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A NEW HERMITE-HADAMARD TYPE INCLUSIONS IN THE SETTING OF INTERVAL-VALUED NON-NEWTONIAN CALCULUS

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Abstract. In this paper, we defined and studied a new notion of logarithmically or multiplicatively interval-valued h-preinvex function. We present some new Hermite-Hadamard type inclusions in the setting of interval-valued non-Newtonian calculus. We also established new Hermite-Hadamard type inclusions for the product of multiplicatively interval-valued h-preinvex functions. Then, we use multiplicative twice differentiable functions and we give two new multiplicative integral identities. Next, we derive some new midpoint and trapezoidal type inclusions using h-preinvexity.

1. Introduction

New definitions of differentiation and integration in which the roles of addition and subtraction move to multiplication and division and introduce a new calculus called multiplicative calculus or non-Newton calculus. This mathematical instrument can be particularly helpful for the study of economics and finance and it was initially explored by Grossman and Katz in [13]. In [10], the authors introduced complex multiplicative calculus and, in [12] and [14], properties of stochastic multiplicative calculus have been studied. After the work of [4], many researchers proved different variants of integral inequalities in the setting of multiplicative calculus. In [8], the authors gave some estimates for the midpoint and trapezoidal inequalities in multiplicative calculus, the HermiteHadamard type inequalities for general multiplicative convex and preinvex functions were treated in [3] and [20]. Many results are also studied for multiplicative s-convex, multiplicative s-preinvex functions and multiplicative h-preinvex functions in [21, 22, 23]. Ali et al., in [6], introduced the notions of multiplicative interval-valued integral and established some new Hermite-Hadamard type inequalities for intervalvalued multiplicative convex functions. J. Xiea et al. established, in [26], some new trapezoidal and midpoint type inequalities for multiplicative twice differentiable multiplicative convex functions. A set-valued analysis is a useful tool for dealing with uncertainties and errors in data

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and computations. An interval-valued integral inequality is used to study functions with outputs determined by intervals rather than sets of arbitrary shapes. Based on this notion, A. A. H. Ahmadini et al., in [2], developed Hermite-Hadamard, weighted Fejer, and trapezium type inequalities. Recently, in [1, 5, 11, 24, 25], the authors treated many problems in relation with Hadamard-Hermite inequalities as global stability for Volterra Hadamard random partial fractional integral equations, boundary value problems of fractional differential equations of variable order, existence results for fractional differential inclusions, Caputo Hadamard boundary value problem with integral boundary condition and Caputo Hadamard fractional derivatives.
Inspired by the ongoing studies, we established new Hermite-Hadamard type inclusions for the product of multiplicatively interval-valued h-preinvex functions. Then, we use h-preinvexity to derive some new midpoint and trapezoidal type inclusions.

2. Preliminaries

We now define some existing definitions and results that may lend support to the main findings presented in the article.
A real valued interval X is bounded, closed subset of $\mathbb{R}$ defined by

$$X = [a, b] = \{t \in \mathbb{R}, a \leq t \leq b\},$$

where $a, b \in \mathbb{R}$ and $a \leq b$. The numbers a and b are called the left and the right endpoints of interval X, respectively. When $a = b$, the interval X is said to be degenerate. Also, we call X positive if $a > 0$ or negative if $b < 0$. The set of all closed intervals of $\mathbb{R}$, the sets of all closed positive intervals of $\mathbb{R}$ and closed negative intervals of $\mathbb{R}$ are denoted by $\mathbb{R}_{\mathcal{I}}, \mathbb{R}_{\mathcal{I}}^{+}$ and $\mathbb{R}_{\mathcal{I}}^{-}$, respectively. The Hausdorff-Pompeiu distance between the intervals $X = [a, b]$ and $Y = [c, d]$ is defined by

$$d(X; Y) = d([a, b], [c, d]) = \max\{|a - c|, |b - d|\}.$$

It is known that $(\mathbb{R}_{\mathcal{I}}; d)$ is a complete metric space (see [7]). For the definitions of basic interval arithmetic operations and algebraic properties, we refer the readers to [7, 15] and the references therein. What's more, one of the set property is the inclusion $\supseteq_{\mathcal{K}_{\mathcal{C}}}$ that is given by

$$[a, b] = X \supseteq_{\mathcal{K}_{\mathcal{C}}} Y = [c, d] \Longleftrightarrow a \leq c \text{ and } d \leq b.$$

Considering together with arithmetic operations and inclusion, one has the following property which is called inclusion isotony of interval operations: Let $\odot$ be the addition, multiplication, subtraction or division. If X, Y, Z and T are intervals such that

$$X \supseteq_{\mathcal{K}_{\mathcal{C}}} Y \text{ and } Z \supseteq_{\mathcal{K}_{\mathcal{C}}} T,$$

then the following relation is valid

$$X \odot Z \supseteq_{\mathcal{K}_{\mathcal{C}}} Y \odot T.$$

In [9], Bashirov et al. introduced the concept of *integral which is denoted by $\int_a^b (F(x))^{dx}$.

Proposition 2.1. *If a positive function F is Riemann integrable on $[a; b]$, then F is *integrable on $[a, b]$ and*

$$\int_a^b (F(x))^{dx} = e^{\int_a^b ln(F(x))dx}.$$

Now we recall the concept of Interval-valued integral given by Moore et al. in [16] and presented in [6]. Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$. The interval-valued Riemann integral of function F is defined by

$$\int_a^b F(x)dx = \int_a^b [\underline{F}(x), \overline{F}(x)]dx.$$

Let's define interval-valued *integral or multiplicative integral (I^*R). A function F is said to be an interval-valued function of t on $[a,b]$ if it assigns a nonempty interval to each $t \in [a,b]$

$$F(t) = [\underline{F}(t), \overline{F}(t)].$$

A partition of $[a,b]$ is any finite ordered subset $\mathcal{P}$ having the form

$$\mathcal{P} : a = t_0 < t_1 < \cdots < t_n = b.$$

The mesh of a partition $\mathcal{P}$ is defined by

$$mesh(\mathcal{P}) = \max\{t_i - t_{i-1}; i = 1, 2, \cdots, n\}.$$

We denote by $\mathcal{P}([a,b])$ the set of all partition of $[a,b]$. Let $\mathcal{P}(\delta, [a,b])$ be the set of all $\mathcal{P}_1 \in \mathcal{P}([a,b])$ such that mesh$(\mathcal{P}_1) < \delta$. Choose arbitrary points ξ_i in interval $[t_{i-1}, t_i], i = 1, 2, \cdots, n$ and we define the product

$$P(F, \mathcal{P}_1, \delta) = \prod_{i=1}^{n} F(\xi_i)^{[t_i - t_{i-1}]},$$

where $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is a positive function. We call $P(F, \mathcal{P}_1, \delta)$ a Riemann product of F corresponding to $\mathcal{P}_1 \in \mathcal{P}(\delta, [a,b])$.

Definition 2.2. *A positive function $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is said to be integrable in multiplicative sense or *integrable (I^*R integrable) on $[a,b]$ if there exists $A \in \mathbb{R}_{\mathcal{I}}$ such that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(P(F, \mathcal{P}_1, \delta), A) < \varepsilon,$$

for every Riemann product P of F corresponding to each $\mathcal{P}_1 \in \mathcal{P}(\delta, [a,b])$ and independent of choice of $\xi_i \in [t_{i-1}, t_i]$ for $1 \le i \le n$. In this case, A is called the I^*R-integral of F on $[a,b]$ and is denoted by

$$A = (I^*R) \int_a^b (F(t))^{dt}.$$

The collection of all functions that are I^*R-integrable on $[a,b]$ will be denoted by $\mathcal{I}^*\mathcal{R}_{([a,b])}$.

The following theorem gives relation between I^*R-integral and multiplicative integral (I^*-integral).

Theorem 2.3. *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be a positive interval-valued function. $F(t) = [\underline{F}(t), \overline{F}(t)] \in \mathcal{I}^*\mathcal{R}_{([a,b])}$ if and only if $\underline{F}(t), \overline{F}(t) \in \mathcal{I}^*_{([a,b])}$ and*

$$(I^*R) \int_a^b (F(t))^{dt} = \left[(I^*) \int_a^b (\underline{F}(t))^{dt}, (I^*) \int_a^b (\overline{F}(t))^{dt} \right],$$

where $\mathcal{I}^*_{([a,b])}$ denotes the all *integrable functions.

It is very easy to notice that if positive function F is interval-valued integrable (IR-integrable), then F is I^*R-integrable and

$$(I^*R)\int_a^b (F(t))^{dt} = e^{\int_a^b (ln\circ F)(t)dt}.$$

Now we give some properties of *integral for interval-valued functions. We consider F and G are positive interval-valued functions then the following equalities hold:

- $\int_a^b (F(t)^p)^{dt} = \left(\int_a^b (F(t))^{dt}\right)^p$.
- $\int_a^b (F(t)G(t))^{dt} = \int_a^b (F(t))^{dt} \int_a^b (G(t))^{dt}$.
- $\int_a^b \left(\frac{F(t)}{G(t)}\right)^{dt} = \frac{\int_a^b (F(t))^{dt}}{\int_a^b (G(t))^{dt}}$.
- $\int_a^b (F(t))^{dt} = \int_a^c (F(t))^{dt} \int_c^b (F(t))^{dt}$, where $a \leq c \leq b$.

Definition 2.4. [9] *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a positive function. The multiplicative derivative of the function f is given by*

$$\frac{d^*f}{dt}(t) = f^*(t) = \lim_{h\to 0}\left(\frac{f(t+h)}{f(t)}\right)^{\frac{1}{h}}.$$

If f has positive values and is differentiable at t, then f^* exists and the relation between f^* and ordinary derivative f' is as follows:

$$f^*(t) = e^{[logf(t)]'} = e^{\frac{f'(t)}{f(t)}}.$$

If, additionally, the second derivative of f at t exists, then by an easy substitution, we obtain

$$f^{**}(t) = e^{[ln\circ f^*(t)]'} = e^{[ln\circ f(t)]''}.$$

Here $(ln\circ f)''(t)$ exists because $f''(t)$ exists. For more details and properties, one can consult [26, 9]. Let $\Theta \subset \mathbb{R}$, and $\xi(.,.) : \Theta\times\Theta \to \mathbb{R}$ is a bifunction.

Definition 2.5. *(See* [18]*) A set Θ is considered to be invex with reference to the bifunction $\xi(.,.)$, iff*

$$a + t\xi(b,a) \in \Theta,$$

for all $a, b \in \Theta$ and $t \in [0,1]$.

Example. *Let $\Theta = [-4,-3]\cup[-2,3]$ be considered to be invex with reference to bifunction $\xi(.,.)$ and defined as:*

$$\xi(a,b) = \begin{cases} a-b \ \textit{if}\ 3\geq a\geq -2,\ 3\geq b\geq -1; \\ a-b \ \textit{if}\ -4\leq a\leq -3,\ -4\leq b\leq -3; \\ -4-b \ \textit{if}\ -2\leq a\leq 3,\ -4\leq b\leq -2; \\ -2-b \ \textit{if}\ -4\leq a\leq -3, -2\leq b\leq 3. \end{cases}$$

Then Θ is considered to be invex with reference to bifunction $\xi(.,.)$.

Definition 2.6. *Let $h : [c,d] \to \mathbb{R}$ be a nonnegative function, $(0,1) \subset [c,d]$ and $h \neq 0$. Let $f : \Theta \to \mathbb{R}_I$ be a nonnegative interval-valued function given by $f = [\underline{f}, \overline{f}]$. Then, f is said to be logarithmically or multiplicatively interval-valued preinvex function with reference to ξ if*

$$f(a+t\xi(b,a)) \supseteq_{\mathcal{K}_\mathcal{C}} (f(a))^{h(1-t)}(f(b))^{h(t)},$$

for all $a, b \in \Theta$ and $t \in (0,1)$.

Remark. *If* $\underline{f} = \overline{f}$ *and* $\xi(b,a) = b - a$, *Definition* 2.6 *reduces to a multiplicatively h-preinvex function (see* [17]*)*

Definition 2.7. *(Condition C (see* [19]*))*
Let $\Theta \subset \mathbb{R}^n$ *be an open invex set with reference to* $\xi(.,.) : \Theta \times \Theta \to \mathbb{R}$. *For all* $a, b \in \Theta$ *and* $\eta \in [0,1]$, *we have*

$$\xi(b, b + \eta\xi(a,b)) = -\eta\xi(a,b),$$

and

$$\xi(a, b + \eta\xi(a,b)) = (1-\eta)\xi(a,b).$$

For any $a, b \in \Theta$, and $\eta_1, \eta_2 \in [0,1]$, from Condition C, we have

$$\xi(b + \eta_2\xi(a,b), b + \eta_1\xi(a,b)) = (\eta_2, \eta_1)\xi(a,b).$$

3. Existence and uniqueness results

The prime objective of this section is to derive and prove several novel Hermite-Hadamard inclusions for multiplicatively interval-valued preinvex function in the framework of multiplicative calculus.

Theorem 3.1. *Let* $\Theta \in \mathbb{R}$ *to be an open invex subset with respect to a bifunction* $\xi : \Theta \times \Theta \to \mathbb{R}$ *and* $a, b \in \Theta$ *with* $\xi(b,a) > 0$. *Let* $h : [0,1] \to \mathbb{R}^+$ *and* $h(\frac{1}{2}) \neq 0$. *Consider* $f : [a, a+\xi(b,a)] \to \mathbb{R}_I^+$ *to be a multiplicatively h-preinvex interval-valued function and* ξ *satisfies Condition C, then*

$$\left[f(\frac{2a+\xi(b,a)}{2})\right]^{\frac{1}{2h(\frac{1}{2})}} \supseteq_{\mathcal{K}_C} \left(\int_a^{a+\xi(b,a)} (f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}} \supseteq_{\mathcal{K}_C} (f(a)f(b))^{\int_0^1 h(t)dt}.$$

Proof. Since f is a multiplicatively interval-valued h-preinvex function, we have, with $t = \frac{1}{2}$

$$f(\frac{2a+\xi(b,a)}{2}) \supseteq_{\mathcal{K}_C} (f(a)f(b))^{h(\frac{1}{2})}.$$

Choosing $\theta_1 = a + t\xi(b,a)$ and $\theta_2 = a + (1-t)\xi(b,a)$, and using the condition C, one has

$$\begin{aligned}
f(\frac{2a+\xi(b,a)}{2}) &= f(\frac{2\theta_1+\xi(\theta_2,\theta_1)}{2}) \\
&= f(a+t\xi(b,a) + \frac{1}{2}\xi(a+(1-t)\xi(b,a), a+t\xi(b,a)) \\
&\supseteq_{\mathcal{K}_C} \left[(f(a+t\xi(b,a)))(f(a+(1-t)\xi(b,a)))\right]^{h(\frac{1}{2})}.
\end{aligned}$$

Taking logarithms of both sides leads to

$$\begin{aligned}
\ln f(\frac{2a+\xi(b,a)}{2}) &\supseteq_{\mathcal{K}_C} \ln\left[(f(a+t\xi(b,a)))(f(a+(1-t)\xi(b,a)))\right]^{h(\frac{1}{2})} \\
&\supseteq_{\mathcal{K}_C} h(\frac{1}{2})(\ln f(a+t\xi(b,a)) + \ln f(a+(1-t)\xi(b,a))).
\end{aligned}$$

Integrating the above inclusion over $(0,1)$, it follows that

$$\begin{aligned}
\int_0^1 \ln f(\frac{2a+\xi(b,a)}{2})dt &\supseteq_{\mathcal{K}_C} h(\frac{1}{2})\int_0^1 (\ln f(a+t\xi(b,a)) + \ln f(a+(1-t)\xi(b,a)))dt \\
&\supseteq_{\mathcal{K}_C} \frac{2h(\frac{1}{2})}{\xi(b,a)}\int_a^{a+\xi(b,a)} \ln f(x)dx.
\end{aligned}$$

Thus

$$\left[f(\frac{2a+\xi(b,a)}{2})\right]^{\frac{1}{2h(\frac{1}{2})}} \supseteq_{\mathcal{K}_{\mathcal{C}}} e^{\left(\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\ln f(x)dx\right)} = \left(\int_a^{a+\xi(b,a)}(f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}},$$

which gives the first inclusion. Now, we have

$$\begin{aligned}\left(\int_a^{a+\xi(b,a)}(f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}} &= \left(e^{\left(\int_a^{a+\xi(b,a)}\ln f(x)dx\right)}\right)^{\frac{1}{\xi(b,a)}}\\ &= e^{\left(\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\ln f(x)dx\right)}\\ &= e^{2h(\frac{1}{2})\left(\int_0^1 \ln(f(a+t\xi(b,a)))dt\right)}\\ &\supseteq_{\mathcal{K}_{\mathcal{C}}} e^{\int_0^1 \ln((f(a))^{h(1-t)}(f(b))^{h(t)})dt}\\ &\supseteq_{\mathcal{K}_{\mathcal{C}}} e^{\ln(f(a)f(b))^{\int_0^1 h(t)dt}}\\ &\supseteq_{\mathcal{K}_{\mathcal{C}}} (f(a)f(b))^{\int_0^1 h(t)dt},\end{aligned}$$

which completes the proof of the theorem. □

Corollary 3.2. *Under the assumptions of Theorem* 3.1, *if we set* $h(t)=t$, *then we have the following Hermite-Hadamard inequality for multiplicatively interval-valued preinvex functions*

$$f(\frac{2a+\xi(b,a)}{2}) \supseteq_{\mathcal{K}_{\mathcal{C}}} \left(\int_a^{a+\xi(b,a)}(f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}} \supseteq_{\mathcal{K}_{\mathcal{C}}} G(f(a),f(b)),$$

where $G(f(a),f(b))$ *referes to the geometric mean of* $f(a)$ *and* $f(b)$.

Now we give a new integral inclusions for a product of multiplicatively h-preinvex positive interval valued functions.

Theorem 3.3. *Let* $\Theta\in\mathbb{R}$ *an open invex subset with respect to a bifunction* $\xi:\Theta\times\Theta\to\mathbb{R}$ *and* $a,b\in\Theta$ *with* $\xi(b,a)>0$. *Let* $h:[0,1]\to\mathbb{R}^+$ *and* $h(\frac{1}{2})\neq 0$. *Consider* $f:[a,a+\xi(b,a)]\to\mathbb{R}_I^+$ *to be a multiplicatively h-preinvex interval-valued function and* ξ *satisfies Condition C, then*

$$\begin{aligned}\left[f(\frac{2a+\xi(b,a)}{2})g(\frac{2a+\xi(b,a)}{2})\right]^{\frac{1}{2h(\frac{1}{2})}} &\supseteq_{\mathcal{K}_{\mathcal{C}}} \left(\int_a^b f(x)^{dx}\int_a^b g(x)^{dx}\right)^{\frac{1}{\xi(b,a)}}\\ &\supseteq_{\mathcal{K}_{\mathcal{C}}} (f(a)f(b)g(a)g(b))^{\int_0^1 h(t)dt}.\end{aligned}$$

Proof. By the definition of multiplicatively interval-valued h-preinvex function, we have

$$f(\frac{2a+\xi(b,a)}{2}) \supseteq_{\mathcal{K}_{\mathcal{C}}} f(a)^{h(\frac{1}{2})}f(b)^{h(\frac{1}{2})},$$

and

$$g(\frac{2a+\xi(b,a)}{2}) \supseteq_{\mathcal{K}_{\mathcal{C}}} f(a)^{h(\frac{1}{2})}f(b)^{h(\frac{1}{2})}.$$

By setting $\theta_1 = a + t\xi(b,a)$ and $\theta_2 = a + (1-t)\xi(b,a)$, and using the condition C, one has

$$f(\frac{2a+\xi(b,a)}{2}) = f(\frac{2\theta_1+\xi(\theta_2,\theta_1)}{2}) \supseteq_{\mathcal{K}_C} f(a+t\xi(b,a))^{h(\frac{1}{2})} f(a+(1-t)\xi(b,a))^{h(\frac{1}{2})},$$

and

$$g(\frac{2a+\xi(b,a)}{2}) = g(\frac{2\theta_1+\xi(\theta_2,\theta_1)}{2}) \supseteq_{\mathcal{K}_C} g(a+t\xi(b,a))^{h(\frac{1}{2})} g(a+(1-t)\xi(b,a))^{h(\frac{1}{2})}.$$

By multiplying the two previous inclusions, we obtain

$$\begin{aligned}
&\ln\left[f(\frac{2a+\xi(b,a)}{2})g(\frac{2a+\xi(b,a)}{2})\right] \\
&\supseteq_{\mathcal{K}_C} h(\frac{1}{2})[\ln f(a+t\xi(b,a)) + \ln f(a+(1-t)\xi(b,a)) \\
&+ \ln g(a+t\xi(b,a)) + \ln g(a+(1-t)\xi(b,a))].
\end{aligned}$$

Integrating the above inclusion over $(0,1)$, it follows that

$$\begin{aligned}
&\ln\left[f(\frac{2a+\xi(b,a)}{2})g(\frac{2a+\xi(b,a)}{2})\right] \\
&\supseteq_{\mathcal{K}_C} h(\frac{1}{2})[\int_0^1 \ln f(a+t\xi(b,a))dt + \int_0^1 \ln f(a+(1-t)\xi(b,a))dt \\
&+ \int_0^1 \ln g(a+t\xi(b,a))dt + \int_0^1 \ln g(a+(1-t)\xi(b,a))dt].
\end{aligned}$$

Consequently,

$$\frac{1}{2h(\frac{1}{2})}\ln\left[f(\frac{2a+\xi(b,a)}{2})g(\frac{2a+\xi(b,a)}{2})\right] \supseteq_{\mathcal{K}_C} \frac{1}{\xi(b,a)}[\int_a^b \ln f(x)dx + \int_a^b \ln g(x)dx],$$

which implies that

$$\left[f(\frac{2a+\xi(b,a)}{2})g(\frac{2a+\xi(b,a)}{2})\right]^{\frac{1}{2h(\frac{1}{2})}} \supseteq_{\mathcal{K}_C} \left(e^{\int_a^b \ln f(x)dx + \int_a^b \ln g(x)dx}\right)^{\frac{1}{\xi(b,a)}}.$$

Hence

$$\left[f(\frac{2a+\xi(b,a)}{2})g(\frac{2a+\xi(b,a)}{2})\right]^{\frac{1}{2h(\frac{1}{2})}} \supseteq_{\mathcal{K}_C} \left(e^{\int_a^b f(x)^{dx} + \int_a^b g(x)^{dx}}\right)^{\frac{1}{\xi(b,a)}}.$$

For the second inclusion, first we note that

$$f(a+t\xi(b,a)) \supseteq_{\mathcal{K}_C} f(a)^{h(t)} f(b)^{h(1-t)},$$

and

$$g(a+t\xi(b,a)) \supseteq_{\mathcal{K}_C} g(a)^{h(t)} g(b)^{h(1-t)}.$$

Then

$$\begin{aligned}
&\ln\left(f(a+t\xi(b,a))g(a+t\xi(b,a))\right) \\
&\supseteq_{\mathcal{K}_C} h(t)[\ln f(a) + \ln g(a)] + h(1-t)[\ln f(b) + \ln g(b)].
\end{aligned}$$

By integrating over $(0,1)$, we get

$$\begin{aligned}
&\int_0^1 \ln\left(f(a+t\xi(b,a))g(a+t\xi(b,a))\right)dt \\
&\supseteq_{\mathcal{K}_\mathcal{C}} [\ln f(a)+\ln g(a)]\int_0^1 h(t)dt \\
&\supseteq_{\mathcal{K}_\mathcal{C}} [\ln f(b)+\ln g(b)]\int_0^1 h(1-t)dt.
\end{aligned}$$

So, it follows that

$$\frac{1}{\xi(b,a)}\int_0^1 \ln(f(x)g(x))dx \supseteq_{\mathcal{K}_\mathcal{C}} \ln(f(a)f(b)g(a)g(b))^{\int_0^1 h(t)dt}.$$

Hence

$$\left(e^{\int_a^b \ln f(x)dx+\int_a^b \ln g(x)dx}\right)^{\frac{1}{\xi(b,a)}} \supseteq_{\mathcal{K}_\mathcal{C}} (f(a)f(b)g(a)g(b))^{\int_0^1 h(t)dt}.$$

Thus

$$\left(\int_a^b f(x)^{dx}\int_a^b g(x)^{dx}\right)^{\frac{1}{\xi(b,a)}} \supseteq_{\mathcal{K}_\mathcal{C}} (f(a)f(b)g(a)g(b))^{\int_0^1 h(t)dt}.$$

□

Corollary 3.4. *Under the assumptions of Theorem* 3.1, *if we set* $h(t)=t$, *then we have the following Hermite-Hadamard inequality for multiplicatively interval-valued preinvex functions*

$$\begin{aligned}
\left[f\left(\frac{2a+\xi(b,a)}{2}\right)g\left(\frac{2a+\xi(b,a)}{2}\right)\right]^{\frac{1}{2h(\frac{1}{2})}} &\supseteq_{\mathcal{K}_\mathcal{C}} \left(e^{\int_a^b f(x)^{dx}+\int_a^b g(x)^{dx}}\right)^{\frac{1}{\xi(b,a)}} \\
&\supseteq_{\mathcal{K}_\mathcal{C}} G(f(a),f(b))G(g(a),g(b)).
\end{aligned}$$

where $G(f(a),f(b))$ *referes to the geometric mean of* $f(a)$ *and* $f(b)$.

4. Multiplicative integral Identities

We begin this section by an integral identity associated with the twice multiplicative differentiable function.

Lemma 4.1. *Let* $\Theta\subset\mathbb{R}$ *an open invex subset with respect to a bifunction* $\xi:\Theta\times\Theta\to\mathbb{R}$ *and* $a,b\in\Theta$ *with* $\xi(b,a)>0$. *Consider* $f:[a,a+\xi(b,a)]\to\mathbb{R}_I^+$ *be a twice differentiable interval-valued function. If* $f^{**}\in L_2[a,a+\xi(b,a)]$, *then the following equality holds:*

$$\sqrt{f(a)f(b)}\left(\int_b^a (f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}} = \left(\int_0^1\left(\left[f^{**}(a+t\xi(b,a))^{t(1-t)}\right]\right)^{dt}\right)^{\frac{(\xi(b,a))^2}{2}}.$$

Proof. By changing the variables of integration and from the fondamental rules of multiplicative integration, we obtain that

$$\begin{aligned}
&\left(\int_0^1 \left([f^{**}(a+t\xi(b,a))^{t(1-t)}]\right)^{dt}\right)^{\frac{(\xi(b,a))^2}{2}}\\
&= e^{\frac{(\xi(b,a))^2}{2}\int_0^1 t(1-t)(ln\circ f)''(a+t\xi(b,a)))dt}\\
&= e^{[\frac{\xi(b,a)}{2}t(1-t)(\ln\circ f)'(a+t\xi(b,a))]_0^1-\frac{\xi(b,a)}{2}\int_0^1(1-2t)(\ln\circ f)'(a+t\xi(b,a))dt}\\
&= e^{-\frac{1}{2}\left[[(1-2t)(\ln\circ f)(a+t\xi(b,a))]_0^1+2\int_0^1(\ln\circ f)(a+t\xi(b,a))dt\right]}\\
&= e^{\frac{1}{2}[\ln(f(a)f(b))]-\int_0^1(\ln\circ f)(a+t\xi(b,a))dt}\\
&= \frac{e^{\ln\sqrt{f(a)f(b)}}}{e^{\int_0^1(\ln\circ f)(a+t\xi(b,a))dt}}\\
&= \sqrt{f(a)f(b)}\left(\int_b^a (f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}}.
\end{aligned}$$

Thus, the proof is completed. □

Lemma 4.2. *Let $\Theta\subset\mathbb{R}$ an open invex subset with respect to a bifunction $\xi:\Theta\times\Theta\to\mathbb{R}$ and $a,b\in\Theta$ with $\xi(b,a)>0$. Consider $f:[a,a+\xi(b,a)]\to\mathbb{R}_I^+$ be a twice differentiable interval-valued function. If $f^{**}\in L_2[a,a+\xi(b,a)]$, then the following equality holds:*

$$\begin{aligned}
\frac{\left(\int_a^b(\ln\circ f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}}}{f(\frac{2a+\xi(b,a)}{2})} &= \left(\int_0^{\frac{1}{2}}\left([f^{**}(a+t\xi(b,a))]^{t^2}\right)^{dt}\right)^{\frac{(\xi(b,a))^2}{2}}\\
&\quad\times\left(\int_{\frac{1}{2}}^1\left([f^{**}(a+t\xi(b,a))]^{(1-t)^2}\right)^{dt}\right)^{\frac{(\xi(b,a))^2}{2}}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
I_1 &= \left(\int_0^{\frac{1}{2}}\left([f^{**}(a+t\xi(b,a))]^{t^2}\right)^{dt}\right)^{\frac{(\xi(b,a))^2}{2}}\\
&= e^{\frac{(\xi(b,a))^2}{2}\int_0^{\frac{1}{2}} t^2(ln\circ f)''(a+t\xi(b,a)))dt}\\
&= e^{\frac{\xi(b,a)}{2}(\ln\circ f)'(\frac{2a+\xi(b,a)}{2})-\frac{1}{2}f(\frac{2a+\xi(b,a)}{2})+\int_0^{\frac{1}{2}}(\ln\circ f)(a+t\xi(b,a))dt},
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \left(\int_{\frac{1}{2}}^1\left([f^{**}(a+t\xi(b,a))]^{(1-t)^2}\right)^{dt}\right)^{\frac{(\xi(b,a))^2}{2}}\\
&= e^{-\frac{\xi(b,a)}{2}(\ln\circ f)'(\frac{2a+\xi(b,a)}{2})-\frac{1}{2}f(\frac{2a+\xi(b,a)}{2})+\int_{\frac{1}{2}}^1(\ln\circ f)(a+t\xi(b,a))dt}.
\end{aligned}$$

From the above two equalities, it follows that

$$\begin{aligned} I_1 \times I_2 &= e^{\int_0^1 \ln\circ f(a+t\xi(b,a))dt - \ln\circ f(\frac{2a+\xi(b,a)}{2})} \\ &= \frac{e^{\frac{1}{\xi(b,a)}\int_a^b \ln\circ f(x)dx}}{f(\frac{2a+\xi(b,a)}{2})} \\ &= \frac{\left(\int_a^b (\ln\circ f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}}}{f(\frac{2a+\xi(b,a)}{2})}. \end{aligned}$$

Thus, we have the result. □

5. Trapezoidal type Inequalities

A new trapezoidal type inclusion in the setting of multiplicative calculus is established in this section.

Theorem 5.1. *Under the assumptions of Lemma* 4.1 *with* $\underline{f} = \overline{f}$. *If* f^{**} *is multiplicative h-preinvex mapping, then the following inequality holds:*

$$\left| \sqrt{f(a)f(b)} \left(\int_b^a (f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}} \right| \leq (f^{**}(a)f^{**}(b))^{\frac{(\xi(b,a))^2\int_0^1 t(1-t)h(t)dt}{2}}.$$

Proof. From Lemma 4.1 and the multiplicative h-preinvexity of f^{**}, we have

$$\begin{aligned} &\left| \sqrt{f(a)f(b)} \left(\int_b^a (f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}} \right| \\ &\leq e^{\left[\frac{(\xi(b,a))^2}{2}\int_0^1 |((ln\circ f)''(a+t\xi(b,a))))^{t(1-t)}|dt\right]} \\ &= e^{\left[\frac{(\xi(b,a))^2}{2}\int_0^1 |t(1-t)(ln\circ f)''(a+t\xi(b,a)))|dt\right]} \\ &\leq e^{\left[\frac{(\xi(b,a))^2}{2}\int_0^1 t(1-t)(h(1-t)(ln\circ f)''(a)+h(t)(ln\circ f)''(b))dt\right]} \\ &\leq (f^{**}(a)f^{**}(b))^{\frac{(\xi(b,a))^2\int_0^1 t(1-t)h(t)dt}{2}}. \end{aligned}$$

□

6. Midpoint Type Inequalities

In the setting of multiplicative calculus, we establish a new midpoint inequality.

Theorem 6.1. *Under the assumptions of Lemma* 4.2 *with* $\underline{f} = \overline{f}$. *If* f^{**} *is multiplicative h-preinvex mapping, then the following inequality holds:*

$$\left| \frac{\left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}}}{f(\frac{2a+\xi(b,a)}{2})} \right| \leq (f^{**}(a)f^{**}(b))^{\frac{(\xi(b,a))^2}{2}\int_0^{\frac{1}{2}} t^2(h(t)+h(1-1))dt}.$$

Proof. From Lemma 4.2 and the multiplicative h-preinvexity of f^{**}, we have

$$\begin{aligned}
&\left|\frac{\left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{\xi(b,a)}}}{f\left(\frac{2a+\xi(b,a)}{2}\right)}\right| \\
&\leq e^{\left[\frac{(\xi(b,a))^2}{2}\left(\int_0^{\frac{1}{2}} t^2 |(ln\circ f)''(a+t\xi(b,a))|dt\right)\right]} \\
&\times e^{\left[\frac{(\xi(b,a))^2}{2}\left(\int_{\frac{1}{2}}^1 (1-t)^2 |(ln\circ f)''(a+t\xi(b,a))|dt\right)\right]} \\
&\leq e^{\left[\frac{(\xi(b,a))^2}{2}\left(\int_0^{\frac{1}{2}} t^2 (h(1-t)(ln\circ f)'')(a)+h(t)(ln\circ f)'')(b))dt\right)\right]} \\
&\times e^{\left[\frac{(\xi(b,a))^2}{2}\left(\int_{\frac{1}{2}}^1 (1-t)^2 (h(1-t)(ln\circ f)'')(a)+h(t)(ln\circ f)'')(b))dt\right)\right]} \\
&\leq (f^{**}(a)f^{**}(b))^{\frac{(\xi(b,a))^2}{2}\int_0^{\frac{1}{2}} t^2(h(t)+h(1-1))dt}.
\end{aligned}$$

□

7. Conclusion

In this article, we showed new Hermite-Hadamard type inclusions, in the setting of interval-valued non-Newtonian calculus, for the product of multiplicatively interval-valued h-preinvex functions. This work is a rafinement of the Hermite-Hadamard inequality for this type of functions. We derived also some new midpoint and trapezoidal type inclusions using h-preinvexity. We hope that our ideas inspired many researchers in this fascinating field.

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