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HAMILTONIAN VECTOR FIELDS ON LOCALLY CONFORMALLY SYMPLECTIC A-MANIFOLDS

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Abstract. In this paper, we consider M to be a paracompact smooth manifold, A a local algebra and, M^A the Weil bundle. We construct the Hamiltonian vector fields on the symplectic A-manifold M^A. Additionally, we investigate and establish the properties of both locally and globally defined Hamiltonian vector fields when M^A is a locally conformally symplectic A-manifold.

1. Introduction

In 1953, **André Weil** [11] introduced the theory of bundles on near points, which has since gained a lot of attention in differential geometry. In the following, a commutative associative unitary real algebra is represented by A. A Weil algebra is a finite-dimensional local algebra of the following form:

$$A = \mathbb{R} \oplus \mathfrak{m} \tag{1.1}$$

where $\mathfrak{m}$ is its unique maximal ideal (see [7]). As an example, we define the algebra $\mathbb{D} = \mathbb{R}[x]/\langle x^2 \rangle$ of dual numbers whose the maximal ideal is $\mathfrak{m} = x\mathbb{R}$.
Let M be a paracompact smooth manifold, $C^\infty(M)$ the algebra of smooth functions on M. Given a Weil algebra A with maximal ideal $\mathfrak{m}$ and basis $a_1, \cdots, a_\alpha$ with $a_1 = \mathbf{1} \in \mathbb{R}$. We recall that an A-point of near to $x \in M$ is a morphism of algebras

$$\xi : C^\infty(M) \longrightarrow A$$

such that

$$\xi(f) = f(x) \cdot a_1 + \lambda = f(x) + \lambda \tag{1.2}$$

for all $x \in M$, where $\lambda \in \mathfrak{m}$. We denote by

$$M^A = \bigcup_{x \in M} M_x^A$$

the manifold of infinitely near points of kind A where $M_x^A \subset Hom_{\mathbb{R}}(C^\infty(M), A)$ is the set of all A-points of M near to x and $\pi : M^A \longrightarrow M$ is the projection such that $\pi\left(M_x^A\right) = x$. The triple $\left(M^A, \pi, M\right)$ defined is a bundle called bundle of infinitely

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near points or simply Weil bundle [6].
If (U,φ) is a local chart of M with coordinate system $(x_1, \cdots, x_n)$, the map

$$\varphi^A : U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), \cdots, \xi(x_n))$$

is a bijection from U^A onto an open set of A^n. In addition, if $(U_i,\varphi_i)_{i\in I}$ is an atlas of M, then $(U_i^A,\varphi_i^A)_{i\in I}$ is also an A-atlas of M^A. Accordingly, M^A is considered an A-manifold, with $\dim M^A = \dim M = n$ (for further information, see [1]).
If M and N are smooth manifolds and $g : M \longrightarrow N$ is a differentiable map of class C^∞, then the map

$$g^A : M^A \longrightarrow N^A, \xi \longmapsto g^A(\xi)$$

such that

$$\left[g^A(\xi)\right](h) = \xi(h \circ g) \tag{1.3}$$

for all $h \in C^\infty(N)$, is differentiable. Thus, for $f \in C^\infty(M)$, the map

$$f^A : M^A \longrightarrow \mathbb{R}^A = A, \xi \longmapsto \left[f^A(\xi)\right](id_{\mathbb{R}}) = \xi(id_{\mathbb{R}} \circ f) = \xi(f) \tag{1.4}$$

is differentiable of class C^∞.
The set $C^\infty(M^A, A)$ of smooth functions on M^A with values in A is a commutative algebra with unit over A and the mapping

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A$$

is an injective homomorphism of algebras. Then, we have the following properties:

$$(f+g)^A = f^A + g^A;$$
$$(\lambda \cdot f)^A = \lambda \cdot f^A;$$
$$(f \cdot g)^A = f^A \cdot g^A,$$

here $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$.

We define $\mathfrak{X}(M^A)$ as the set of all smooth sections of TM^A.
According to [2], the set $\mathfrak{X}(M^A)$ is a module of vector fields on M^A over $C^\infty(M^A)$ and $C^\infty(M^A, A)$.
The theory of prolongation of some geometric structures on Weil bundles has been in the last decades developed in different directions by many researchers (see [1], [2], [3] and [10]). In [2], the author defines and studies the notions of Jacobi structures on M^A regarded as A-manifold. In [9], the authors give a characterization of Hamiltonian vector fields on M^A in the case of Poisson manifolds and symplectic manifolds. The author of [4] characterizes in terms of Lie-Rinehart-Jacobi algebras on the $C^\infty(M)$-module of vector fields $\mathfrak{X}(M)$ the locally and globally Hamiltonian vector fields and gives their properties.
In this paper, we consider M^A as an A-manifold, we discuss the construction of Hamiltonian vector fields on the symplectic A-manifold M^A. We also study and establish the properties of locally and globally Hamiltonian vector fields when M^A is a locally conformally symplectic manifold on Weil bundles.

2. Generalities and basic notions

In this section, we recall some constructions of A-structures.

2.1. Vector fields on Weil bundles.

Theorem 2.1. [10] *The following assertions are equivalent:*

1) *A vector field on M^A is a differentiable section of the tangent bundle (TM^A, π_{M^A}, M^A).*
2) *A vector field on M^A is a derivation of $C^\infty(M^A)$.*
3) *A vector field on M^A is a derivation of $C^\infty(M^A, A)$ which is A-linear.*
4) *A vector field on M^A is a linear map*

$$Y : C^\infty(M) \longrightarrow C^\infty(M^A, A)$$

such that

$$Y(f \cdot g) = Y(f) \cdot g^A + f^A \cdot Y(g), \tag{2.1}$$

for all $f, g \in C^\infty(M)$.

We note by $\mathcal{D}er_A[C^\infty(M^A, A)]$ the $C^\infty(M^A, A)$-module of derivations of $C^\infty(M^A, A)$ which are A-linear.

2.2. Differential forms and d^A-cohomology on Weil bundles.

An A-covector field at $\xi \in M^A$ is a linear form on the A-module $T_\xi M^A$. The set, $T^*_\xi M^A$, of A-covectors at $\xi \in M^A$ is an A-free module of dimension n and

$$T^*M^A = \bigcup_{\xi \in M^A} T^*_\xi M^A$$

is an A-manifold of dimension $2n$. The set, $\Lambda^1(M^A, A)$, of differential sections of T^*M^A is a $C^\infty(M^A, A)$-module and we say that $\Lambda^1(M^A, A)$ is the $C^\infty(M^A, A)$-module of differential A-forms of degree $+1$.
For $p \in \{0\} \cup \mathbb{N}$ and for $\xi \in M^A$, we note $\mathcal{L}^p_{sks}(T_\xi M^A, A)$ the A-module of skew-symmetric multilinear forms of degree p on the A-module $T_\xi M^A$. We have, $\mathcal{L}^0_{sks}(T_\xi M^A, A) = A$. For two integers p and q, we define the wedge product

$$\wedge : \mathcal{L}^p_{sks}(T_\xi M^A, A) \times \mathcal{L}^q_{sks}(T_\xi M^A, A) \longrightarrow \mathcal{L}^{p+q}_{sks}(T_\xi M^A, A), (\alpha,\, \beta) \longmapsto \alpha \wedge \beta.$$

The set,

$$A^p(T^*_\xi M^A,\, A) = \bigcup_{\xi \in M^A} \mathcal{L}^p_{sks}(T_\xi M^A, A),$$

is an A-manifold of dimension $n + \mathcal{C}^p_n$. The set, $\Lambda^p(M^A, A)$, of differential sections of $A^p(T^*M^A,\, A)$ is a $C^\infty(M^A, A)$-module. We say that $\Lambda^p(M^A, A)$ is the $C^\infty(M^A, A)$-module of A-differential forms of degree p on M^A and

$$\Lambda^\bullet(M^A, A) = \bigoplus_{p=0}^{n} \Lambda^p(M^A, A),$$

is the algebra of differential A-forms on M^A. The algebra $\Lambda^\bullet(M^A, A)$ of differential A-forms on M^A is canonically isomorphic to $A \otimes \Lambda^\bullet(M^A)$. We have $\Lambda^0(M^A, A) = C^\infty(M^A, A)$.

Theorem 2.2. *If η is a differential form of degree p on M (according to* [2]*), then there exists a unique differential A-form of degree p,*

$$\eta^A : \mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \times \cdots \times \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A)$$

such that

$$\eta^A(f_1^A\theta_1^A,\cdots,f_p^A\theta_p^A)=f_1^A\cdots f_p^A[\eta(\theta_1,\cdots,\theta_p)]^A, \tag{2.2}$$

for all $\theta_1,\cdots,\theta_p\in\mathfrak{X}(M)$ *and* $f_1,\cdots,f_p\in C^\infty(M)$.

The mapping $\Lambda^\bullet(M)\longrightarrow\Lambda^\bullet(M^A,A),\omega\longmapsto\omega^A$, is a morphism of graded $\mathbb{R}$-algebras, and if

$$d:\Lambda^\bullet(M)\longrightarrow\Lambda^\bullet(M)$$

is an exterior differential operator, following [2], we note

$$d^A:\Lambda^\bullet(M^A,A)\longrightarrow\Lambda^\bullet(M^A,A)$$

the cohomology operator associated with the representation

$$\mathfrak{X}(M^A)\longrightarrow\mathcal{D}er_A[C^\infty(M^A,A)],X\longmapsto X.$$

The mapping d^A is A-linear and verifies

$$d^A(\omega^A)=(d\omega)^A,\ \forall\,\omega\in\Lambda^\bullet(M).$$

2.3. Lie-Rinehart-Jacobi algebra structure on Weil bundles.

2.3.1. *Differential operators of order ≤ 1 on Weil bundles.*

Definition 2.3. *We have the following definitions.*

1) *An application δ is called differential operator of order ≤ 1 on M^A if*

$$\delta:C^\infty(M^A)\longrightarrow C^\infty(M^A)$$

is $\mathbb{R}$-linear such that

$$\delta(\varphi\cdot\psi)=\delta(\varphi)\cdot\psi+\varphi\cdot\delta(\psi)-\varphi\cdot\psi\cdot\delta(1_{C^\infty(M^A)}), \tag{2.3}$$

for all $\varphi,\psi\in C^\infty(M^A)$.

We note by $\mathcal{D}^{[1]}_{\mathbb{R}}(M^A)$ the $C^\infty(M^A)$-module of differential operators of order ≤ 1 on M^A.

2) *An application $\partial:C^\infty(M^A,A)\longrightarrow C^\infty(M^A,A)$ is called A-differential operator of order ≤ 1 if ∂ is A-linear such that*

$$\partial(\varphi_1\cdot\varphi_2)=\partial(\varphi_1)\cdot\varphi_2+\varphi_1\cdot\partial(\varphi_2)-\varphi_1\cdot\varphi_2\cdot\partial(1_{C^\infty(M^A,A)}) \tag{2.4}$$

for any $\varphi_1,\varphi_2\in C^\infty(M^A,A)$.

We note $\mathcal{D}^{[1]}_{A}(M^A)$ the set of differential operators of order ≤ 1 on $C^\infty(M^A,A)$. When $\partial(1_{C^\infty(M^A,A)})=0$, we say that ∂ is an A-derivation on $C^\infty(M^A,A)$.

Theorem 2.4. [8] *The following statements are equivalent:*

1) *A differential operator of order ≤ 1 on M^A is a $\mathbb{R}$-linear map*

$$\delta:C^\infty(M^A)\longrightarrow C^\infty(M^A)$$

such that

$$\delta(\varphi\cdot\psi)=\delta(\varphi)\cdot\psi+\varphi\cdot\delta(\psi)-\varphi\cdot\psi\cdot\delta(1_{C^\infty(M^A)}),\forall\,\varphi,\psi\in C^\infty(M^A). \tag{2.5}$$

2) *A differential operator of order ≤ 1 on M^A is an A-linear map*

$$\partial:C^\infty(M^A,A)\longrightarrow C^\infty(M^A,A)$$

such that

$$\partial(\varphi\cdot\psi)=\partial(\varphi)\cdot\psi+\varphi\cdot\partial(\psi)-\varphi\cdot\psi\cdot\partial(1_{C^\infty(M^A,A)}),\forall\,\varphi,\psi\in C^\infty(M^A,A). \tag{2.6}$$

3) *A differential operator of order* ≤ 1 *on* M^A *is a* $\mathbb{R}$*-linear map from* $C^\infty(M)$ *into* $C^\infty(M^A, A)$

$$\sigma : C^\infty(M) \longrightarrow C^\infty(M^A, A)$$

such that

$$\sigma(f \cdot g) = \sigma(f) \cdot g^A + f^A \cdot \sigma(g) - f^A \cdot g^A \cdot \sigma(1_{C^\infty(M)}), \forall\, f,g \in C^\infty(M). \quad (2.7)$$

Theorem 2.5. [8] *The application*

$$[\cdot,\cdot] : \mathcal{D}_A^{[1]}(M^A) \times \mathcal{D}_A^{[1]}(M^A) \longrightarrow \mathcal{D}_A^{[1]}(M^A), (\boldsymbol{\partial}_1, \boldsymbol{\partial}_2) \longmapsto \boldsymbol{\partial}_1 \circ \boldsymbol{\partial}_2 - \boldsymbol{\partial}_2 \circ \boldsymbol{\partial}_1 \quad (2.8)$$

is skew-symmetric A-bilinear and defines a structure of Lie A-algebra on the A-module $\mathcal{D}_A^{[1]}(M^A)$.
Moreover, we have, for all $\boldsymbol{\partial}_1, \boldsymbol{\partial}_2 \in \mathcal{D}_A^{[1]}(M^A)$*, for all* $\varphi \in C^\infty(M^A, A)$*,*

$$[\boldsymbol{\partial}_1, \varphi \cdot \boldsymbol{\partial}_2] = \Big(\boldsymbol{\partial}_1(\varphi) - \varphi \cdot \boldsymbol{\partial}_1(\mathbf{1}_{C^\infty(M^A,A)})\Big) \cdot \boldsymbol{\partial}_2 + \varphi \cdot [\boldsymbol{\partial}_1, \boldsymbol{\partial}_2]. \quad (2.9)$$

2.3.2. *Lie-Rinehart algebra structure on Weil bundles.*

Definition 2.6. *A Lie-Rinehart algebra structure on* M^A *is the anchor of morphism*

$$\rho : \mathfrak{X}(M^A) \to \mathcal{D}_A^{[1]}(M^A)$$

both of Lie A-algebras and $C^\infty(M^A, A)$*-modules such that*

$$[X, \varphi \cdot Y] = \big(\rho(X)(\varphi) - \varphi \cdot \rho(X)(1_{C^\infty(M^A,A)})\big) \cdot Y + \varphi \cdot [X, Y] \quad (2.10)$$

for all vector fields X, Y on M^A *and* $\varphi \in C^\infty(M^A, A)$*.*
Then, we say that the pair $(\mathfrak{X}(M^A), \rho)$ *is a Lie-Rinehart algebra.*

We put

$$\mathcal{L}_{sks}(\mathfrak{X}(M^A), C^\infty(M^A, A)) = \bigoplus_{p\in\mathbb{N}} \mathcal{L}_{sks}^p(\mathfrak{X}(M^A), C^\infty(M^A, A)),$$

where $\mathcal{L}_{sks}^p(\mathfrak{X}(M^A), C^\infty(M^A, A))$ is the module of skew-symmetric A-multilinear maps of degree p from $\mathfrak{X}(M^A)$ to $C^\infty(M^A, A)$. Finally

$$d_\rho^A : \mathcal{L}_{sks}(\mathfrak{X}(M^A), C^\infty(M^A, A)) \longrightarrow \mathcal{L}_{sks}(\mathfrak{X}(M^A), C^\infty(M^A, A))$$

is the cohomology operator associated with the representation ρ. In differential geometry, d_ρ^A is the generalization of the differential operator of Lichnerowicz with 1-form

$$d_\rho^A(1_{C^\infty(M^A,A)}) : \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A),$$

and the pair $\big(\mathcal{L}_{sks}(\mathfrak{X}(M^A), C^\infty(M^A, A)), d_\rho^A\big)$ is a differential algebra.

Definition 2.7. *We call canonic form* $\boldsymbol{\alpha} \in \Lambda^1(M^A)$ *associated with the structure of Lie-Rinehart algebra* $(\mathfrak{X}(M^A), \rho)$ *on* M^A*, the 1-form*

$$\boldsymbol{\alpha} : \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A), X \longmapsto \rho(X)(1_{C^\infty(M^A,A)}). \quad (2.11)$$

Theorem 2.8. [8] *Let* $(\mathfrak{X}(M^A), \rho)$ *be a Lie-Rinehart algebra on* M^A*. There exists a differential A-form of degree $+1$ on* M^A*,*

$$\boldsymbol{\alpha} : \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A)$$

such that

$$\rho\,(X)\,(\varphi) = X(\varphi) + \varphi \cdot \boldsymbol{\alpha}(X). \quad (2.12)$$

Proposition 2.9. [8] *If d^A denotes the differential operator of degree $+1$ and of square 0 associated with the representation*

$$\mathfrak{X}(M^A) \longrightarrow \mathcal{D}er_A[C^\infty(M^A, A)], X \longmapsto X,$$

then the 1-form $\boldsymbol{\alpha}$ is d^A-closed, that is, $d^A\boldsymbol{\alpha} = 0$.

We end this subsection to give the following consequence.

Corollary 2.10. *Let $\boldsymbol{\alpha}$ be a differential A-form of degree $+1$ on M^A and let a representation*

$$\rho_{\boldsymbol{\alpha}} : \mathfrak{X}\left(M^A\right) \longrightarrow \mathcal{D}_A^{[1]}\left(M^A\right)$$

such that

$$\rho_{\boldsymbol{\alpha}}(X)(\varphi) = \varphi \cdot \boldsymbol{\alpha}(X) + X(\varphi) \tag{2.13}$$

for any $\varphi \in C^\infty\left(M^A, A\right)$. The pair $\left(\mathfrak{X}\left(M^A\right), \rho_{\boldsymbol{\alpha}}\right)$ is an A-Lie-Rinehart algebra if and only if $d^A\boldsymbol{\alpha} = 0$.

2.3.3. *Lie-Rinehart-Jacobi algebra structure on Weil bundles.*

Definition 2.11. *A Lie-Rinehart-Jacobi algebra structure on Lie-Rinehart algebra $(\mathfrak{X}\left(M^A\right), \rho_{\boldsymbol{\alpha}})$ is defined by a skew-symmetric A-bilinear form*

$$\mu : \mathfrak{X}\left(M^A\right) \times \mathfrak{X}\left(M^A\right) \longrightarrow C^\infty\left(M^A, A\right)$$

such that

$$d^A_{\rho_{\boldsymbol{\alpha}}}\mu = 0. \tag{2.14}$$

We say that $(\mathfrak{X}\left(M^A\right), \rho_{\boldsymbol{\alpha}}, \mu)$ is a Lie-Rinehart-Jacobi algebra on Weil bundles.

3. Hamiltonian vector fields on the symplectic A-manifold M^A

Theorem 3.1. [2], [5] *If*

$$\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$$

is the nondegenerate 2-form, then so is

$$\omega^A : \mathfrak{X}\left(M^A\right) \times \mathfrak{X}\left(M^A\right) \longrightarrow C^\infty\left(M^A, A\right).$$

Corollary 3.2. *When (M, ω) is the symplectic manifold, then $\left(M^A, \omega^A\right)$ is also a symplectic A-manifold.*

Definition 3.3. *Assume that X belongs to $\mathfrak{X}\left(M^A\right)$.*

1) *A vector field X is said to be locally Hamiltonian if $\mathcal{L}_X\omega^A = 0$, where $\mathcal{L}_X$ is the Lie derivative in the direction of a vector field X.*
2) *X is globally Hamiltonian if $i_X\omega^A$ is exact.*
3) *The Hamiltonian of $\psi \in C^\infty\left(M^A, A\right)$ is the unique vector field X_ψ such that*

$$i_{X_\psi}\omega^A = -d^A\psi. \tag{3.1}$$

Proposition 3.4. *Let X be a vector field on M^A. The following assertions are equivalent:*

1) *X is a locally Hamiltonian vector field.*
2) *$d^A\left(i_X\omega^A\right) = 0$.*

Proof. The proof derives from the definition 3.3. □

According to [5], we have the following theorem.

Theorem 3.5. *(Theorem of Darboux version bundles on near points).*
Let (M^A,ω^A) be a symplectic A-manifold of dimension $2n$; for all point ξ of M^A, there exists a system of coordinates $\left(x_1^A,\cdots,x_n^A,y_1^A,\cdots,y_n^A\right)$ on an open U^A containing ξ such that

$$\omega^A=\sum_{i=1}^{n} d^A x_i^A\wedge d^A y_i^A.$$

A such coordinate system is called a canonic coordinate system.

3.1. **Local expressions in canonical coordinates.** quad
Let $\left(x_1^A,\cdots,x_n^A,y_1^A,\cdots,y_n^A\right)$ be the canonical coordinate system with

$$\omega^A=\sum_{i=1}^{n} d^A x_i^A\wedge d^A y_i^A.$$

For all function $\varphi\in C^\infty(M^A,A)$, we have

$$\begin{aligned} i_{X_\varphi}\omega^A &= \sum_{i=1}^{n}\left(d^Ax_i^A(X_\varphi)d^Ay_i^A-d^Ay_i^A(X_\varphi)d^Ax_i^A\right) &(3.2)\\ &= -d^A\varphi &(3.3)\\ &= -\sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial x_i^A}d^Ax_i^A+\frac{\partial\varphi}{\partial y_i^A}d^Ay_i^A\right). &(3.4)\end{aligned}$$

We deduce that

$$d^Ax_i^A(X_\varphi)=-\frac{\partial\varphi}{\partial y_i^A} \tag{3.5}$$

and

$$d^Ay_i^A(X_\varphi)=\frac{\partial\varphi}{\partial x_i^A}. \tag{3.6}$$

Then, it comes that

$$X_\varphi=\sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial x_i^A}\frac{\partial}{\partial y_i^A}-\frac{\partial\varphi}{\partial y_i^A}\frac{\partial}{\partial x_i^A}\right) \tag{3.7}$$

and

$$\begin{aligned}\{\varphi,\psi\}_{\omega^A} &= X_\varphi(\psi) &(3.8)\\ &= \sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial x_i^A}\frac{\partial\psi}{\partial y_i^A}-\frac{\partial\varphi}{\partial y_i^A}\frac{\partial\psi}{\partial x_i^A}\right). &(3.9)\end{aligned}$$

Proposition 3.6. *Let (M^A,ω^A) be a symplectic A-manifold.*

1) *The set of locally Hamiltonian vector fields equipped with the Lie bracket is a Lie algebra $\mathfrak{L}(M^A,\omega^A)$ which an ideal is the set of Hamiltonian vector fields $\mathcal{H}(M^A,\omega^A)$. In addition, the Lie bracket of two locally Hamiltonian vector fields is a globally Hamiltonian vector field.*
2) *$\mathfrak{L}(M^A,\omega^A)=\mathcal{H}(M^A,\omega^A)$ if and only if $H^1_{dR}(M^A)=\{0\}$, where $H^1_{dR}(M^A)$ is the first cohomology group of **de Rham**.*

Proof. 1) Firstly, the application

$$\mathfrak{L}(M^A,\omega^A) \longrightarrow \Lambda^2(M^A), X \longmapsto \mathcal{L}_X$$

is linear and $\mathfrak{L}(M^A,\omega^A)$ is a vector espace over $\mathbb{R}$. We have

$$\begin{aligned} \mathcal{L}_{[X,Y]}\omega^A &= \mathcal{L}_X\left(\mathcal{L}_Y\omega^A\right) - \mathcal{L}_Y\left(\mathcal{L}_X\omega^A\right) \\ &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{L}(M^A,\omega^A)$, then $[X,Y] \in \mathfrak{L}(M^A,\omega^A)$. So, $\mathfrak{L}(M^A,\omega^A)$ is then a Lie sub-algebra of $\mathfrak{X}(M^A)$.
On the other hand, the application

$$\mathcal{H}(M^A,\omega^A) \longrightarrow \Lambda^1(M^A), X \longmapsto i_X\omega^A$$

is linear, involves that $\mathcal{H}(M^A,\omega^A)$ is a vectorial $\mathbb{R}$-sub-space of $\mathfrak{L}(M^A,\omega^A)$.
We have

$$\begin{aligned} i_{[X,Y]}\omega^A &= \mathcal{L}_X\left(i_Y\omega^A\right) - i_Y\left(\mathcal{L}_X\omega^A\right) \\ &= \mathcal{L}_X\left(i_Y\omega^A\right) \\ &= d^A\omega^A(X,Y). \end{aligned}$$

for all $X, Y \in \mathfrak{L}(M^A,\omega^A)$. Therefore, $[\mathfrak{L}(M^A,\omega^A),\mathfrak{L}(M^A,\omega^A)] \subset \mathcal{H}(M^A,\omega^A)$. Particularly, $[X,Y] \in \mathcal{H}(M^A,\omega^A)$, for all $X,Y \in \mathcal{H}(M^A,\omega^A)$. So, $\mathcal{H}(M^A,\omega^A)$ is then a Lie sub-algebra of $\mathfrak{L}(M^A,\omega^A)$. From inclusion $[\mathfrak{L}(M^A,\omega^A),\mathfrak{L}(M^A,\omega^A)] \subset \mathcal{H}(M^A,\omega^A)$, we conclude that $[\mathfrak{L}(M^A,\omega^A),\mathcal{H}(M^A,\omega^A)] \subset \mathcal{H}(M^A,\omega^A)$. Thus $\mathcal{H}(M^A,\omega^A)$ is an ideal of $\mathfrak{L}(M^A,\omega^A)$.

2) Let us put $\Omega^1(M^A) = \{\alpha \in \Lambda^1(M^A)/d^A\alpha = 0\}$.
We define an equivalence relation on $\Omega^1(M^A)$ by: $\alpha,\beta \in \Omega^1(M^A)$, $\alpha \sim \beta$ if and only if there exists $\varphi \in C^\infty(M^A,A)$ such that $\alpha - \beta = d^A\varphi$. The first group of cohomology of **de Rham** is

$$H^1_{dR}(M^A) = \Omega^1(M^A)/\sim .$$

Then, we have $\mathfrak{L}(M^A,\omega^A) = \mathcal{H}(M^A,\omega^A)$ if and only if $i_X\omega^A$ is exact for all $X \in \mathfrak{L}(M^A,\omega^A)$ if and only if all closed form is exact, that is, if and only if $H^1_{dR}(M^A) = \{0\}$.

This completes the proof. □

Remark. *Generally,* $\mathfrak{L}(M^A,\omega^A) \neq \mathcal{H}(M^A,\omega^A)$.

Proposition 3.7. (M^A,ω^A) *being a symplectic A-manifold, the application*

$$\theta : C^\infty(M^A,A) \longrightarrow \mathcal{H}(M^A,\omega^A), \varphi \longmapsto X_\varphi$$

is a homomorphism of Lie algebras whose the kernel is the set of functions, locally constant.
Moreover if M^A *is related then,* $\ker\theta = A$.

Proof. Since $X_{\{\varphi,\psi\}} = [X_\varphi, X_\psi]$ either $\theta([\varphi,\psi]) = [\theta(\varphi),\theta(\psi)], \ker\theta = \{\varphi \in C^\infty(M^A,A)/X_\varphi = 0\}$. In a system of canonic coordinates $\left(x_1^A,\cdots,x_n^A,y_1^A,\cdots,y_n^A\right)$

$$X_\varphi = 0 \Longleftrightarrow \frac{\partial\varphi}{\partial x_i^A} = 0 \text{ and } \frac{\partial\varphi}{\partial y_i^A} = 0$$

for all $i = 1,...,n$. We deduce that φ is constant on $U^A \Longleftrightarrow \varphi$ is locally constant. If M^A is related, all locally constant function is constant, then $\ker\theta = A$. □

4. Locally conformally symplectic structure on Weil bundles

Definition 4.1. .[4] *A smooth manifold M is said to be a locally conformally symplectic manifold if there exist a nondegenerate 2-form*

$$\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

and a closed 1-form α such that

$$d\omega = -\alpha \wedge \omega,$$

where d is the operator of exterior differentiation.

Remark. *If $\alpha = 0$, then M is a symplectic manifold.*

Proposition 4.2. *When (M, ω, α) is a locally conformally symplectic manifold, there exists the 1-form*

$$\boldsymbol{\alpha} : \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A),$$

such that

$$d^A \omega^A = (d\omega)^A = -\boldsymbol{\alpha} \wedge \omega^A.$$

Then the triple $\left(M^A, \omega^A, \boldsymbol{\alpha}\right)$ is said to be a locally conformally symplectic A-manifold.

If $d^A_{\rho_{\boldsymbol{\alpha}}}$ is a differential operator of cohomology associated with the representation $\rho_{\boldsymbol{\alpha}} : \mathfrak{X}(M^A) \longrightarrow \mathcal{D}^{[1]}_A(M^A)$ and if d^A is an operator of cohomology associated with the representation

$$\mathfrak{X}(M^A) \longrightarrow \mathcal{D}er_A[C^\infty(M^A, A)], X \longmapsto X,$$

then

$$d^A_{\rho_{\boldsymbol{\alpha}}} \eta = d^A \eta + \boldsymbol{\alpha} \wedge \eta$$

for all $\eta \in \mathcal{L}_{sks}(\mathfrak{X}(M^A), C^\infty(M^A, A))$. Thus, we conclude that

$$d^A_{\rho_{\boldsymbol{\alpha}}} = d^A_{\boldsymbol{\alpha}}.$$

Proposition 4.3. *Let $f^A : M^A \longrightarrow M^A$ be a diffeomorphism and*

$$\left(f^A\right)^* : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), \varphi \longmapsto \left(f^A\right)^* (\varphi) = \varphi \circ f^A,$$

its pull-back. We have the following assertions:

1) *The 1-form $\boldsymbol{\alpha}$ is $d^A_{\boldsymbol{\alpha}}$-closed if and only if $\boldsymbol{\alpha}$ is d^A-closed.*
2) *If $\boldsymbol{\alpha}$ is closed, then $\left(f^A\right)^* \boldsymbol{\alpha}$ is closed. Moreover, $\left(f^A\right)^* \circ d^A_{\boldsymbol{\alpha}} = d^A_{(f^A)^*\boldsymbol{\alpha}} \circ \left(f^A\right)^*$.*

Proof. Consider a diffeomorphism f^A on M^A with pull-back $\left(f^A\right)^*$. It is obvious that $d^A_{\boldsymbol{\alpha}} \boldsymbol{\alpha} = d^A \boldsymbol{\alpha} + \boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = d^A \boldsymbol{\alpha}$, then $d^A_{\boldsymbol{\alpha}} = 0$ if and only if $d^A \boldsymbol{\alpha} = 0$. Also if $\boldsymbol{\alpha}$ is d^A-closed, we have $d^A \left[\left(f^A\right)^* \boldsymbol{\alpha}\right] = \left(\left(f^A\right)^* \circ d^A\right)(\boldsymbol{\alpha}) = \left(f^A\right)^* \left(d^A \boldsymbol{\alpha}\right) = 0$. Moreover for any $\eta \in \mathcal{L}_{sks}(\mathfrak{X}(M^A), C^\infty(M^A, A))$, we have $\left[\left(f^A\right)^* \circ d^A_{\boldsymbol{\alpha}}\right](\eta) = \left(f^A\right)^* \left(d^A \eta + \boldsymbol{\alpha} \wedge \eta\right) = \left(d^A_{(f^A)^*\boldsymbol{\alpha}} \circ \left(f^A\right)^*\right)(\eta)$. As η is arbitrary, we have $\left(f^A\right)^* \circ d^A_{\boldsymbol{\alpha}} = d^A_{(f^A)^*\boldsymbol{\alpha}} \circ \left(f^A\right)^*$. This completes the proof of the assertion. □

Proposition 4.4. *If $(M^A, \omega^A, \boldsymbol{\alpha})$ designates a locally conformally symplectic A-manifold, then the triple $(\mathfrak{X}(M^A), \rho_{\boldsymbol{\alpha}}, \omega^A)$ is a sympletic Lie-Rinehart-Jacobi A-algebra.*

Being given $(M^A, \omega^A, \boldsymbol{\alpha})$ a locally conformally symplectic A-manifold. For $\varphi \in C^\infty(M^A, A)$, there exists a unique vector X_φ such that $i_{X_\varphi}\omega^A = d^A_{\boldsymbol{\alpha}}\varphi = d^A\varphi + \varphi\cdot\boldsymbol{\alpha}$.

Proposition 4.5. *The map*

$$\{\cdot,\cdot\}_{\omega^A} : C^\infty(M^A, A) \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), (\varphi, \psi) \longmapsto -\omega^A(X_\varphi, X_\psi)$$

defines a structure of a Jacobi A-algebra on $C^\infty(M^A, A)$.

We remark that when $(\omega^A, \boldsymbol{\alpha})$ is a locally conformally symplectic A-structure, since ω^A is nondegenerate, then there exists a unique vector field $X_{1_{C^\infty(M^A,A)}}$ such that

$$i_{X_{1_{C^\infty(M^A,A)}}}\omega^A = \boldsymbol{\alpha},$$

i.e., for any vector field $X \in \mathfrak{X}(M^A)$, we have

$$\omega^A(X_{1_{C^\infty(M^A,A)}}, X) = \boldsymbol{\alpha}(X).$$

Proposition 4.6. *For all $\varphi, \psi \in C^\infty(M^A, A)$, we get*

1) $\mathcal{L}_{X_\varphi}\omega^A = 0$.
2) $[X_\varphi, X_\psi] = X_{\{\varphi,\psi\}_{\omega^A}}$.
3) $i_{X_\varphi}\boldsymbol{\alpha} = \{\varphi, 1_{C^\infty(M^A,A)}\}_{\omega^A}$.
4) $\mathcal{L}_{X_\varphi}\boldsymbol{\alpha} = d^A_{\boldsymbol{\alpha}}\{\varphi, 1_{C^\infty(M^A,A)}\}_{\omega^A}$.

Proof. For all $\varphi, \psi \in C^\infty(M^A, A)$, we find

1) $\mathcal{L}_{X_\varphi}\omega^A = [i_{X_\varphi}, d^A_{\boldsymbol{\alpha}}]\omega^A = \left(d^A_{\boldsymbol{\alpha}}\right)^2(\varphi) = 0$.
2) $i_{[X_\varphi, X_\psi]}\omega^A = [\mathcal{L}_{X_\varphi}, i_{X_\psi}]\omega^A = d^A_{\boldsymbol{\alpha}}\omega^A(X_\psi, X_\varphi) = d^A_{\boldsymbol{\alpha}}\{\varphi,\psi\}_{\omega^A} = i_{X_{\{\varphi,\psi\}_{\omega^A}}}\omega^A$. Since ω^A is nondegenerate, then $[X_\varphi, X_\psi] = X_{\{\varphi,\psi\}_{\omega^A}}$.
3) $i_{X_\varphi}\boldsymbol{\alpha} = \boldsymbol{\alpha}(X_\varphi) = \omega^A\left(X_{1_{C^\infty(M^A,A)}}, X_\varphi\right) = \left\{\varphi, 1_{C^\infty(M^A,A)}\right\}_{\omega^A}$.
4) $\mathcal{L}_{X_\varphi}\boldsymbol{\alpha} = [i_{X_\varphi}, d^A_{\boldsymbol{\alpha}}]\boldsymbol{\alpha} = d^A_{\boldsymbol{\alpha}} i_{X_\varphi}\boldsymbol{\alpha} = d^A_{\boldsymbol{\alpha}}\{\varphi, 1_{C^\infty(M^A,A)}\}_{\omega^A}$.

This is precisely the assertion of the proposition. □

Remark. *$\mathcal{L}_{X_\varphi}\boldsymbol{\alpha}$ is the differential of $i_{X_\varphi}\boldsymbol{\alpha}$.*

4.1. Locally and globally Hamiltonian vector fields on locally conformally symplectic A-manifold M^A.

Definition 4.7. *We have the following definitions.*

1) *A vector field X on a locally conformally symplectic A-manifold M^A is said to be locally Hamiltonian if $\mathcal{L}_X\omega^A = 0$.*
2) *A vector field X on a locally conformally symplectic A-manifold M^A is said to be globally Hamiltonian if $i_X\omega^A$ is exact, i.e., there exists a differentiable application $\Phi \in C^\infty(M^A, A)$ such that $i_X\omega^A = d^A_{\boldsymbol{\alpha}}\Phi$.*

The function Φ is said to be a Hamiltonian of X.

Proposition 4.8. *Let $(M^A, \omega^A, \boldsymbol{\alpha})$ be a locally conformally symplectic A-manifold and X a vector field on M^A. The following conditions are equivalent:*

1) *X is a locally Hamiltonian vector field,*

2) $d^A_{\boldsymbol{\alpha}}\left(i_X\omega^A\right)=0$.

Remark. *A globally Hamiltonian vector field is locally Hamiltonian. Indeed,* $i_X\omega^A = d^A_{\boldsymbol{\alpha}}\Phi$*, then* $d^A_{\boldsymbol{\alpha}} i_X\omega^A = d^A_{\boldsymbol{\alpha}}\left(d^A_{\boldsymbol{\alpha}}\Phi\right) = \left(d^A_{\boldsymbol{\alpha}}\right)^2(\Phi) = 0$.

Proposition 4.9. *The bracket of two locally Hamiltonian vector fields is a globally Hamiltonian vector field.*

Proof. Let X and Y be two locally Hamiltonian vector fields, i.e., $d^A_{\boldsymbol{\alpha}}(i_X\omega^A)=0$ and $d^A_{\boldsymbol{\alpha}}(i_Y\omega^A)=0$. Since $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$, then $[\mathcal{L}_X, i_Y]\,\omega^A = d^A_{\boldsymbol{\alpha}}\left[i_X(i_Y\omega^A)\right] = d^A_{\boldsymbol{\alpha}}\omega^A(X,Y)$. We conclude that $[X,Y]$ is a globally Hamiltonian vector field. □

Remark. *The map* $C^\infty(M^A, A) \longrightarrow \mathfrak{X}\left(M^A\right), \varphi \longmapsto X_\varphi$ *is a morphism of Lie A-algebras and a differential operator of order* ≤ 1.

Theorem 4.10. *For all smooth function φ on a locally conformally symplectic A-manifold M^A, the Lie derivation of Hamiltonian vector field X_φ preserves φ if and only if* $X_\varphi(\varphi) = \widetilde{X}(\varphi)$.

Proof. Let $\varphi \in C^\infty(M^A, A)$ and let X_φ be the Hamiltonian vector field on M^A. We obtain $\mathcal{L}_{X_\varphi}(\varphi) = \left[i_{X_\varphi}, d^A_{\boldsymbol{\alpha}}\right](\varphi) = i_{X_\varphi}\left(d^A\varphi + \varphi\boldsymbol{\alpha}\right) = X_\varphi(\varphi) - \varphi\boldsymbol{\alpha}(X_\varphi)$ so $\mathcal{L}_{X_\varphi}(\varphi) = X_\varphi(\varphi) - \varphi\widetilde{X}(\varphi)$. Thus $\mathcal{L}_{X_\varphi}(\varphi) = 0 \Longleftrightarrow X_\varphi(\varphi) = \varphi\widetilde{X}(\varphi)$. □

Proposition 4.11. *Any Hamiltonian vector field on a locally conformally symplectic A-manifold M^A has the following properties:*

1) $X_\varepsilon = \varepsilon\widetilde{X}, \varepsilon \in A$;
2) $X_{-\varphi} = -X_\varphi$, *for any* $\varphi \in C^\infty(M^A, A)$;
3) $X_\varphi(\varphi^n) = n\varphi^n \cdot \widetilde{X}(\varphi)$, *for any* $\varphi \in C^\infty(M^A, A)$.

Proof. Let $\varepsilon \in A$; $\varphi \in C^\infty(M^A, A)$. Since ω^A is nondegenerate, we have

1) $i_{X_\varepsilon}\omega^A = d^A_{\boldsymbol{\alpha}}(\varepsilon) = \varepsilon d^A_{\boldsymbol{\alpha}}(1_{C^\infty(M^A,A)}) = \varepsilon i_{\widetilde{X}}\omega^A = i_{\varepsilon\widetilde{X}}\omega^A$. We conclude that $X_\varepsilon = \varepsilon\widetilde{X}$.
2) $X_{-\varphi}\omega^A = d^A_{\boldsymbol{\alpha}}(-\varphi) = -d^A_{\boldsymbol{\alpha}}(\varphi) = -i_{X_\varphi}\omega^A = i_{(-X_\varphi)}\omega^A$, that is, $X_{-\varphi} = -X_\varphi$.
3) Making use of the theorem 4.10 together with the fact that X_φ is a derivation.

This is the desired conclusion. □

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