$, we have

$$S_2(j) \leq \sup\left|g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}, \tau\right)\right| \left(\frac{|K_\varepsilon(p)|}{j}\right) \leq M \frac{p!\,|K_\varepsilon(p)|}{(\lambda_j)^p} \to 0$$

as $j \to \infty$, since $\delta_\lambda(K(\varepsilon)) = 0$. Hence

$$\left\{\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right\} \to \tau[V, \lambda, g_{\mathbf{Q}}].$$

As a result, we conclude that $\tau_j \to \tau[V, \lambda, g_{\mathbf{Q}}]$.
(iii) Follows from (i) and (ii). □

Theorem 3.5. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. Then*

i) If $\tau_j \to \tau\left([C, 1, g_{\mathbf{Q}}]\right)$, then $\tau_j \to \tau\left(g_{\mathbf{Q}}(st - \lambda)\right)$.

ii) If $(X, g_{\mathbf{Q}})$ is bounded and $\tau_j \to \tau\left(g_{\mathbf{Q}}(st - \lambda)\right)$, then $\tau_j \to \tau\left([C, 1, g_{\mathbf{Q}}]\right)$.

Proof. The proof has been omitted since it follows by the similar way used in proving Theorem 3.4. □

It is easily seen that $\tau_j \to \tau\left(g_{\mathbf{Q}}(st - \lambda)\right) \subseteq \tau_j \to \tau\left(g_{\mathbf{Q}}(st)\right)$ for all λ, since $\frac{(\lambda_j)^p}{j^p}$ is bounded by 1. Therefore, we establish the following relation.

Theorem 3.6. $g_{\mathbf{Q}}(st) \subseteq g_{\mathbf{Q}}(st - \lambda)$ *iff* $\liminf \frac{(\lambda_j)^p}{j^p} > 0$.

Proof. For given $\varepsilon > 0$, we see that

$$\begin{aligned}
&\left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \right. \\
&\qquad \left. \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \geq |q| = \varepsilon\right\} \\
&\subset \left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \right. \\
&\qquad \left. \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \geq |q| = \varepsilon\right\},
\end{aligned}$$

where $q = \frac{\varepsilon}{2} + i\frac{\varepsilon}{2} + j\frac{\varepsilon}{2} + k\frac{\varepsilon}{2}$. This gives

$$\begin{aligned}
&\frac{p!}{j^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \right.\right. \\
&\qquad \left.\left. \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \geq |q| = \varepsilon\right\}\right| \\
&\geq \frac{p!}{j^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \right.\right. \\
&\qquad \left.\left. \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \geq |q| = \varepsilon\right\}\right| \\
&\geq \frac{(\lambda_j)^p}{j^p} \cdot \frac{p!}{(\lambda_j)^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \right.\right. \\
&\qquad \left.\left. \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_p}\right)\right| \geq |q| = \varepsilon\right\}\right|.
\end{aligned}$$

Taking the limits as $j \to \infty$ and employ the reality that $\liminf \frac{(\lambda_j)^p}{j^p} > 0$, we get

$$\tau_j \to \tau\left(g_{\mathbf{Q}}(st)\right) \Rightarrow \tau_j \to \tau\left(g_{\mathbf{Q}}(st - \lambda)\right).$$

Conversely, suppose that $\liminf \frac{(\lambda_j)^p}{j^p} = 0$. As in [11, p. 510], we can choose a subsequence $(j\,(r))_{r=1}^{\infty}$ such that $\frac{(\lambda_j(r))^p}{(j(r))^p} < \frac{1}{r}$. Let $X = \mathbb{R}$ and $g_{\mathbf{Q}}$ be the quaternion valued metric as Example 1. Let $\tau_j = \left\{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right\}$ be a sequence defined as

$$\tau_j = \begin{cases} 1, & \text{if } i_1, i_2, ..., i_p \in I_{j(r)}^p \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\tau \in [C, 1, g_{\mathbf{Q}}]$, and so, by [7, Theorem 2.1], $\tau \in g_{\mathbf{Q}}(st)$. On the other hand, $\tau \notin [V, \lambda, g_{\mathbf{Q}}]$ and Theorem 3.4(ii) gives that $\tau \notin g_{\mathbf{Q}}(st - \lambda)$. Hence, $\liminf \frac{(\lambda_j)^p}{j^p} > 0$ is necessary. □

Let σ be a mapping of the positive integers into themselves. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a σ-mean if the following conditions hold:

(i) $\varphi\,(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, ...)$ and

(iii) $\varphi\left(x_{\sigma(n)}\right) = \varphi(x_n)$ for all $x \in \ell_\infty$, where ℓ_∞ denotes the set of bounded sequences.

The mapping σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $n, m \in \mathbb{Z}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, φ extends the limit functional on c, the space of convergent sequences, in the sense that $\varphi(x_n) = \lim x_n$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit. The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } m \right\}.$$

In [42], Schaefer proved that a bounded sequence $x = (x_k)$ of real numbers is σ-convergent to L if and only if

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} x_{\sigma^i(m)} = L,$$

uniformly in m. A sequence $x = (x_k)$ is said to be strongly σ-convergent to L if there exists a number L such that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} d\left(x_{\sigma^i(m)}, L\right) = 0,$$

uniformly in m. We write $[V_\sigma]$ as the set of all strong σ-convergent sequences. A sequence $x = (x_n) \in \ell_\infty$ is said to be almost convergent of all of its Banach limits coincide. The spaces $\widehat{c}$ and $[\widehat{c}]$ of almost convergent sequences and strongly almost convergent sequences are defined respectively by

$$\widehat{c} = \left\{ x \in \ell_\infty : \lim_m t_{mn}(x) \text{ exists uniformly in } n \right\}$$

and

$$[\widehat{c}] = \left\{ x \in \ell_\infty : \lim_m t_{mn}(|x - le|) \text{ exists uniformly in } n \text{ for some } l \in \mathbb{C} \right\}$$

where $t_{mn}(x) = \frac{x_n + x_{n+1} + ... + x_{n+m}}{m+1}$ and $e = (1, 1, ...)$. Taking $\sigma(m) = m + 1$, we obtain $[V_\sigma] = [\widehat{c}]$.

Definition 15. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) σ-statistically converges to τ if for every $q \prec \mathbf{Q}$ with $0 \prec q$ such that*

$$\lim_{n\to\infty} \tfrac{p!}{j^p} \left|\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \geq |q| \right\}\right| = 0,$$

uniformly in m. In this case, we write $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st} - \sigma)\right)$.

Before presenting the promised inclusion relations, we will introduce a new definition.

Definition 16. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) (σ, λ)-statistically converges to τ if for every $q \prec \mathbf{Q}$ with $0 \prec q$ such that*

$$\lim_{j\to\infty} \tfrac{p!}{(\lambda_j)^p} \left|\left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \left|g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \geq |q|\right\}\right| = 0,$$

uniformly in m. In this case, we write $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st} - \sigma - \lambda)\right)$.

Definition 17. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. A sequence (τ_j) is said to be strongly (σ, λ)-summable to limit τ in $(X, g_{\mathbf{Q}})$ if*

$$\lim_j \frac{j!}{(\lambda_j)^j} \sum_{(i_1, i_2, ..., i_p) \in I_j} g\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right) = 0,$$

uniformly in $m = 1, 2, 3, ...$
This form of convergence is represented as $g_{\mathbf{Q}}\left[\widehat{V} - \sigma - \lambda\right]$-$\lim \tau_j = \tau$ or $\tau_j \to \tau\left(g_{\mathbf{Q}}\left(\widehat{V} - \sigma - \lambda\right)\right)$.

Now we give some inclusion relations between $g_{\mathbf{Q}}(\widehat{st} - \sigma - \lambda)$ and $g_{\mathbf{Q}}\left(\widehat{V} - \sigma - \lambda\right)$.

Theorem 3.7. *The following statements hold:*

(i) If $\tau_j \to \tau\left(g_{\mathbf{Q}}\left(\widehat{V} - \sigma - \lambda\right)\right)$, then $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st} - \sigma - \lambda)\right)$,

(ii) If $x \in \ell_\infty^{\mathbf{Q}}$ and $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st} - \sigma - \lambda)\right)$, then $\tau_j \to \tau\left(g_{\mathbf{Q}}\left(\widehat{V} - \sigma - \lambda\right)\right)$, and hence $\tau_j \to \tau\left([C, 1, g_{\mathbf{Q}}]\right)$.

Proof. (i) Let

$$K_\varepsilon(p) = \left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \leq j\ (j \in \mathbb{N}) : \left|g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}, \tau\right)\right| \geq \varepsilon\right\},$$

Now since $\tau_j \to \tau\left(g_{\mathbf{Q}}\left(\widehat{V} - \sigma - \lambda\right)\right)$ then, for all $0 \prec q \in \mathbf{Q}$, we have

$$\begin{aligned}
0 \longleftarrow & \frac{p!}{(\lambda_j)^p} \sum_{(i_1, i_2, ..., i_p) \in I_j^p} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}, \tau\right) \\
\geq & \frac{p!}{(\lambda_j)^p} \left\{ \sum_{\substack{i_1, i_2, ..., i_p = 1 \\ (i_1, i_2, ..., i_p) \notin K_\varepsilon(p)}}^{j} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}, \tau\right) \right. \\
& \left. + \sum_{\substack{i_1, i_2, ..., i_p = 1 \\ (i_1, i_2, ..., i_p) \in K_\varepsilon(p)}}^{j} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}, \tau\right) \right\} \\
\geq & \frac{p!}{(\lambda_j)^p} \varepsilon \left|K_\varepsilon(p)\right|, \text{ as } n \to \infty.
\end{aligned}$$

Hence, we have $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st}-\sigma-\lambda)\right)$.

(ii) Suppose that $\left\{\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right\}$ is bounded and (σ,λ)-statistically $g_{\mathbf{Q}}$-convergent to τ in $(X,g_{\mathbf{Q}})$. Since $\left\{\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right\}\in\ell_\infty^{\mathbf{Q}}$, there exists $M>0$ such that $\left|g_{\mathbf{Q}}\left(\tau_{i_1},\tau_{i_2},...,\tau_{i_p},l\right)\right|^P\le M$ for all p and m.We have

$$\begin{aligned}
&\frac{p!}{(\lambda_j)^p}\sum_{i=1}^{j} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)},\tau\right)\\
&=\frac{p!}{(\lambda_j)^p}\sum_{\substack{i_1,i_2,...,i_p=1\\(i_1,i_2,...,i_p)\notin K_\varepsilon(p)}}^{j} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)},\tau\right)\\
&+\frac{p!}{(\lambda_j)^p}\sum_{\substack{i_1,i_2,...,i_p=1\\(i_1,i_2,...,i_p)\in K_\varepsilon(p)}}^{j} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)},\tau\right)\\
&=S_1(j)+S_2(j).
\end{aligned}$$

Now if $\{i_1,i_2,\dots,i_p\}\notin K_\varepsilon(p)$ then $S_1(j)<\varepsilon$. For $\{i_1,i_2,\dots,i_p\}\in K_\varepsilon(p)$, we have

$$S_2(j)\le\sup\left|g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)},\tau\right)\right|\left(\frac{|K_\varepsilon(p)|}{j}\right)\le M\frac{p!\,|K_\varepsilon(p)|}{(\lambda_j)^p}\to 0$$

as $j\to\infty$. As a result, we get $\tau_j\to\tau\left(g_{\mathbf{Q}}\left(\widehat{V}-\sigma-\lambda\right)\right)$. □

Theorem 3.8. *If*

$$\liminf\frac{(\lambda_j)^p}{j^p}>0 \tag{3.1}$$

then

$$\tau_j\to\tau\left(g_{\mathbf{Q}}(st-\lambda)\right)\ \textit{implies}\ \tau_j\to\tau\left(g_{\mathbf{Q}}(\widehat{st}-\sigma-\lambda)\right).$$

Proof. For given $\varepsilon>0$, we see that

$$\begin{aligned}
&\Big\{(i_1,i_2,\dots,i_p)\in I_j^p,\ i_1,i_2,\dots,i_p\le j\ (j\in\mathbb{N}):\\
&\qquad\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|=\varepsilon\Big\}\\
&\supset\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p,\ i_1,i_2,\dots,i_p\le j\ (j\in\mathbb{N}):\\
&\qquad\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|=\varepsilon\Big\}
\end{aligned}$$

where $q=\frac{\varepsilon}{2}+i\frac{\varepsilon}{2}+j\frac{\varepsilon}{2}+k\frac{\varepsilon}{2}$. This gives

$$\begin{aligned}
&\tfrac{p!}{j^p}\left|\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p,\ i_1,i_2,\dots,i_p\le j\ (j\in\mathbb{N}):\right.\\
&\qquad\left.\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|=\varepsilon\Big\}\right|\\
&\ge\tfrac{p!}{j^p}\left|\Big\{(i_1,i_2,\dots,i_p)\in I_j^p,\ i_1,i_2,\dots,i_p\le j\ (j\in\mathbb{N}):\right.\\
&\qquad\left.\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|=\varepsilon\Big\}\right|\\
&\ge\tfrac{(\lambda_j)^p}{j^p}.\tfrac{p!}{(\lambda_j)^p}\left|\Big\{(i_1,i_2,\dots,i_p)\in I_j^p,\ i_1,i_2,\dots,i_p\le j\ (j\in\mathbb{N})\right.\\
&\qquad\left.\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},...,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|=\varepsilon\Big\}\right|.
\end{aligned}$$

Taking the limits as $j\to\infty$ and employ the reality that $\liminf\frac{(\lambda_j)^p}{j^p}>0$, we get

$$\tau_j\to\tau\left(g_{\mathbf{Q}}(st-\lambda)\right)\Rightarrow\tau_j\to\tau\left(g_{\mathbf{Q}}(\widehat{st}-\sigma-\lambda)\right).$$

This finalizes the proof. □

If we take $\sigma(j) = j+1$ in the Definitions 15, 16 and 17, then we have the following:

Definition 18. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) σ-statistically converges to τ if for every $q \prec \mathbf{Q}$ with $0 \prec q$ such that*

$$\lim_{j\to\infty} \tfrac{p!}{j^p} \left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le j\ (j \in \mathbb{N}) : \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q| \right\}\right| = 0,$$

uniformly in m. In this case, we write $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st} - \sigma)\right)$.

Definition 19. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) almost statistically converges to τ provided that for every $q \prec \mathbf{Q}$ with $0 \prec q$,*

$$\lim_{j\to\infty} \tfrac{p!}{j^p} \left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le j\ (j \in \mathbb{N}) : \left| g_{\mathbf{Q}}\left(\tau_{i_1+m}, \tau_{i_2+m}, \ldots, \tau_{i_p+m}, \tau\right)\right| \ge |q| \right\}\right| = 0,$$

uniformly in m. In this case, we write $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st})\right)$.

Definition 20. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) almost λ-statistically converges to τ provided that for every $q \prec \mathbf{Q}$ with $0 \prec q$,*

$$\lim_{j\to\infty} \tfrac{p!}{(\lambda_j)^p} \left|\left\{(i_1, i_2, \ldots, i_p) \in I_j^p,\ i_1, i_2, \ldots, i_p \le j\ (j \in \mathbb{N}) : \left| g_{\mathbf{Q}}\left(\tau_{i_1+m}, \tau_{i_2+m}, \ldots, \tau_{i_p+m}, \tau\right)\right| \ge |q| \right\}\right| = 0,$$

uniformly in m. In this case, we express $\tau_j \to \tau\left(g_{\mathbf{Q}}(\widehat{st} - \lambda)\right)$.

If $\lambda_j = j$, for all j, then $g_{\mathbf{Q}}(\widehat{st} - \lambda)$ is same as $g_{\mathbf{Q}}(\widehat{st})$.

Definition 21. *Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $(\tau_j) \subseteq X$ be a sequence. (τ_j) is said to be strongly almost λ-summable to a number τ if*

$$\lim_{j\to\infty} \frac{p!}{(\lambda_j)^p} \sum_{(i_1, i_2, \ldots, i_p) \in I_j^p} g_{\mathbf{Q}}\left(\tau_{i_1+m}, \tau_{i_2+m}, \ldots, \tau_{i_j+m}, \tau\right) = 0,$$

uniformly in $m = 1, 2, 3, \ldots$, (denoted by $\tau_j \to \tau\left(\left[g_{\mathbf{Q}}(\widehat{V} - \lambda)\right]\right)$).

Remark. *Similar inclusions to Theorems 3.7 and 3.8 hold between strongly λ-almost statistically convergent and almost λ-statistically convergent in $(X, g_{\mathbf{Q}})$.*

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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Ömer Kişi
Department of Mathematics, Bartin University Bartin, Turkey
E-mail address: okisi@bartin.edu.tr

Burak Çakal
Department of Mathematics, Bartin University Bartin, Turkey
E-mail address: burakcakal@gmail.com

Mehmet Gürdal
Department of Mathematics, Suleyman Demirel University, 32260, Isparta, Turkey
E-mail address: gurdalmehmet@sdu.edu.tr