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BLOW-UP PHENOMENA FOR COUPLED PSEUDO-PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS

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Abstract. The initial boundary value problem of a class of coupled pseudo-parabolic equations is considered. Using a differential inequality technique, We demonstrate that, at a finite time T, the solutions become unbounded, and find an upper bound for this time with negative initial energy. A lower bound for the blow-up time is also established.

1. Introduction

Let Ω represent a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$. We consider the following coupled pseudo-parabolic equations

$$\begin{cases} u_t - \mu_1 \Delta u_t - \operatorname{div}(A(x,t)\, |\nabla u|^{m(x)-2} \nabla u) = |uv|^{p(x)-2} uv^2 & \text{in } Q_T \\ v_t - \mu_2 \Delta v_t - \operatorname{div}(B(x,t)\, |\nabla v|^{n(x)-2} \nabla v) = |uv|^{p(x)-2} u^2 v & \text{in } Q_T \\ u(t,x) = v(t,x) = 0 & \text{on } \partial Q_T \\ u(0,x) = u_0(x), \qquad v(0,x) = v_0(x) & x \in \Omega \end{cases} \tag{1}$$

where $Q_T = \Omega \times (0,T)$, $\partial Q_T = \partial\Omega \times (0,T)$, $\mu_1, \mu_2 \geq 0$ are a contants and $\operatorname{div}(A(x,t)\, |\nabla u|^{m(x)-2} \nabla u)$, $\operatorname{div}(B(x,t)\, |\nabla v|^{n(x)-2} \nabla v$ are the so-called $m(x), n(x)$-Laplace operators with the presence of a matrices $A(x,t)$, $B(x,t)$ respectively. The terms with a variable exponent $|uv|^{p(x)-2} uv^2$, $|uv|^{p(x)-2} u^2 v$ play the role of a source, and the dissipative terms Δu_t and Δv_t are a linear strong damping term.

The matrices $A = (a_{ij}(x,t))_{i,j}$ and $B = (b_{ij}(x,t))_{i,j}$ where $a_{ij}(x,t)$ and $b_{ij}(x,t)$ are a function of class $C^1(\overline{\Omega} \times [0, \infty[)$ such that for constants $a_0, b_0 > 0$ and all $\xi \in \mathbb{R}^n$,

$$A\xi.\xi \geq a_0 \, |\xi|^2, \qquad B\xi.\xi \geq b_0 \, |\xi|^2 \tag{2}$$

$$A^{'}\xi.\xi \leq 0, \qquad B^{'}\xi.\xi \leq 0 \tag{3}$$

where $A^{'} = \frac{\partial A}{\partial t}(.,t)$ and $B^{'} = \frac{\partial A}{\partial t}(.,t)$.

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The exponents $m(.), n(.)$ and $p(.)$ are continuous functions defined on $\overline{\Omega}$ and satisfies

$$2 \leq m_{-} \leq m(x) \leq m_{+} \leq n_{-} \leq n(x) \leq n_{+} < p_{-} \leq p(x) \leq p_{+} < \infty \tag{4}$$

where

$$\begin{aligned} m_{-} &= ess\ \inf m(x), \quad m_{+} = ess\ \sup m(x) \\ n_{-} &= ess\ \inf\ n(x), \quad n_{+} = ess\ \sup n(x) \\ p_{-} &= ess\ \inf\ p(x), \quad p_{+} = ess\ \sup p(x) \end{aligned}$$

and the Zhikov–Fan condition:

$$p(x) - p(y) \leq \frac{-a}{\log|x-y|} \quad \text{for all } x, y \in \Omega, \text{with } |x-y| < \delta, a > 0 \text{ and } 0 < \delta < 1. \tag{5}$$

Equations or systems with variable exponents like (1) apear in the study of various problem of the hydrodynamics, thermodynamics, filtration theory etc. (see [1, 2, 3, 4]). Obviously, if $\mu_1 = \mu_2 = 0$, $A = B = I_n$, $m(x) = n(x) = 2, p(x) = p =$constant, then Equations (1) reduce to the following parabolic system

$$\begin{cases} u_t - \Delta u = |uv|^{p-2} uv^2 & \text{in } \Omega \times\ (0,T) \\ v_t - \Delta v = |uv|^{p-2} u^2 v & \text{in } \Omega \times\ (0,T) \end{cases} \tag{6}$$

For system (6), many results have been obtained, such as the existence and uniqueness in [5, 9], global extence in [10, 11], asymptotic behavior in [10, 12] and so on. In [7] Xu et al. studied the coupled parabolic systems

$$\begin{cases} \upsilon_t - \Delta \upsilon = \left(|\upsilon|^{2p} + |\nu|^{p+1} |\upsilon|^{p-1}\right) \upsilon \\ \nu_t - \Delta \nu = \left(|\nu|^{2p} + |\upsilon|^{p+1} |\nu|^{p-1}\right) \nu \end{cases} \tag{7}$$

with Dirichlet boundary conditions. By introducing a family of potential wells, the whole study is conducted by considering the following three cases according to initial energy: low, critical, and high initial energy cases. Under the condition $J(u_0, v_0) < d$, where d is a depth of potential well associated with the energy functional

$$J(u,v) = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - \frac{1}{2(p+1)} \|u\|_{2p+2}^{2p+2} + \|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}$$

they obtained the global existence and finite time blowup of the solution for the problem (7). On the other side, if $J(u_0, v_0) = d$ they proved the global solution, blow-up solution, and asymptotic behavior of the problem (7). With the high initial energy level $J(u_0, v_0) > d$, by adopting the comparison principle of the coupled parabolic systems, they gave sufficient conditions to obtain the finite time blow-up anf global existence of the solution. In the presence of the damping ($\mu_1, \mu_2 > 0$), Qi, Chen and Wang [6] considered the following coupled pseudo-parabolic equations

$$\begin{cases} \upsilon_t - \mu_1 \Delta \upsilon_t - \operatorname{div}(|\nabla \upsilon|^{m(x)-2} \nabla \upsilon) = |\upsilon\nu|^{p(x)-2} \upsilon\nu^2 & \text{in } \Omega \times\ (0,T) \\ \nu_t - \mu_2 \Delta \nu_t - \operatorname{div}(|\nabla \nu|^{n(x)-2} \nabla \nu) = |\upsilon\nu|^{p(x)-2} \upsilon^2 \nu & \text{in } \Omega \times\ (0,T) \end{cases} \tag{8}$$

which is just the $A = B = I_n$ case of (1). By using differential inequality technique, they proved that any solutions to (8) with $m_+ \geq n_-, n_+ \geq m_-$, $p_- > \max\{m_+, n_+\}$ and $\min\{m_-, n_-\} \geq 2$ blow up in finite time in $H^1(\Omega)$-norm, and also they obtained an upper bound and a lower bound for a blow-up time of the solution if the initial energy $E(0)$ is negative. On the other hand, Messaoudi et al [7] sudies the blowing up of a solution to the problem

$$\begin{cases} \upsilon_{tt} - \operatorname{div}(A(x,t)\nabla\upsilon) + |\upsilon_t|^{m(x)-2}\upsilon_t = f_1(x,\upsilon,\nu) & \text{in } \Omega\times\ (0,T) \\ \nu_{tt} - \operatorname{div}(B(x,t)\nabla\nu) + |\nu_t|^{r(x)-2}\nu_t = f_2(x,\upsilon,\nu) & \text{in } \Omega\times\ (0,T) \end{cases} \tag{9}$$

where $f_1(x,\upsilon,\nu) = \frac{\partial}{\partial\upsilon}F(x,\upsilon,\nu)$, $f_1(x,\upsilon,\nu) = \frac{\partial}{\partial\nu}F(x,\upsilon,\nu)$ with $F(x,\upsilon,\nu) = \alpha\,|\upsilon+\nu|^{p(x)+1} + 2b\,|\upsilon\nu|^{\frac{p(x)+1}{2}}$. According to [7], if A and B satisfy (2) (3) and if $m(x), r(x) \geq 2$ for $n = 1, 2; m(x), r(x) \in [2,6]$ for $n = 3$, then the solution for nontrivial initial data blows up in finite time with positive initial energy. In addition, we refer to [13, 17, 18, 20] for other result concerning the theory of our type equation.

It is worth mentioning that equations with nonstandard growth conditions are equations (or systems) that have nonlinearities of variable exponent type. This type of equation arises in the mathematical representation of different physical phenomena, such as the movement of electrorheological fluids, nonlinear viscoelasticity, fluids with viscosity that depends on temperature, filtration processes through porous media, and image processing, among others (see [8, 21, 22, 28]).

Based on the above-mentioned work and motivated by [6, 7], this paper aims to find an upper bound for blow-up time if the variable exponents $m(.), n(.), p(.)$, the initial data and the matrices $A(.,t), B(.,t)$ satisfy some conditions. Also, we will give the lower bounds on blow-up time under some other conditions for the problem (1).

The outline of this paper is as follows. In section 2, we recall the definitions of the variable exponent Lebesgue spaces $L^{p(.)}(\Omega)$, the Sobolev spaces $W^{1,p(.)}(\Omega)$, as well as some of their properties. In Section 3 and Section 4, we give a study of the blow-up of solutions to the problem under consideration.

2. **Essential tools**

The $L^p(\Omega)$ norm for $1 \leq p \leq \infty$ is denoted by $\|.\|_p$Throughout this paper, and the inner product on Hilbert space $L^2(\Omega)$ is denoted by $(.,.)$. We will equip $H_0^1(\Omega)$ with the norm $\|u\|_{H_0^1(\Omega)} = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2}$ and the inner product $(u,v)_{H_0^1(\Omega)} = (u,v) + (\nabla u, \nabla v), \forall u, v \in H_0^1(\Omega)$. Firstly, let us recall some definitions, properties, and important lemmas related to Lebesgue and Sobolev space with a variable exponent to state the main results of this paper. Let Ω be a domain of $\mathbb{R}^n$ and $p : \Omega \longrightarrow [1,\infty)$ be a measurable function. The Lebesgue space $L^{p(.)}(\Omega)$, with variable exponent $p(.)$ is defined by

$$L^{p(.)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} | u \text{ is measurable and } \int_\Omega |\lambda u(x)|^{p(x)}\,dx < \infty \text{ for some } \lambda > 0\right\}.$$

The Luxemburg-type norm is given by

$$\|u\|_{p(.)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces bear many similarities to classical Lebesgue spaces, including being reflexive if $1 < p(x) < \infty$ being Banach spaces, and adhering to the Hölder inequality. $W^{1,p(.)}(\Omega)$ is the variable exponent Sobolev space defined by

$$W^{1,p(.)}(\Omega)=\left\{u\in L^{p(.)}(\Omega):\nabla u \text{ exists and } \nabla u\in L^{p(.)}(\Omega)\right\}.$$

Considering the norm

$$\|u\|_{W^{1,p(.)}(\Omega)}=\|u\|_{p(.)}+\|\nabla u\|_{p(.)},$$

this is a Banach space.

The space $W_0^{1,p(.)}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. The definition of the space $W_0^{1,p(.)}(\Omega)$ in the constant exponent case is usually different. However, under condition (5) both definitions coincide (See [16]|). The dual space $W_0^{-1,p'(.)}(\Omega)$ of $W_0^{1,p(.)}(\Omega)$ is defined in the same way as in the classical Sobolev spaces, where

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1$$

Lemma 1. *(Poincaré's inequality)* [16]. *Suppose that p(.) satisfies (5); then,*

$$\|u\|_{p(.)}\leq C\|\nabla u\|_{p(.)},\quad u\in W_0^{1,p(.)}(\Omega)$$

where $p(.)$ and Ω are the only variables that determine the constant $C>0$.

Lemma 2. *(Embedding Proprety)*[16]. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. If $q\in C(\overline{\Omega})$ such that $q\geq 2$ and $q(x)<2^*$ in $\overline{\Omega}$ with*

$$2^*=\begin{cases}\frac{2n}{n-2}, & \text{if } n>2;\\ \infty, & \text{if } n\leq 2,\end{cases}$$

then we have continuous and compact embedding $H_0^1(\Omega)\hookrightarrow L^{q(.)}(\Omega)$. So, there exists $C>0$ such that

$$\|u\|_{L^{q(.)}(\Omega)}\leq C\|u\|_{H_0^1(\Omega)}$$

Next, we give a precise definition of a weak solution to the problem (1).

Definition 1. *Let $(u_0,v_0)\in\left(W_0^{1,m(.)}\cap L^{p(.)}(\Omega)\right)\times\left(W_0^{1,n(.)}\cap L^{p(.)}(\Omega)\right)$. Any pair of functions (u,v) such that*

$$\begin{cases} u\in L^\infty([0,T_0],W_0^{1,m(.)}\cap L^{p(.)}(\Omega)), & u_t\in L^2([0,T_0],H_0^1(\Omega)),\\ v\in L^\infty([0,T_0];W_0^{1,n(.)}(\Omega)\cap L^{p(.)}(\Omega)), & v_t\in L^2([0,T_0];W_0^{1,2}(\Omega)),\end{cases}\tag{10}$$

is called a weak solution of (1) on $[0,T)$, if

$$(u_t,\Psi)+(\nabla u_t,\nabla\Psi)+(A(x,t)\left|\nabla u\right|^{m(x)-2}\nabla u,\nabla\Psi) = (|u|^{p(x)-2}uv^2,\Psi),\tag{11}$$

$$(v_t,\Phi)+(\nabla v_t,\nabla\Phi)+(B(x,t)\left|\nabla v\right|^{n(x)-2}\nabla v,\nabla\Phi) = (|u|^{p(x)-2}u^2v,\Phi)\tag{12}$$

$$u(x,0)=u_0(x),v(x,0)=v_0(x),\tag{13}$$

for a.e. $t\in(0,T)$ and all test functions $\Psi,\Phi\in W_0^{1,m(.)}(\Omega)\cap L^{p(.)}(\Omega),W_0^{1,n(.)}(\Omega)\cap L^{p(.)}(\Omega)$ respectively

Remark 1. *It is easy to see, under the condition (4) that all the terms in formula (11), (12)make sense. Furthermore, frome (10), we also deduce that* $u, v \in C([0,T_0], H_0^1(\Omega))$. *This fact implies that* u, v *have a pointwise meaning on time, so also (13) satisfies in the usual sense.*

3. Upper bound for blow-up time

Firstly, we establish the existence and uniqueness of a local solution for problem(1). This can be achieved through the application of Faedo-Galerkin techniques, as demonstrated in ([7], Theorem 3.2). Here, the proof is thus omitted.

Theorem 3. *Let* $(u_0, v_0) \in \left(W_0^{1,m(.)} \cap L^{p(.)}(\Omega)\right) \times \left(W_0^{1,n(.)} \cap L^{p(.)}(\Omega)\right)$ *be given. Assume that the conditions listed in section 1 for* $m(.), n(.), p(.), A,$ *and* B *hold. Then, according to definition 1, problem (1) has a unique local weak solution* (u, v) *on* $[0, T)$. *Moreover, either* (u, v) *can be extended to the whole of* $[0, \infty)$ *or there is* $T < \infty$ *such that* $\lim_{t \longrightarrow T} \|u\|_{H_0^1(\Omega)} + \|v\|_{H_0^1(\Omega)} = \infty$

The decay of the energy of the system (1) is given in the following Lemma

Lemma 4. *For* $(u_0, v_0) \in \left(W_0^{1,m(.)} \cap L^{p(.)}(\Omega)\right) \times \left(W_0^{1,n(.)} \cap L^{p(.)}(\Omega)\right)$. *The energy functional* E *of the problem (1) is decreasing function. Here*

$$\begin{aligned} E(t) &= \int_\Omega \frac{1}{m(x)} A \left|\nabla u\right|^{m(x)-2} \nabla u.\nabla u dx + \int_\Omega \frac{1}{n(x)} B \left|\nabla v\right|^{n(x)-2} \nabla v.\nabla v dx \\ &\quad - \int_\Omega \frac{1}{p(x)} \left|uv\right|^{p(x)} dx. \end{aligned} \tag{14}$$

Proof. multiplying the first differential equations in (1) by u_t, the second one by v_t, integrating the two equation over Ω, adding the two results,

$$\begin{aligned} &\int_\Omega \left(\left|u_t\right|^2 + \Delta u_t u_t - \operatorname{div}(A \left|\nabla u\right|^{m(x)-2} \nabla u) u_t\right) dx \\ &+ \int_\Omega \left(\left|v_t\right|^2 + \Delta v_t v_t - \operatorname{div}(\left|\nabla v\right|^{n(x)-2} \nabla v) v_t\right) dx \\ = &\int_\Omega \left(|uv|^{p(x)-2} u u_t v^2 + |uv|^{p(x)-2} u^2 v v_t\right) dx \end{aligned}$$

Then, we use the generalized Green formula and the boundary conditions, to find

$$\int_\Omega \left(\left|u_t\right|^2 + \left|\nabla u_t\right|^2 + A \left|\nabla u\right|^{m(x)-2} \nabla u.\nabla u_t\right) dx$$

$$+ \int_\Omega \left(\left|v_t\right|^2 + \left|\nabla v_t\right|^2 + B \left|\nabla u\right|^{n(x)-2} \nabla v.\nabla v_t\right) dx = \frac{d}{dt} \int_\Omega \frac{1}{p(x)} \left|uv\right|^{p(x)} dx$$

This implies that

$$\int_{\Omega}\left(|u_t|^2+|\nabla u_t|^2\right)dx+\frac{d}{dt}\int_{\Omega}\frac{1}{m(x)}A\,|\nabla u|^{m(x)-2}\,\nabla u.\nabla u dx$$

$$-\int_{\Omega}\frac{1}{m(x)}A'\,|\nabla u|^{m(x)-2}\,\nabla u.\nabla u dx+\int_{\Omega}\left(|v_t|^2+|\nabla v_t|^2\right)dx$$

$$+\frac{d}{dt}\int_{\Omega}\frac{1}{n(x)}B\,|\nabla v|^{n(x)-2}\,\nabla v.\nabla v dx-\int_{\Omega}\frac{1}{n(x)}B'\,|\nabla v|^{n(x)-2}\,\nabla v.\nabla v dx$$

$$=\frac{d}{dt}\int_{\Omega}\frac{1}{p(x)}\,|uv|^{p(x)}\,dx$$

so

$$\begin{aligned}E'(t) &= -\|u_t\|^2_{H_0^1(\Omega)}-\|v_t\|^2_{H_0^1(\Omega)}+\int_{\Omega}\frac{1}{m(x)}A'\,|\nabla u|^{m(x)-2}\,\nabla u.\nabla u dx\\ &\quad+\int_{\Omega}\frac{1}{n(x)}B'\,|\nabla v|^{n(x)-2}\,\nabla v.\nabla v dx\end{aligned}$$

Taking into account condition (3) on A' and B', we find

$$E'(t)\leq-\|u_t\|^2_{H_0^1(\Omega)}-\|v_t\|^2_{H_0^1(\Omega)}\leq 0.$$

∎

Theorem 5. *Assume that (2)- (5) hold. Let (u,v) be a solution of (1) and assume that $(u_0,v_0)\in\left(W_0^{1,m(.)}\cap L^{p(.)}(\Omega)\right)\times\left(W_0^{1,n(.)}\cap L^{p(.)}(\Omega)\right)$ satisfy*

$$\int_{\Omega}\left(\frac{1}{p(x)}\,|u_0v_0|^{p(x)}-\frac{1}{m(x)}A\,|\nabla u_0|^{m(x)-2}\,\nabla u_0.\nabla u_0\right.$$
$$\left.-\frac{1}{n(x)}B\,|\nabla v_0|^{n(x)-2}\,\nabla v_0.\nabla v_0\right)dx\geq 0, \tag{15}$$

then the solution (u,v) blows up at finite time $T_{\max}>0$ in $H_0^1(\Omega)$-norm. Furthermore, there exists an upper bound for the time is given by

$$T_{\max}\leq\frac{2\,(G(0))^{\left(\frac{2-m_-}{2}\right)}}{(m_--2)\,K} \tag{16}$$

where K is a suitable positive constant is given later and the constant $G(0)=\|u_0\|^2_{H_0^1(\Omega)}+\|v_0\|^2_{H_0^1(\Omega)}$.

Proof. Let us define the auxiliary function

$$G(t)=\|u\|^2_{H_0^1(\Omega)}+\|v\|^2_{H_0^1(\Omega)}. \tag{17}$$

Our objective is to demonstrate that G leads to a blow up in finite time by satisfying a differential inequality. Multiply u and v by the first and second equations in (1), integrate the two equations over Ω, and add the two results to obtain

$$\begin{aligned}&\int_{\Omega}(uu_t+\nabla u\nabla u_t+vv_t+\nabla v\nabla v_t)\,dx\\ =&\int_{\Omega}\left(2\,|uv|^{p(x)}-A\,|\nabla u|^{m(x)-2}\,\nabla u.\nabla u-B\,|\nabla v|^{n(x)-2}\,\nabla v.\nabla v\right)dx\end{aligned} \tag{18}$$

By differentiating $G(t)$ with respect to t, we get

$$\begin{aligned}
G^{'}(t) &= 2\int_{\Omega}(uu_t+\nabla u\nabla u_t+vv_t+\nabla v\nabla v_t)\,dx\\
&= 2\int_{\Omega}\left(2\,|uv|^{p(x)}-A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u-B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v\right)dx\\
&= 4\int_{\Omega}p(x)\left[\frac{|uv|^{p(x)}}{p(x)}-\frac{A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u}{m(x)}-\frac{B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v}{n(x)}\right]dx\\
&\quad +4\int_{\Omega}\left[\begin{array}{c} p(x)\left(\frac{1}{m(x)}-\frac{1}{p(x)}\right)A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u\\ +p(x)\left(\frac{1}{n(x)}-\frac{1}{p(x)}\right)B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v\end{array}\right]dx\\
&\quad +2\int_{\Omega}A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u dx+2\int_{\Omega}B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v dx \qquad (19)
\end{aligned}$$

By (15) and the fact that $E(t)\leq E(0)$ ($E^{'}(t)\leq 0$) (See **Lemma** 4), we have

$$\begin{aligned}
&\int_{\Omega}p(x)\left[\frac{|uv|^{p(x)}}{p(x)}-\frac{A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u}{m(x)}-\frac{B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v}{n(x)}\right]dx\\
&\geq \int_{\Omega}p(x)\left[\frac{|u_0v_0|^{p(x)}}{p(x)}-\frac{A\,|\nabla u_0|^{m(x)-2}\nabla u_0.\nabla u_0}{m(x)}-\frac{B\,|\nabla v_0|^{n(x)-2}\nabla v_0.\nabla v_0}{n(x)}\right]dx\\
&\geq p_-\int_{\Omega}\frac{|u_0v_0|^{p(x)}}{p(x)}-\frac{A\,|\nabla u_0|^{m(x)-2}\nabla u_0.\nabla u_0}{m(x)}-\frac{B\,|\nabla v_0|^{n(x)-2}\nabla v_0.\nabla v_0}{n(x)}dx\geq 0. \qquad (20)
\end{aligned}$$

By (19), (20) and 4, we see

$$\begin{aligned}
G^{'}(t) &\geq 4\int_{\Omega}\left[p_-\left(\frac{1}{m_+}-\frac{1}{p_-}\right)A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u\right.\\
&\qquad\left.+\,p_-\left(\frac{1}{n_+}-\frac{1}{p_-}\right)B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v\right]dx\\
&\quad +2\int_{\Omega}A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u dx+2\int_{\Omega}B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v dx\\
&= \int_{\Omega}\left(4p_-\left(\frac{1}{m_+}-\frac{1}{p_-}\right)+2\right)A\,|\nabla u|^{m(x)-2}\nabla u.\nabla u dx\\
&\quad +\int_{\Omega}\left(4p_-\left(\frac{1}{n_+}-\frac{1}{p_-}\right)+2\right)B\,|\nabla v|^{n(x)-2}\nabla v.\nabla v dx.
\end{aligned}$$

Using condition (2) on A and B, we obtain

$$G^{'}(t)\geq a_0C_1\int_{\Omega}|\nabla v|^{m(x)}dx+b_0C_2\int_{\Omega}|\nabla v|^{n(x)}dx. \qquad (21)$$

where $C_1 = \left(4p_- \left[\frac{1}{m_+} - \frac{1}{p_-}\right] + 2\right) > 0$ and $C_2 = \left(4p_- \left[\frac{1}{n_+} - \frac{1}{p_-}\right] + 2\right) > 0$.

Now we define the sets:

$$\Omega_+ = \{x \in \Omega : |\nabla u| \geq 1, |\nabla v| \geq 1\} \text{ and } \Omega_- = \{x \in \Omega : |\nabla u| < 1, |\nabla v| < 1\},$$

so we get

$$\begin{aligned} G'(t) &\geq a_0 C_1 \left(\int_{\Omega_-} |\nabla u|^{m_+} dx + \int_{\Omega_+} |\nabla u|^{m_-} dx\right) \\ &\quad + b_0 C_2 \left(\int_{\Omega_-} |\nabla v|^{n_+} dx + \int_{\Omega_+} |\nabla v|^{n_-} dx\right) \\ &\geq C_3 \left[\left(\int_{\Omega_-} |\nabla u|^2 dx\right)^{\frac{m_+}{2}} + \left(\int_{\Omega_+} |\nabla u|^2 dx\right)^{\frac{m_-}{2}}\right. \\ &\quad \left. + \left(\int_{\Omega_-} |\nabla v|^2 dx\right)^{\frac{n_+}{2}} + \left(\int_{\Omega+} |\nabla v|^2 dx\right)^{\frac{n_-}{2}}\right], \end{aligned}$$

using the fact that $\|\nabla u\|_2 \leq C \|\nabla u\|_r$ for all $r \geq 2$. This implies that

$$\begin{cases} \left(G'(t)\right)^{\frac{2}{m_+}} \geq C_4 \int_{\Omega_-} |\nabla u|^2 dx, & \left(G'(t)\right)^{\frac{2}{m_-}} \geq C_5 \int_{\Omega_+} |\nabla u|^2 dx, \\ \left(G'(t)\right)^{\frac{2}{n_+}} \geq C_6 \int_{\Omega_-} |\nabla v|^2 dx, \quad \text{and} & \left(G'(t)\right)^{\frac{2}{n_-}} \geq C_7 \int_{\Omega_+} |\nabla v|^2 dx. \end{cases} \tag{22}$$

The Poincaré inequality gives $\|\nabla u\|_2^2 \geq \lambda \|u\|_2^2$ and $\|\nabla v\|_2^2 \geq \lambda \|v\|_2^2$, where λ is the first eigenvalue of $-\Delta$ with zero Dirichlet conditions. Therefore, we get

$$\|\nabla u\|_2^2 = \frac{1}{1+\lambda} \|\nabla u\|_2^2 + \frac{\lambda}{1+\lambda} \|\nabla u\|_2^2 \geq \frac{\lambda}{1+\lambda} \|u\|_{H_0^1(\Omega)}^2 \tag{23}$$

and

$$\|\nabla v\|_2^2 = \frac{1}{1+\lambda} \|\nabla v\|_2^2 + \frac{\lambda}{1+\lambda} \|\nabla v\|_2^2 \geq \frac{\lambda}{1+\lambda} \|v\|_{H_0^1(\Omega)}^2 . \tag{24}$$

It follows from (22), (23) and (24) that

$$\left(G'(t)\right)^{\frac{2}{m_+}} + \left(G'(t)\right)^{\frac{2}{m_-}} + \left(G'(t)\right)^{\frac{2}{n_+}} + \left(G'(t)\right)^{\frac{2}{n_-}} \geq C_8 \int_\Omega |\nabla u|^2 dx + C_9 \int_\Omega |\nabla v|^2 dx$$

$$\begin{aligned} &\geq \frac{\lambda}{1+\lambda}\left(C_8 \|u\|_{H_0^1}^2 + C_9 \|v\|_{H_0^1}^2\right) \\ &\geq C_{10} G(t), \end{aligned} \tag{25}$$

or

$$\left(G^{'}(t)\right)^{\frac{2}{m_-}}\left(1+\left(G^{'}(t)\right)^{2\left(\frac{1}{m_+}-\frac{1}{m_-}\right)}\right) + \left(G^{'}(t)\right)^{\frac{2}{n_-}}\left(1+\left(G^{'}(t)\right)^{2\left(\frac{1}{n_+}-\frac{1}{n_-}\right)}\right) \geq C_{10} G(t), \tag{26}$$

where $C_8 = \min(C_4, C_5)$, $C_9 = \min(C_6, C_7)$ and $C_{10} = \min(C_8, C_9)$.

By (25) and the fact that $G(t) \geq G(0) > 0$ ($G^{'}(t) \geq 0$) (see 21), we have either

$$\begin{cases} \left(G^{'}(t)\right)^{\frac{2}{m_+}} \geq \frac{C_{10}}{4} G(t) \geq \frac{C_{10}}{4} G(0) \quad \text{or} \quad \left(G^{'}(t)\right)^{\frac{2}{m_-}} \geq \frac{C_{10}}{4} G(t) \geq \frac{C_{10}}{4} G(0) \\ \left(G^{'}(t)\right)^{\frac{2}{n_+}} \geq \frac{C_{10}}{4} G(t) \geq \frac{C_{10}}{4} G(0) \quad \text{or} \quad \left(G^{'}(t)\right)^{\frac{2}{n_-}} \geq \frac{C_{10}}{4} G(t) \geq \frac{C_{10}}{4} G(0), \end{cases} \tag{27}$$

this implies that

$$\begin{cases} G^{'}(t) \geq C_{11}\left(G(0)\right)^{\frac{m_+}{2}} \quad \text{or} \quad G^{'}(t) \geq C_{12}\left(G(0)\right)^{\frac{m_-}{2}} \quad \text{or} \\ G^{'}(t) \geq C_{13}\left(G(0)\right)^{\frac{n_+}{2}} \quad \text{or} \quad G^{'}(t) \geq C_{14}\left(G(0)\right)^{\frac{n_-}{2}} \end{cases} \tag{28}$$

Therefore , we have that

$$G^{'}(t) \geq \alpha, \tag{29}$$

where $\alpha = \min\left\{C_{11}\left(G(0)\right)^{\frac{m_+}{2}}, C_{12}\left(G(0)\right)^{\frac{m_-}{2}}, C_{13}\left(G(0)\right)^{\frac{n_+}{2}}, C_{14}\left(G(0)\right)^{\frac{n_-}{2}}\right\}$.
Furthermore , from $\left(\frac{1}{m_+} - \frac{1}{m_-}\right) \leq 0$, $\left(\frac{1}{n_+} - \frac{1}{n_-}\right) \leq 0$ and (26), we get

$$\left(G^{'}(t)\right)^{\frac{2}{m_-}}\left(1+\alpha^{2\left(\frac{1}{m_+}-\frac{1}{m_-}\right)}\right) + \left(G^{'}(t)\right)^{\frac{2}{n_-}}\left(1+\alpha^{2\left(\frac{1}{n_+}-\frac{1}{n_-}\right)}\right) \geq C_8 G(t) \tag{30}$$

which implies that

$$\left(G^{'}(t)\right)^{\frac{2}{m_-}} + \left(G^{'}(t)\right)^{\frac{2}{n_-}} \geq \frac{C_{10}}{\beta_1} G(t), \tag{31}$$

where the constant $\beta_1 = \max\left[\left(1+\alpha^{2\left(\frac{1}{m_+}-\frac{1}{m_-}\right)}\right), \left(1+\alpha^{2\left(\frac{1}{n_+}-\frac{1}{n_-}\right)}\right)\right]$.

Then

$$\left(G^{'}(t)\right)^{\frac{2}{m_-}}\left(1+\left(G^{'}(t)\right)^{2\left(\frac{1}{n_-}-\frac{1}{m_-}\right)}\right) \geq \frac{C_{10}}{\beta_1} G(t). \tag{32}$$

From (4), we observe that $2\left(\frac{1}{n_-}-\frac{1}{m_-}\right)\leq 0$, and by using (29), we get

$$\left(G^{'}(t)\right)^{\frac{2}{m_-}}\left(1+\alpha^{2\left(\frac{2}{n_-}-\frac{1}{m_-}\right)}\right)\geq\frac{C_8}{\beta_1}G(t),$$

then

$$G^{'}(t)\geq K\left(G(t)\right)^{\frac{m_-}{2}}, \tag{33}$$

where $K=\left(\frac{C_8}{\beta_1\left(1+\alpha^{2\left(\frac{2}{n_-}-\frac{1}{m_-}\right)}\right)}\right)^{\frac{m_-}{2}}$ is a positive constant. Integrating (33) from 0 to t, we get

$$G(t)\geq\frac{1}{\left((G(0))^{\left(1-\frac{m_-}{2}\right)}+\frac{(2-m_-)Kt}{2}\right)^{\frac{2}{m_--2}}} \tag{34}$$

which implies that $G(t)\longrightarrow\infty$ as $t\longrightarrow T_{\max}$ in $H_0^1(\Omega)$, where

$$T_{\max}\leq\frac{2\left(G(0)\right)^{\left(\frac{2-m_-}{2}\right)}}{\left(m_--2\right)K}.$$

Consequently, the solution to the problem (1) blows up in $H_0^1(\Omega)$-norm in finite time. Hence the proof is completed. ∎

4. Lower bound for blow-up time

In this section, we determine a lower bound for the blow-up time of the problem (1).

Theorem 6. *Suppose that the conditions on $s(x), r(x), A$, and B, given in section 1, hold. Furthermore assume that $2<s_+<\infty$ if $n\leq 2$, $2<s_+\leq\frac{2n}{n-2}$ if $n>2$, $(u_0,v_0)\in\left(W_0^{1,m(.)}\cap L^{p(.)}(\Omega)\right)\times\left(W_0^{1,n(.)}\cap L^{p(.)}(\Omega)\right)$ and (u,v) be a blow-up solution of problem (1), then a lower bound for blow-up time $T_{\min}$ can be estimated in the form*

$$T_{\min}\geq\int_{G(0)}^{\infty}\frac{d\xi}{2\left(B_+^{2P_+}\xi^{P_+}+B_-^{2P_-}\xi^{P_-}\right)} \tag{35}$$

where B_-, B_+ are the corresponding embedding constants satisfying $\|w\|_{L^{2p_-}}\leq B_-\|\nabla w\|_2$ and $\|w\|_{L^{2p_+}}\leq B_+\|\nabla w\|_2$, where $w=u$ or v.

Proof. Consider $G(t)$ as in (17)

$$G(t)=\|u\|^2_{H^1_0(\Omega)}+\|v\|^2_{H^1_0(\Omega)}.$$

Multiply the first equation in (1) by u the second one by v integrating the two equation over Ω, adding the two results to get

$$\int_\Omega (uu_t+\nabla u\nabla u_t+vv_t+\nabla v\nabla v_t)\,dx = \int_\Omega \Big(2\,|uv|^{p(x)}-A\,|\nabla u|^{m(x)-2}\,\nabla u.\nabla u -B\,|\nabla v|^{n(x)-2}\,\nabla v.\nabla v\Big)\,dx$$

A direct differentiation of $G(t)$ yields

$$\begin{aligned} G^{'}(t) &= 2\int_\Omega (uu_t+\nabla u\nabla v+vv_t+\nabla v\nabla v_t)\,dx \\ &= 2\left[\int_\Omega \Big(2\,|uv|^{p(x)}-A\,|\nabla u|^{m(x)-2}\,\nabla u.\nabla u-B\,|\nabla v|^{n(x)-2}\,\nabla v.\nabla v\Big)\,dx\right].\end{aligned}$$

Taking into account condition (2) on A and B, we find

$$G'(t)\le 4\int_\Omega |uv|^{p(x)}\,dx. \tag{36}$$

Defining the sets $\Omega_+=\{x\in\Omega:|u|\ge 1,|v|\ge 1\}$ and $\Omega_-=\{x\in\Omega:|u|<1,|v|<1\}$, and by simple use of Young's inequality, we get

$$\begin{aligned}\int_\Omega |uv|^{p(x)}\,dx &\le \int_{\Omega_+}|uv|^{p_+}\,dx+\int_{\Omega_-}|uv|^{p_-}\,dx \\ &\le \frac12\left(\int_{\Omega_+}|u|^{2P_+}\,dx+\int_{\Omega_+}|v|^{2P_+}\,dx+\int_{\Omega_-}|u|^{2P_-}\,dx+\int_{\Omega_-}|v|^{2P_-}\,dx\right) \\ &\le \frac12\left(\int_\Omega\Big(|u|^{2P_+}+|v|^{2P_+}\Big)\,dx+\int_\Omega\Big(|u|^{2P_-}+|v|^{2P_-}\Big)\,dx\right),\end{aligned}$$

now, by Sobolev embeddings (Lemma 2), we have

$$\begin{aligned}\int_\Omega |uv|^{p(x)}\,dx &\le \frac12 B_+^{2P_+}\left(\left(\int_\Omega|\nabla u|^2\,dx\right)^{P_+}+\left(\int_\Omega|\nabla v|^2\,dx\right)^{P_+}\right) \\ &\quad +\frac12 B_-^{2P_-}\left(\left(\int_\Omega|\nabla u|^2\,dx\right)^{P_-}+\left(\int_\Omega|\nabla v|^2\,dx\right)^{P_-}\right) \\ &\le \frac12\left[B_+^{2P_+}\left(\int_\Omega\Big(|\nabla u|^2+|\nabla v|^2\Big)\,dx\right)^{P_+} +B_-^{2P_-}\left(\int_\Omega\Big(|\nabla u|^2+|\nabla v|^2\Big)\,dx\right)^{P_-}\right] \\ &\le \frac12\left[B_+^{2P_+}\,(G(t))^{P_+}+B_-^{2P_-}\,(G(t))^{P_-}\right],\end{aligned} \tag{37}$$

where B_-, B_+ are the corresponding embedding constants satisfying $\|w\|_{L^{2p_-}}\le B_-\,\|\nabla w\|_2$ and $\|w\|_{L^{2p_+}}\le B_+\,\|\nabla w\|_2$, where $w=u,v$. Therefore (36) becomes

$$G^{'}(t) \leq 2\left(B_{+}^{2P_{+}}\left(G(t)\right)^{P_{+}} + B_{-}^{2P_{-}}\left(G(t)\right)^{P_{-}}\right) \tag{38}$$

By integrating both sides of the inequality (38) over $(0,T)$, we obtain

$$\int_{G(0)}^{G(t)} \frac{d\xi}{2\left(B_{+}^{2P_{+}}\xi^{P_{+}} + B_{-}^{2P_{-}}\xi^{P_{-}}\right)} \leq T$$

If (u,v) blows up in H_0^1-norm, then we obtain a lower bound for $T_{\min}$ given by

$$T_{\min} \geq \int_{G(0)}^{\infty} \frac{d\xi}{2\left(B_{+}^{2P_{+}}\xi^{P_{+}} + B_{-}^{2P_{-}}\xi^{P_{-}}\right)}. \tag{39}$$

The integral (39) is bound since exponents $p_{+} \geq p_{-} > 2$. This completes the proof of Theorem6. ∎

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References

[1] A.B. Al'shin, M.O. Korpusov and A.G. Sveshnikov, A. G. Blow-up in nonlinear Sobolev type equations, (Vol. 15). Walter de Gruyter, (2011).

[2] L. Songzhe, W. Gao, C. Cao et al, Study of the solutions to a model porous medium equation with variable exponent of nonlinearity. J Math Anal Appl. 2008;342:27–38.

[3] O. Korpusov, A.G. Sveshnikov, Three-dimensional nonlinear evolution equations of pseudo-parabolic type in problems of mathematical physics, Zh. Vychist. Math. Fiz. 43 (12) (2003) 1835-1869.

[4] O. Korpusov, A.G. Sveshnikov, Three-dimensional nonlinear evolution equations of pseudo-parabolic type in problems of mathematical physics, II. Vychisl. Math. Math. Fiz. 44 (11) (2004) 2041-2048.

[5] B. Rebiai, S. Benachour, Global Classical Solutions for Reaction–diffusion Systems With Nonlinearities of Exponential Growth, J. Evol. Equ. 10 (2010), 511–527

[6] Q. Qi, Y. Chen and Q. Wang, Blow-up phenomena for a pseudo-parabolic system with variable exponents, Electronic Journal of Qualitative Theory of Differential Equations, 2017, No. 36, 1-9.

[7] S. A. Messaoudi, O. Bouhoufani, I. Hamchi, M. Alahyane, Existence and blow up in a system of wave equations with nonstandard nonlinearities, Electronic Journal of Di§erential Equations, Vol. 2021 (2021), No. 91, pp. 1ñ33

[8] S.N. Antontsev, J.I. Diaz, S. Shmarev, Energy methods for free boundary problems: applications to non linear PDEs and fluid mechanics, in : Progress in Nonlinear Differential Equations and Their Applications, Vol 48. Bikhäuser, Boston 2.200

[9] S. Kouachi, Existence of global solutions to reaction-diffusion systemswith nonhomogeneous boundary conditions via a Lypunov functional,Electron. J. Diff. Equ, 2002(88) (2002), 1-13.

[10] K. Masuda, On the global existence and asymptotic behaviour of solutionsof reaction-diffusion equations, Hokkaido Math. J., 12(3) (1983), 303-308

[11] M. Mebarki, A. Moumeni, Global Solution of System Reaction Diffusion With Full Matrix, Glob. J. Math. Anal. (2015), 04-25

[12] N.D. Alikakos, L^p -Bounds of Solutions of Reaction-Diffusion Equations, Commun. Part. Differ. Equ. 4 (1979), 827–868.

[13] H. Di,Y. Shang, X. Peng, Blow-up phenomena for a pseudo-parabolic equation with variable exponents, Aplied Mathematics Lettres. 64 (2017) 67-73

[14] K. Masuda, On the Global Existence and Asymptotic Behavior of Solutions of Reaction-Diffusion Equations, Hokkaido Math. J. 12 (1983), 360-370..

[15] X.Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. 263, 424–446 (2001)

[16] L. Diening, H. Petteri, P. Hasto, et al, Lebesgue and Sobolev spaces with variable exponents. In: Lecture Notes Mathematics. Vol. 2017, 2011
[17] N. Lakshmipriya, S. Gnanavel, K. Balachandran, Yong-Ki Ma, Existence and blow up of weak solutions of a pseudo-parabolic equation with logarithmic nonlinearity. Boudary Value Problems (2022) 2022:30.
[18] W. Lian, J. Wang , R. Xu, Global existence and blow up of solutions for pseudo parabolic equation with singular potential. J. Differ. Equ. 269, 4914-4959 (2020).
[19] S.N. Antontsev, J.I. Diaz, S. Shmarev, Energy methods for free boundary problems: applications to non linear PDEs and fluid mechanics, in : Progress in Nonlinear Differential Equations and Their Applications, Vol 48. Bikhäuser, Boston 2.200
[20] M. Liao, B. Guo, Q. Li, global existence and energy decay estimates for weak solutions to the pseudo parabolic equation with variable exponents. Math. Methods. Appl. Sci. 43, 2516-2527 (2020).
[21] M. Rúžička, Electroheological fluids, modeling and mathematical theory, Lecture Notes in Mathematics, vol 1748 Springer, Berlin, 2000.
[22] Aboulaich R, Meskine D, Souissi A. New diffusion models in image processing. Comput Math Appl. 2008;56: 874–882.
[23] R. Z. Xu, J. Su, Global existence and finite time blow up for a class of semilinear pseudo-parabolic equations, J. Funct. Anal.264 (2013) 2732-2763.
[24] A.Toualbia, Upper-Lower Bounds For Blow-up Time In Initial Value Boundary Problems For A Class Of Pseudo-Parabolic Equations, Applied Mathematics E-Notes, 24(2024), 228-236
[25] A. Toualbia, N.Barrouk, Blow-up phenomena for pseudo-parabolic Kirchhoff equations with logarithmic nonlinearity, Results in Nonlinear Analysis 7 (2024) No. 2, 115-126
[26] P. Luo, Blow-up phenomena for a pseudo parabolic equation . Math. Methods Appl. Sci.38 (2015) 2636-2641.
[27] R. Pan, Y. Gao , and Q. Meng, Properties of Weak Solutions for a Pseudo-parabolic Equation with Logarithmic Nonlinearity of Variable Exponent, Hindawi Journal of Mathematics. Volume 2023, 1-9.
[28] Songzhe L, Gao W, Cao C, et al. Study of the solutions to a model porous medium equation with variable exponent of nonlinearity. J Math Anal Appl. 2008;342:27–38.

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