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BOUNDARY VALUE PROBLEMS FOR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS ASSOCIATED WITH ATANGANA-BALEANU INTEGRAL

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Abstract. The aim of the present work is to investigate the existence and uniqueness of solutions to fractional differential equations involving the Caputo Hadamard fractional operator of order $1 < \lambda \leq 2$ with impulsive boundary conditions and Atangana-Baleanu(AB) fractional integral. To establish our main results the Banach Contraction Principle and Schauder's fixed point theorem are used. Also, some examples are given to illustrate our main results.

1. Introduction

A more extensive and comprehensive type of differential equation theory is known as fractional differential equation theory. There are several uses for boundary value problems of fractional order in applied physics, biology, engineering, and chemical background. By using initial and boundary conditions a lot of research has been done by reseachers to the differential equation of arbitrary order. For more details see ([1],[2],[6],[10],[12]).

Several researchers have recently studied the differential equations of fractional order using the Riemann-Liouville and Caputo fractional derivatives ([3], [20], [18]). Furthermore to the Riemann-Liouville and Caputo derivatives, the Hadamard fractional derivative is another kind of fractional derivative that has been addressed in the literature. It includes an arbitrary logarithmic function which makes it different from the previous ones; see ([12],[13],[14],[18]). By using Caputo and Hadamard fractional derivatives, Jarad et al. [11] defined the Caputo-Hadamard fractional derivative. For detailed study of the Hadamard integral and derivative see ([8],[14],[22]).

The past few decades, it has been seen an evolution in the research of impulsive boundary value problems. Few of the established outcomes from integral boundary value problems involving fractional derivatives of the Caputo type have been given

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by Tian and Bai [19]. The existence and uniqueness results have been developed using the fixed point theorem. There are several research papers related to impulsive boundary value problems available in the literature; see ([5],[7],[9],[14],[20],[21]).

The latest study has been focused on impulsive differential equations with Hadamard and Caputo-Hadamard derivatives. The Caputo-Hadamard fractional differential equation in the following form with the impulsive boundary condition was examined by the authors of [16]:

$$\begin{cases} {}^{CH}D^{\lambda}_{\sigma_k}\varphi(\sigma) = f(\sigma, \varphi(\sigma)),\ \sigma, \sigma_k \in [1, e],\ \sigma \neq \sigma_k \\ \Delta\varphi(\sigma_k) = I_k(\varphi(\sigma_k)),\ k = 1, 2, \cdots, m \\ \Delta\delta\varphi(\sigma_k) = \bar{I}_k(\varphi(\sigma_k)) \\ \varphi(1) = h(\varphi),\ \varphi(e) = g(\varphi). \end{cases}$$

The Hadamard fractional differential equation for the impulsive multi-order was studied by W. Yukunthorn et.al. in [20]:

$$\begin{cases} {}^{C}D^{\lambda_k}_{\sigma_k}\varphi(\sigma) = f(\sigma, \varphi(\sigma)),\ \sigma, \sigma_k \in [1, e],\ \sigma \neq \sigma_k \\ \Delta_1\varphi(\sigma_k) = \phi I_k(\varphi(\sigma_k)),\ k = 1, 2, \cdots, m \\ \alpha_1\varphi(t_0) + \beta_1\varphi(T) = \sum_{i=0}^{M} \gamma 1_i F^{R_i}_{\sigma}{}_i\varphi(\sigma_{i+1}). \end{cases}$$

The Caputo fractional differential equation involving the Riemann-Liouville integral was studied by the authors in [17] :

$$\begin{cases} {}^{C}D^{\alpha}_{a+}\varphi(\sigma) = f(\sigma, \varphi(\sigma), I^{\alpha}_{a+}\varphi(\sigma)),\ t \in [a, b], \\ \varphi(a) = \varphi_a. \end{cases}$$

In [15] by using non instantaneous impulsive boundary conditions the authors described the ψ-Caputo fractional integro differential equations of the form

$$\begin{cases} {}^{C}D^{p:\psi}\varphi(\sigma) = f(\sigma, \varphi(\sigma), \beta\varphi(\sigma)),\ \sigma \in (s_i, \sigma_{i+1}],\ 0 < p < 1, \\ \varphi(\sigma) = H_i(\sigma, \varphi(\sigma)),\ k = 1, 2, \cdots, m \\ \alpha_1\varphi(t_0) + \beta_1\varphi(T) = c. \end{cases}$$

Another extended form of the Caputo-Hadamard fractional differential equation with impulsive boundary conditions is studied by the authors in [4]:

$$\begin{cases} {}^{CH}D^{p}\varphi(\sigma) = f(\sigma, \varphi(\sigma), \beta\varphi(\sigma)),\ \sigma \in [1, T],\ 0 < p < 1, \\ w(\sigma_K^+) = w(\sigma_K^-) + y_k,\ y_k \in \mathbb{R},\ k = 1, 2, \cdots, m \\ w(1) = 0, \alpha_H F^2 w(n) + \beta^G_H D^R w(T) = \lambda,\ 2, \gamma \in (0, 1]. \end{cases}$$

Motivated by the above results, here we have considered the Caputo-Hadamard fractional differential equation involving Atangana-Baleanu fractional derivative with impulsive boundary conditions:

$$\begin{cases} {}^{CH}D^{\lambda}_{\sigma_k}\varphi(\sigma) = f(\sigma, \varphi(\sigma), {}^{AB}I^{\mu}_{i}(\varphi(\sigma))),\ \sigma, \sigma_k \in [t_0, T],\ \sigma \neq \sigma_k \\ \Delta\varphi(\sigma_k) = I_k(\varphi(\sigma_k)),\ k = 1, 2, \cdots, m \\ \Delta\delta\varphi(\sigma_k) = \bar{I}_k(\varphi(\sigma_k)) \\ \varphi(t_0) = h(\varphi),\ \varphi(T) = g(\varphi), \end{cases} \tag{1.1}$$

where ${}^{CH}D^{\lambda}_{\sigma_k}$ is the Caputo-Hadamard fractional derivative of order $1 < \lambda \leq 2$ and ${}^{AB}I^{\mu}_{i}$ is the Atangana-Baleanu fractional integral which is defined as

$$({}^{AB}I^{\mu}_{i}w)(\varphi) = \big(\frac{1-\mu}{\Delta\mu}\big)w(\varphi) + \big(\frac{\mu}{\Delta\mu}\big)^{RL}I^{\mu}_{i}w.$$

Also, let $f : [t_0, T] * \mathbb{R} * \mathbb{R} \to \mathbb{R}$ be a given continuous function and $\Delta\varphi(\sigma_k) = \varphi(\sigma_k^+) - \varphi(\sigma_k^-)$, $\Delta\delta\varphi(\sigma_k) = \delta\varphi(\sigma_k^+) - \delta\varphi(\sigma_k^-)$, where $\delta = \sigma\frac{d}{d\sigma}$. Also, $\varphi(\sigma_k^+)$ and $\varphi(\sigma_k^-)$ are the right limit and left limit of $\varphi(\sigma)$ at $\sigma = \sigma_k$ respectively.

This manuscript is organized as follows. In Section 2, some definitions and lemmas are given which are used in the main results. The results based on the theorems Banach Contraction Principle and Schauder's fixed point are given in section 3. To illustrate the application of our main results some examples are explained in section 4. In section 5 conclusions are presented.

2. Preliminaries

In this section, some basic definitions and lemmas are given which are used in our main results.

Let us consider $J = [t_0, T],\ t_0 < t_1 < t_2 \cdots < t_m < t_{m+1} = T,\ J_0 = [t_0, t_1]$, $J_1 = [t_1, t_2], \cdots\ J_m = [t_m, T]$, and the Banach space

$$PC(J, \mathbb{R}) = \{u : J \to \mathbb{R}; u(.) \in C((t_k.t_{k+1}], \mathbb{R}), k = 0, 1, 2, \cdots m\}$$

and $u(t_k^+),\ u(t_k^-)$ exist with $u(t_k^-) = u(t_k),\ k = 1, 2, \cdots, m$, with the norm $||u||_{PC} := \sup\{|u(t)| : t \in J\}$.

Definition 2.1. *The Riemann-Liouville fractional derivative of order $\eta > 0$ for a continuous function φ is defined by*

$$D^{\eta}_{0^+}\varphi(\sigma) = \frac{1}{\Gamma(\mu-\eta)}\left(\frac{\delta}{\delta t}\right)^{\mu}\int_0^{\sigma}(\sigma - s)^{\mu-\eta-1}f(s)\delta s,\ \mu - 1 < \eta < \mu.$$

Definition 2.2. *The Riemann-Liouville fractional integral of order $\eta > 0$ for a continuous function φ is defined by*

$$I^{\eta}_{0^+}\varphi(\sigma) = \frac{1}{\Gamma(\eta)}\int_0^{\sigma}(\sigma - s)^{\eta-1}f(s)\delta s,$$

where Γ is defined by $\Gamma(\eta) = \int_0^{\infty} e^{-x}x^{\eta-1}dx$.

Definition 2.3. *The Caputo derivative of order η for the function $\varphi : [0, \infty) \to \mathbb{R}$, is defined by*

$${}^{c}D^{\eta}\varphi(\sigma) = \frac{1}{\Gamma(\mu-\eta)}\int_0^{\sigma}\frac{\varphi^{\mu}(s)}{(\sigma - s)^{\eta+1-\mu}}\delta s,\ \sigma > 0,\ \mu - 1 < \eta < \mu.$$

Definition 2.4. *The Hadamard derivative of fractional order η for the function $\varphi : [a, b] \to \mathbb{R},\ 0 < a < b$ is defined as*

$${}^{H}D^{\eta}_{a}\varphi(\sigma) = \frac{1}{\Gamma(\mu-\eta)}\left(\sigma\frac{\delta}{\delta\sigma}\right)^{\mu}\int_a^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\mu-\eta-1}\frac{\varphi(s)}{s}\delta s,\ \mu{-}1 < \eta \leq \mu,\ \mu = [\eta]{+}1,$$

where $[\eta]$ represents the integer part of the real number η.

Definition 2.5. *The Hadamard integral of fractional order* η *for the function* $\varphi : [a,b] \to \mathbb{R}$ *is defined as*

$$^{H}I_{a}^{\eta}\varphi(\sigma) = \frac{1}{\Gamma(\eta)}\int_{a}^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\eta-1}\frac{\varphi(s)}{s}\delta s.$$

Definition 2.6. *The Caputo-Hadamard derivative of fractional order* η *for the function* $\varphi : [a,b] \to \mathbb{R}$, $0 < a < b$ *is defined as*

$$^{CH}D_{a}^{\eta}\varphi(\sigma) = \frac{1}{\Gamma(\mu-\eta)}\int_{a}^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\mu-\eta-1}\delta^{\mu}\frac{g(s)}{s}ds,\ \ \mu-1<\eta<\mu,\ \ \mu=[\eta]+1,$$

where $\delta = (\sigma\frac{d}{d\sigma})$, *and* $[\eta]$ *represents the integer part of the real number* η.

Lemma 2.1. [11] *Let* $y \in AC_{\delta}^{n}[t_0,T]$ *or* $C_{\delta}^{n}[t_0,T]$ *and* $\alpha \in \mathbb{C}$. *Then*

$$^{H}I_{a}^{\alpha}(^{CH}D_{a}^{\alpha}y(x)) = y(x) - \sum_{k=0}^{n-1}c_k\left(\log\frac{x}{a}\right)^{k}$$

Lemma 2.2. *Let* $1 < \lambda \leq 2$ *and* $a \in C(J,\mathbb{R})$. *Then the nonlinear system*

$$\begin{cases} ^{CH}D_{\sigma_k}^{\lambda}\varphi(\sigma) = a(\sigma),\ \ \sigma,\sigma_k \in [t_0,T],\ \ \sigma \neq \sigma_k \\ \Delta\varphi(\sigma_k) = I_k(u(\sigma_k)), k = 1,2,\cdots,m \\ \Delta\delta\varphi(\sigma_k) = \bar{I}_k(u(\sigma_k)) \\ \varphi(t_0) = h(\varphi),\ \ \varphi(T) = g(\varphi) \end{cases}$$

is equivalent to the following integral equation

$$\varphi(\sigma) = \begin{cases} c\ \log(\sigma) + h(\varphi) +\ ^{H}I_{\sigma_0}^{\lambda}a(\sigma);\ \ \sigma \in J_0 \\ c\ \log(\sigma) + h(\varphi) +^{H} I_{\sigma_k}^{\lambda}a(\sigma) + \sum_{j=1}^{k}{}^{H}I_{\sigma_{j-1}}^{\lambda}a(\sigma_j) \\ +\sum_{j=1}^{k}I_j(\varphi(\sigma_j)) + \sum_{j=1}^{k}\left(\log\frac{\sigma}{\sigma_j}\right)^{H}I_{\sigma_{j-1}}^{\lambda-1}a(\sigma_j) \\ +\sum_{j=1}^{k}\left(\log\frac{\sigma}{\sigma_j}\right)\bar{I}_j(\varphi(\sigma_j));\ \ \sigma \in J_k,\ \ k = 1,2,\cdots,m \end{cases}$$

where

$$c = \frac{1}{\left(\log\frac{T}{\sigma_0}\right)}\Bigg(g(\varphi) - h(\varphi) - \sum_{j=1}^{m+1}{}^{H}I_{\sigma_{j-1}}^{\lambda}a(\sigma_j) - \sum_{j=1}^{m}I_j(\varphi(\sigma_j))$$
$$-\sum_{j=1}^{m}\left(\log\frac{T}{\sigma_j}\right)^{H}I_{\sigma_{j-1}}^{\lambda-1}a(\sigma_j) - \sum_{j=1}^{m}\left(\log\frac{T}{\sigma_j}\right)\bar{I}_j(\varphi(\sigma_j))\Bigg).$$

Proof. By Lemma 2.1, for $\sigma \in J_0 = [\sigma_0,\sigma_1]$, we have

$$^{H}I_{\sigma_0}^{\lambda}\big(^{CH}D_{\sigma_0}^{\lambda}\varphi(\sigma)\big) = \varphi(\sigma) - \sum_{k=0}^{1}c_k\left(\log\frac{\sigma}{\sigma_0}\right)^{k} = \varphi(\sigma) - c_0 - c_1\ \log\big(\frac{\sigma}{\sigma_0}\big)$$

$$\varphi(\sigma) = {}^{H}I_{\sigma_0}^{\lambda}a(\sigma) + c_0 + c_1\ \log(\frac{\sigma}{\sigma_0}) = \frac{1}{\Gamma(\lambda)}\int_{\sigma_0}^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\lambda-1}a(s)\frac{ds}{s} + c_0 + c_1\ \log(\frac{\sigma}{\sigma_0})$$

and

$$\delta\varphi(\sigma) = \sigma\frac{d}{d\sigma}[\varphi(\sigma)] = \sigma\left[\frac{1}{\Gamma(\lambda-1)}\int_{\sigma_0}^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\lambda-2}a(s)\frac{ds}{s}\frac{1}{\sigma} + c_1\frac{1}{\sigma}\right] = {}^{H}I_{\sigma_0}^{\lambda-1}a(\sigma) + c_1.$$

As $\varphi(1) = h(\varphi)$ implies $c_0 = h(\varphi)$, it follows that

$$\varphi(\sigma) = {}^{H}I^{\lambda}_{\sigma_0}a(\sigma) + h(\varphi) + c_1 \log(\sigma) = c\ \log(\sigma) + h(\varphi) + {}^{H}I^{\lambda}_{\sigma_0}a(\sigma),$$

where $c = c_1$.

Now for $\sigma \in J_1 = (\sigma_1, \sigma_2]$, we have

$$ {}^{H}I^{\lambda}_{\sigma_1}a(\sigma)({}^{CH}D^{\lambda}_{\sigma_1})\varphi(\sigma) = \varphi(\sigma) - d_0 - d_1\bigg(\log\frac{\sigma}{\sigma_1}\bigg),$$

$$\varphi(\sigma) = {}^{H}I^{\lambda}_{\sigma_1}a(\sigma) + d_0 + d_1\bigg(\log\frac{\sigma}{\sigma_1}\bigg)$$

and

$$\delta\varphi(\sigma) = \sigma\frac{d}{d\sigma}(\varphi(\sigma)) = \sigma\left[\frac{1}{\sigma}\frac{1}{\Gamma(\lambda-1)}\int_{\sigma_1}^{\sigma}\bigg(\log\frac{\sigma}{s}\bigg)^{\lambda-2}a(s)\frac{ds}{s} + \frac{d_1}{\sigma}\right] = {}^{H}I^{\lambda-1}_{\sigma_1}a(\sigma) + d_1.$$

Also, $\varphi(\sigma_1^+) - \varphi(\sigma_1^-) = \Delta\varphi(\sigma_1) = I_1(\varphi(\sigma_1))$ implies

$$I_1(\varphi(\sigma_1)) = d_0 - {}^{H}I^{\lambda}_{\sigma_0}a(\sigma_1) - h(\varphi) - c\ \log(\sigma_1)$$

$$d_0 = {}^{H}I^{\lambda}_{\sigma_0}a(\sigma_1) + h(\varphi) + c\ \log(\sigma_1) + I_1(\varphi(\sigma_1))$$

and $\delta\varphi(\sigma_1^+) - \delta\varphi(\sigma_1^-) = \Delta\delta\varphi(\sigma_1) = \bar{I}_1(\varphi(\sigma_1))$ implies

$$\bar{I}_1(\varphi(\sigma_1)) = d_1 - {}^{H}I^{\lambda-1}_{\sigma_0}a(\sigma_1) - c,$$

$$d_1 = \bar{I}_1(\varphi(\sigma_1)) + {}^{H}I^{\lambda-1}_{\sigma_0}a(\sigma_1) + c.$$

Applying the above arguments, we get

$$\varphi(\sigma) = {}^{H}I^{\lambda}_{\sigma_1}a(\sigma) + {}^{H}I^{\lambda}_{\sigma_0}a(\sigma_1) + \bigg(\log\tfrac{\sigma}{\sigma_1}\bigg){}^{H}I^{\lambda-1}_{\sigma_0}a(\sigma_1) + \bigg(\log\tfrac{\sigma}{\sigma_1}\bigg)\bar{I}_1(\varphi(\sigma_1)) + h(\varphi)$$
$$+ I_1(\varphi(\sigma_1)) + c\ \log(\sigma).$$

On repeating the same process, for $\sigma \in J_m = (\sigma_m, T]$, we get

$$\varphi(\sigma) = c\ \log(\tfrac{\sigma}{\sigma_0}) + h(\varphi) + \textstyle\sum_{j=1}^{m} I_j(\varphi(\sigma_j)) + \sum_{j=1}^{m}\bigg(\log\tfrac{\sigma}{\sigma_1}\bigg)\bar{I}_1(\varphi(\sigma_1)) + {}^{H}I^{\lambda}_{\sigma_m}a(\sigma)$$
$$+ \textstyle\sum_{j=1}^{m} {}^{H}I^{\lambda}_{\sigma_{j-1}}a(\sigma_j) + \sum_{j=1}^{m}\bigg(\log\tfrac{\sigma}{\sigma_j}\bigg){}^{H}I^{\lambda-1}_{\sigma j-1}a(\sigma_j),$$

using the condtion $\varphi(T) = g(\varphi)$, we have

$$g(\varphi) = c\log(\tfrac{T}{\sigma_0}) + h(\varphi) + \textstyle\sum_{j=1}^{m} I_j(u(\sigma_j)) + \sum_{j=1}^{m}\bigg(\log\tfrac{T}{\sigma_j}\bigg)\bar{I}_j(\varphi(\sigma_j)) + {}^{H}I^{\lambda}_{\sigma_m}a(T)$$
$$+ \textstyle\sum_{j=1}^{m} {}^{H}I^{\lambda}_{\sigma_{j-1}}a(\sigma_j) + \sum_{j=1}^{m}\bigg(\log\tfrac{T}{\sigma_j}\bigg){}^{H}I^{\lambda-1}_{\sigma j-1}a(\sigma_j).$$

This implies that

$$c = \frac{1}{\left(\log\frac{T}{\sigma_0}\right)}\bigg(g(\varphi) - h(\varphi) - \textstyle\sum_{j=1}^{m} I_j(\varphi(\sigma_j)) - \sum_{j=1}^{m}\bigg(\log\tfrac{T}{\sigma_j}\bigg)\bar{I}_j(\varphi(\sigma_j))$$
$$+ \textstyle\sum_{j=1}^{m+1} {}^{H}I^{\lambda}_{\sigma_{j-1}}a(\sigma_j) - \sum_{j=1}^{m}\bigg(\log\tfrac{T}{\sigma_j}\bigg){}^{H}I^{\lambda-1}_{\sigma j-1}a(\sigma_j)\bigg). \qquad \square$$

3. Main Results

Now we construct an operator $\aleph : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ such that

$$\aleph(\varphi(\sigma)) = a\ \log(\sigma) + (1-\log(\sigma))h(\sigma) + \log(\sigma)g(\varphi) + \sum_0^{\sigma_k} I_k(\varphi(\sigma_k))$$
$$+ \sum_0^{\sigma_k}\left(\log\frac{\sigma}{\sigma_k}\right)\bar{I}_k(\varphi(\sigma_k)) + \sum_0^{\sigma_k}\frac{1}{\Gamma(\lambda)}\int_{\sigma_k-1}^{\sigma_k}\left(\log\frac{\sigma}{s}\right)^{\lambda-1} f(s,\varphi(s),^{AB} I_i^{\mu}(\varphi(s)))\frac{ds}{s}$$
$$+ \sum_0^{\sigma_k}\frac{1}{\Gamma(\lambda)}\int_{\sigma_k}^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\lambda-1} f(s,\varphi(s),^{AB} I_i^{\mu}(\varphi(s)))\frac{ds}{s}$$
$$+ \sum_0^{\sigma_k}\left(\log\frac{\sigma}{\sigma_j}\right)\frac{1}{\Gamma(\lambda-1)}\int_{\sigma_{k-1}}^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\lambda-2} f(s,\varphi(s),^{AB} I_i^{\mu}(\varphi(s)))\frac{ds}{s}$$

where

$$a = \frac{1}{\left(\log\frac{T}{\sigma_0}\right)}\Bigg(-\sum_{k=1}^{m} I_k(\varphi(\sigma_k)) - \sum_{k=1}^{m}\left(\log\frac{T}{\sigma_k}\right)\frac{1}{\Gamma(\lambda-1)}\int_{\sigma_{k-1}}^{\sigma_k}\left(\log\frac{\sigma}{s}\right)^{\lambda-2} f(s,\varphi(s),^{AB} I_i^{\mu}(\varphi(s)))\frac{ds}{s}$$
$$-\sum_{k=1}^{m+1}\frac{1}{\Gamma(\lambda)}\int_{\sigma_{k-1}}^{\sigma_k}\left(\log\frac{\sigma_k}{s}\right)^{\lambda-2} f(s,\varphi(s),^{AB} I_i^{\mu}(\varphi(s)))\frac{ds}{s} - \sum_{k=1}^{m}\left(\log\frac{T}{\sigma_k}\right)\bar{I}_k(\varphi(\sigma_k))\Bigg).$$

Here, we make a few suitable assumptions for establishing our main outcomes.

(H_1) Suppose $P_1 > 0$ is a constant and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(\sigma, s_1, r_1) - f(\sigma, s_2, r_2)| \le P_1|s_1 - s_2| + P_2|r_1 - r_2|,$$

for all $\sigma \in J$ and all $s_1, r_1, s_2, r_2 \in \mathbb{R}$.

(H_2) Also let $P_3, P_4 > 0$ be constants such that

$$|I_k(\varphi_1) - I_k(\varphi_2)| \le P_3|\varphi_1 - \varphi_2|,\ |\bar{I}_k(\varphi_1) - \bar{I}_k(\varphi_2)| \le P_4|\varphi_1 - \varphi_2|,$$

for $\varphi_1, \varphi_2 \in \mathbb{R},\ k = 1, 2, \cdots, m$.

(H_3) There exist constants $P_5, P_6 > 0$ such that

$$|g(\varphi_1) - g(\varphi_2)| \le P_5||\varphi_1 - \varphi_2||_{PC},\ |h(\varphi_1) - h(\varphi_2)| \le P_6||\varphi_1 - \varphi_2||_{PC}.$$

Now, we first apply the Banach Contraction Principle to demonstrate the existence and uniqueness result.

Theorem 3.1. *Suppose that the conditions of* $(H_1), (H_2)$ *and* (H_3) *hold and*

$$\left((P_1 + P_2)\frac{4m+2}{\Gamma(\lambda)} + 2m(P_3 + P_4) + P_5 + P_6\right) < 1.$$

Then the problem given by (1) has a unique solution on $[t_0, T]$.

Proof. Let $\sup_{\sigma\in J}|f(\sigma,0,0)| = Q_1$, $\max_k |I_K(0)| = Q_2$, $\max_k |\bar{I}_K(0)| = Q_3$, $|g(0)| = Q_4$, $|h(0)| = Q_5$ and

$$U_0 = \Big\{\varphi(\sigma) \in PC(J,\mathbb{R}); ||u||_{PC} \le R_0\Big\},$$

where

$$R_0 = \frac{(4m+2)Q_1 + 2m\Gamma(\lambda)(Q_2 + Q_3) + \Gamma(\lambda)(Q_4 + Q_5)}{\Gamma(\lambda) - (4m+2)(P_1 + P_2) - 2m\Gamma(\lambda)(P_3 + P_4) - (P_5 + P_6)}.$$

Firstly we prove that $\aleph$ maps U_0 into itself

$$|a| \le \frac{1}{\Gamma(\lambda)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s} \big)^{\lambda-1} |f(s,\varphi(s),^{AB} I_i^{\mu}(\varphi(s))) - f(s,0,0)| \frac{ds}{s}$$

$$+ \frac{1}{\Gamma(\lambda)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s} \big)^{\lambda-2} |f(s,0,0)| \frac{ds}{s}$$

$$+ \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) \frac{1}{\Gamma(\lambda-1)} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma}{s} \big)^{\lambda-2} |f(s,\varphi(s),^{AB} I_i^{\mu}(\varphi(s))) - f(s,0,0)| \frac{ds}{s}$$

$$+ \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) \frac{1}{\Gamma(\lambda-1)} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma}{s} \big)^{\lambda-2} |f(s,0,0)| \frac{ds}{s}$$

$$+ \sum_{k=1}^{m} |I_k(\varphi(\sigma_k)) - I_k(\varphi(0))| + \sum_{k=1}^{m} |I_k(\varphi(0))|$$

$$+ \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) |\bar{I}_k(\varphi(\sigma_k)) - \bar{I}_k(\varphi(0))| + \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) |\bar{I}_k(\varphi(0))|$$

$$\le \sum_{k=1}^{m+1} \frac{P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1}{\Gamma(\lambda+1)} + \sum_{k=1}^{m+1} \frac{P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1}{\Gamma(\lambda)}$$

$$+ m(P_3||\varphi||_{PC} + Q_2) + m(P_4||\varphi||_{PC} + Q_3)$$

$$\le \frac{2m+1}{\Gamma(\lambda)} (P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1) + m(P_3||\varphi||_{PC} + P_4||\varphi||_{PC} + Q_2 + Q_3).$$

Again

$$|\aleph(\varphi(\sigma))| \le |a| + |h(\varphi)| + |g(\varphi)| + m(P_3||\varphi||_{PC} + Q_2) + m(P_4||\varphi||_{PC} + Q_3)$$

$$+ \frac{P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1}{\Gamma(\lambda+1)} + \sum_{k=1}^{m} \frac{P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1}{\Gamma(\lambda+1)} + \sum_{k=1}^{m} \frac{P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1}{\Gamma(\lambda)}$$

$$\le \frac{2m+1}{\Gamma(\lambda)} (P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1) + 2m(P_3||\varphi||_{PC} + Q_2) + 2m(P_4||\varphi||_{PC} + Q_3)$$

$$+ (P_5||\varphi||_{PC} + Q_4) + (P_6||\varphi||_{PC} + Q_5) + \frac{2m+1}{\Gamma(\lambda)} (P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1)$$

$$\le \frac{4m+2}{\Gamma(\lambda)} (P_1||\varphi||_{PC} + P_2||\varphi||_{PC} + Q_1) + 2m(P_3||\varphi||_{PC} + Q_2) + 2m(P_4||\varphi||_{PC} + Q_3)$$

$$+ (P_5||\varphi||_{PC} + Q_4) + (P_6||\varphi||_{PC} + Q_5)$$

$$\le R_0.$$

This shows the operator $\aleph$ maps U_0 into itself.

Now to prove the map $\aleph$ is a contraction. Consider $\varphi_1, \varphi_2 \in PC(J, \mathbb{R})$ such that for any $\sigma \in J_k$, $K = 1, 2, \cdots, m$, we have

$$|\aleph(\varphi_1(\sigma)) - \aleph(\varphi_2(\sigma))| \le \left(\frac{2m+1}{\Gamma(\lambda+1)} (P_1 + P_2) + \frac{2m+1}{\Gamma(\lambda)} (P_1 + P_2) + 2mP_3 + 2mP_4 + P_5 + P_6 \right) \times ||\varphi_1 - \varphi_2||_{PC}$$

$$\le \left(\frac{4m+2}{\Gamma(\lambda)} (P_1 + P_2) + 2mP_3 + 2mP_4 + P_5 + P_6 \right) ||\varphi_1 - \varphi_2||_{PC}.$$

This shows that the mapping $\aleph$ is a contraction map. So by the Banach fixed point theorem it is clear that the mapping $\aleph$ has a unique fixed point. Hence the problem given by (1.1) has a unique solution on $[t_0, T]$. $\square$

Theorem 3.2. *Suppose f is a continuous function and $a(.) \in L(J, \mathbb{R})$, $s_1 \geq 0$ are any constants such that $|f(\sigma, \varphi, ^{AB} I_i^{\mu}(\varphi(s)))| \leq a(\sigma) + s_1|\varphi|^{\sigma} + s_2|(^{AB} I_i^{\mu}(\varphi(s)))|^{\sigma}$. Also let $I_k, \bar{I}_k, g, h$ be continuous functions such that $s_i > 0$ for i=3,4,5,6 s.t., $|I_k(\varphi)| \leq s_3|\varphi|^{\mu}$, $|\bar{I}_k(\varphi)| \leq s_4|\varphi|^{\nu}$, $|g(\varphi)| \leq s_5|\varphi|^{\lambda}$, $|h(\varphi)| \leq s_6|\varphi|^{\gamma}$, k=1,2,$\cdots$,m, for any $\varphi \in \mathbb{R}$ and for some $0 < \sigma, \mu, \nu, \lambda, \gamma \leq 1$. If*

$$\frac{2(2m+1)(s_1+s_2)}{\Gamma(\lambda)} + 2ms_3 + 2ms_4 + s_5 + s_6 < 1,$$

then the problem given by (1.1) has at least one solution on $[t_0, T]$.

Proof. Firstly we show that the operator $\aleph$ is continuous. Since the functions $f, g, h, I_k, \bar{I}_k$; $k = 1, 2, \cdots, m$, are continuous functions, so the operator $\aleph$ is continuous.

Now we will show that the operator $\aleph$ maps a bounded set into uniformly bounded sets.
Let

$$U_0 = \left\{ \varphi(\sigma) \in PC(J, \mathbb{R}); ||\varphi||_{PC} \leq R_0 \right\},$$

where

$$R_0 = \frac{(4m+2)Q_1 + 2m\Gamma(\lambda)(Q_2+Q_3) + \Gamma(\lambda)(Q_4+Q_5)}{\Gamma(\lambda) - (4m+2)(P_1+P_2) - 2m\Gamma(\lambda)(P_3+P_4) - (P_5+P_6)}.$$

For any arbitrary $\varphi \in U$, we have

$$|a| \leq \sum_{k=1}^{m} |I_k(\varphi(\sigma_k))| + \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) \frac{1}{\Gamma(\lambda-1)} \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma}{s} \right)^{\lambda-2} |f(s, \varphi(s), ^{AB} I_i^{\mu}(\varphi(s)))| \frac{ds}{s}$$

$$+ \sum_{k=1}^{m+1} \frac{1}{\Gamma(\lambda)} \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma_k}{s} \right)^{\lambda-2} |f(s, \varphi(s), ^{AB} I_i^{\mu}(\varphi(s)))| \frac{ds}{s} + \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) |\bar{I}_k(\varphi(\sigma_k))|$$

$$\leq ms_3R^{\mu} + ms_4R^{\nu} + \frac{1}{\Gamma(\lambda)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma_k}{s} \right)^{\lambda-1} a(s) \frac{ds}{s}$$

$$+ \frac{(s_1+s_2)R^{\sigma}}{\Gamma(\lambda)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma_k}{s} \right)^{\lambda-1} \frac{ds}{s} + \frac{1}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma}{s} \right)^{\lambda-2} a(s) \frac{ds}{s}$$

$$+ \frac{(s_1+s_2)R^{\sigma}}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) \frac{1}{\Gamma(\lambda-1)} \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma}{s} \right)^{\lambda-2} \frac{ds}{s}$$

$$\leq \frac{1}{\Gamma(\lambda)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma_k}{s} \right)^{\lambda-1} a(s) \frac{ds}{s}$$

$$+ \frac{1}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \left(\log \frac{T}{\sigma_k} \right) \int_{\sigma_{k-1}}^{\sigma_k} \left(\log \frac{\sigma}{s} \right)^{\lambda-2} a(s) \frac{ds}{s} + \frac{(m+1)(s_1+s_2)R^{\sigma}}{\Gamma(\lambda+1)}$$

$$+ \frac{(ms_1+s_2)R^{\sigma}}{\Gamma(\lambda)} + ms_3R^{\mu} + ms_4R^{\nu}$$

$$\leq \frac{1}{\Gamma(\lambda-1)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \left(\left(\log \frac{\sigma_k}{s} \right)^{\lambda-2} + \left(\log \frac{\sigma_k}{s} \right)^{\lambda-2} \right) a(s) \frac{ds}{s}$$

$$+ \frac{(2m+1)(s_1+s_2)R^{\sigma}}{\Gamma(\lambda)} + ms_3R^{\mu} + ms_4R^{\nu}$$

$= \frac{2}{\Gamma(\lambda-1)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} a(s) \frac{ds}{s} + \frac{((2m+1)(s_1+s_2)R^{\sigma}}{\Gamma(\lambda)} + ms_3R^{\mu} + ms_4R^{\nu}$

and

$|\aleph(\varphi(\sigma))| \leq |a| + |g(\varphi)| + |h(\varphi)| + \sum_{j=1}^{m} |I_k(\varphi(\sigma_k))| + \sum_{j=1}^{m} \Bigg(\log \frac{\sigma}{\sigma_k}\Bigg) |\bar{I}_k(\varphi(\sigma_k))|$

$+ \frac{1}{\Gamma(\lambda)} \int_{\sigma_k}^{\sigma} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} a(s) \frac{ds}{s} + \frac{(s_1+s_2)R^{\sigma}}{\Gamma(\lambda)} \int_{\sigma_k}^{\sigma} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} a(s) \frac{ds}{s}$

$+ \frac{1}{\Gamma(\lambda)} \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} a(s) \frac{ds}{s} + \frac{(s_1+s_2)R^{\sigma}}{\Gamma(\lambda)} \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} \frac{ds}{s}$

$+ \frac{1}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \big(\log \frac{\sigma}{\sigma_k}\big) \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma}{s}\big)^{\lambda-2} a(s) \frac{ds}{s}$

$+ \frac{(s_1+s_2)R^{\sigma}}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \big(\log \frac{\sigma}{\sigma_k}\big) \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma}{s}\big)^{\lambda-2} a(s) \frac{ds}{s}$

$\leq |a| + ms_5R^{\lambda} + ms_6R^{\gamma} + ms_3R^{\mu} + ms_4R^{\nu} + \frac{(m+1)(s_1+s_2)R^{\sigma}}{\Gamma(\lambda+1)} + \frac{m(s_1+s_2)R^{\sigma}}{\Gamma(\lambda+1)}$

$+ \frac{1}{\Gamma(\lambda)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} a(s) \frac{ds}{s}$

$+ \frac{1}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \Bigg(\log \frac{\sigma}{\sigma_k}\Bigg) \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} a(s) \frac{ds}{s}$

$\leq \frac{2}{\Gamma(\lambda-1)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} a(s) \frac{ds}{s} + \frac{(2m+1)(s_1+s_2)R^{\sigma}}{\Gamma(\lambda+1)} + ms_3R^{\mu}$

$+ ms_4R^{\nu} + ms_5R^{\lambda} + ms_6R^{\gamma} + ms_3R^{\mu} + ms_4R^{\nu} + \frac{(m+1)(s_1+s_2)R^{\sigma}}{\Gamma(\lambda+1)}$

$+ \frac{m(s_1+s_2)R^{\sigma}}{\Gamma(\lambda+1)} + \frac{2}{\Gamma(\lambda-1)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} a(s) \frac{ds}{s}$

$\leq \frac{4}{\Gamma(\lambda-1)} \sum_{k=1}^{m+1} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} a(s) \frac{ds}{s} + \frac{2(2m+1)(s_1+s_2)R}{\Gamma(\lambda+1)}$

$+ 2ms_3R + 2ms_4R + ms_5R + ms_6R$

$\leq R.$

This shows that $\aleph$ maps U into itself.

Also to show that the operator $\aleph$ maps bounded sets into equicontinuous sets.

Let $U \subset PC(J, \mathbb{R})$ be any arbitrary bounded set. Also consider $N_f = max|\{f(\sigma, \varphi) : \sigma \in J, \varphi \in U\}| + 1$, $\varphi \in U$ such that $\sigma_1, \sigma_2 \in [t_0, T]$ where $\sigma_1 < \sigma_2$. So we obtain

$|a| \leq ms_3R^{\mu} + ms_4R^{\nu} + \frac{N_f}{\Gamma(\lambda)} \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} \frac{ds}{s}$

$+ \frac{N_f}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \big(\log \frac{T}{\sigma_k}\big) \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} |f(s, \varphi(s), {}^{AB}I_i^{\mu}(\varphi(s)))| \frac{ds}{s}$

$+ \leq \frac{N_f(2m+1)}{\Gamma(\lambda)} + ms_3R^{\mu} + ms_4R^{\nu},$

and

$|\aleph(\varphi(\sigma_1)) - \aleph(\varphi(\sigma_2))| \leq |a|(\log(\sigma_1) - \log(\sigma_2)) + s_5R^{\lambda}(\log(\sigma_1) - \log(\sigma_2)) + s_6R^{\lambda}(\log(\sigma_1) - \log(\sigma_2))$

$+ \frac{N_f}{\Gamma(\lambda)} \big| \int_{\sigma_k}^{\sigma_1} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} \frac{ds}{s} - \int_{\sigma_k}^{\sigma_2} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-1} \frac{ds}{s} \big|$

$+ \frac{|\log(\sigma_1) - \log(\sigma_2)| N_f}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} \frac{ds}{s} + (\log(\sigma_1) - \log(\sigma_2)) ms_4R^{\nu}$

$\leq (|a| + s_5R^{\lambda} + s_6R^{\lambda} + ms_6R^{\nu})(\log(\sigma_1) - \log(\sigma_2))$

$+ \frac{N_f}{\Gamma(\lambda)}\left|\left| \int_{\sigma_k}^{\sigma_1} \big(\log(\sigma_1) - \log(s)\big)^{\lambda-1} \frac{ds}{s} - \int_{\sigma_k}^{\sigma_2} \big(\log(\sigma_2) - \log(s)\big)^{\lambda-1} \frac{ds}{s}\right|\right|_{PC}$

$+ \frac{|\log(\sigma_1)-\log(\sigma_2)|N_f}{\Gamma(\lambda-1)} \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} \big(\log \frac{\sigma_k}{s}\big)^{\lambda-2} \frac{ds}{s}$

$\leq (|a| + s_5R^{\lambda} + s_6R^{\lambda} + ms_4R^{\nu})(\log(\sigma_1) - \log(\sigma_2))$

$+ \frac{(\log(\sigma_1)-\log(\sigma_2))N_f}{\Gamma(\lambda)} \sum_{k=1}^{m} \big(\log(\sigma_k) - \log(\sigma_{k-1})\big)^{\lambda-1}$

$+ \frac{N_f}{\Gamma(\lambda+1)}\Big[\big(log(\sigma_1) - \log(\sigma_k)\big)^{\lambda} - \big(\log(\sigma_2) - \log(\sigma_k)\big)^{\lambda}\Big],$

$\to 0$ as $\sigma_1 \to \sigma_2$. This shows that the operator $\aleph$ maps bounded sets into equicontinuous sets of $PC(J, \mathbb{R})$. On combining the complete proof we get $\aleph$ is a completely continuous operator. Therefore by the Schauder's fixed point theorem the operator $\aleph$ has at least one fixed point and hence the problem given by (1.1) has at least one solution in $[t_0, T]$. □

4. Numerical examples

Example 1. Consider the following non-linear boundary value problem

$$\begin{cases} {}^{CH}D_{\sigma_k}^{\frac{3}{2}}\varphi(\sigma) = \frac{\sin(\varphi(\sigma))-\varphi(\sigma)}{2\sigma+1} + \frac{1}{1+\sigma^2}, \ \sigma \in [\frac{3}{2}, 3], \ \sigma \neq \frac{9}{5} \\ \Delta\varphi(\frac{9}{5}) = \frac{1}{10}\varphi(\frac{9}{5}), \\ \Delta\delta\varphi(\frac{9}{5}) = \frac{1}{30}u(\frac{9}{5}), \\ \varphi(\frac{3}{2}) = h(\varphi); \ \varphi(3) = g(\varphi), \end{cases} \tag{4.1}$$

where
$h(\varphi) = \sum_{i=1}^{n} \alpha_i\varphi(\xi_i), \ g(\varphi) = \sum_{i=1}^{n} \beta_i\varphi(\eta_i), \ \xi_i, \eta_i \neq \frac{9}{5} \in (\frac{3}{2}, 3)$

and

$$\sum_{i=1}^{n} \alpha_i < \frac{1}{10}, \ \sum_{i=1}^{n} \beta_i < \frac{1}{10}.$$

Here

$$f(\sigma, \varphi) = \frac{\sin(\varphi(\sigma)) - \varphi(\sigma)}{2\sigma + 1} + \frac{1}{1 + \sigma^2}$$

We can easily see that,

$$|f(\sigma, \varphi_1, {}^{AB}I_i^{\mu}(\varphi_1(s))) - f(\sigma, \varphi_2, {}^{AB}I_i^{\mu}(\varphi_2(s)))| \leq \frac{1}{24}|\varphi_1 - \varphi_2| + \frac{1}{20}|\varphi_1 - \varphi_2|.$$

So $P_1 = \frac{1}{24}, P_2 = \frac{1}{24}$. Similarly $P_3 = \frac{1}{10}$, $P_4 = \frac{1}{30}$, $P_5 = \frac{1}{10}$, $P_6 = \frac{1}{10}$. Also, here $\lambda = \frac{3}{2}$, $\sigma_1 = \frac{9}{5}$, $m = 1$ and therefore

$$\left((P_1 + P_2)\frac{4m + 2}{\Gamma(\lambda)} + 2m(P_3 + P_4) + P_5 + P_6\right) = 0.749 < 1.$$

As all the conditions of Theorem 3.1 are satisfied. So the above boundary value problem has a unique solution on $[t_0, T]$.

Example 2. Consider the following boundary value problem

$$\begin{cases} {}^{CH}D^{\frac{3}{2}}_{\sigma_k}\varphi(\sigma) = \frac{2e^{\varphi(\sigma)}+\sigma^2}{10e^{\sigma}+8}, \ \sigma \in [\frac{4}{3},3], \ \sigma \neq \frac{5}{3} \\ \Delta\varphi(\frac{5}{3}) = \frac{\varphi(\frac{5}{3})}{13+\varphi(\frac{5}{3})}, \\ \Delta\delta\varphi(\frac{5}{7}) = \frac{\varphi(\frac{5}{3})}{25+\varphi(\frac{5}{3})}, \\ \varphi(\frac{4}{3}) = h(\varphi); \ \varphi(3) = g(\varphi), \end{cases} \tag{4.2}$$

where

$$h(\varphi) = \min \frac{(\varphi(\lambda_j))^{\frac{1}{4}}}{15+\varphi(\lambda_j)}, \ g(\varphi) = \max \frac{(\varphi(\beta_j))^{\frac{1}{4}}}{15+\varphi(\beta_j)} \tag{4.3}$$

with $j = 1, 2, \cdots, 10$, $\lambda_j, \beta_j \neq \frac{5}{3} \in (\frac{4}{3}, 3)$. Here $s_1 = \frac{1}{10e^4}$, $s_2 = \frac{1}{10e^4}$, $s_3 = \frac{1}{13}$, $s_4 = \frac{1}{25}$, $s_5 = \frac{1}{15}$, $s_6 = \frac{1}{15}$ and $m = 1$. Clearly

$$\left(\frac{2(2m+1)(s_1+s_2)}{\Gamma(\lambda)} + 2ms_3 + 2ms_4 + s_5 + s_6\right) = 0.368 < 1. \tag{4.4}$$

As all the conditions of Theorem 3.2 are satisfied. So the above boundary value problem has at least one solution on $[t_0, T]$.

5. Conclusion

The investigation of IBVP has advanced in the past few decades. It has also been extremely useful to develop a variety of applied mathematical models of actual processes in applied sciences and engineering. Recently, much attention has been paid to the existence of solutions for fractional differential equations due to its wide application in engineering, economics and other fields. In this work the Banach Contraction Principle and Schauder's fixed point theorem are used to establish the existence and uniqueness of solutions to fractional differential equations involving the Caputo Hadamard fractional operator with impulsive boundary conditions and Atangana-Baleanu(AB) fractional integral. Also to validate our main results some examples are given. In future works, it could be apply for the more complicated fractional systems and uses some other kinds of operators.

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