Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 17 Issue 1(2025), Pages 77-99
https://doi.org/10.54671/BMAA-2025-1-7

ON THE ROBUST STABILITY OF POSITIVE DELAY SYSTEMS UNDER TIME-VARYING PERTURBATIONS IN BANACH LATTICES

ASSMA DERDOUKH, MAISSA KADA

Abstract. A generalisation of an explicit robust stability bound from finite to infinite dimensional case is proved. The studied systems are positive linear time delay differential systems (PLTDDS (s)) which are subject to time varying structured perturbations in Banach lattices. The semigroup method, Perron-Frobenius Theorem and the principle of comparison are used. In addition, the coincidence of complex, real and positive stability radii for a number of special cases is proved, and an explicit formula for the radius is derived.

1. Introduction

In modeling many biological and physical phenomena, a more realistic model would contain some past information of the system. This characteristic is called a time delay, which is one of the most important causes of instability that leads to poor control over system performance, and a system with a time delay is called a Time-Delay Differential System (TDDS). Basic theories describing such systems were established in the 1950s and 1960s; it was concerned with the existence and uniqueness of solutions, stability of trivial solutions, etc. These works formed the basic building block for the analysis and design of time-delay systems [19, 22]. Due to its importance and wide spreadness, (TDDS (s)) have attracted the attention of many researchers, which makes it a rapidly growing and valuable field of research. Bellman and Danskin are some of the first researchers who pointed out the diverse applications of systems that depend on past states to other areas such as biology and economics [12]. And with the ability to intervene to make changes on the design and control of real-world systems, new applications also continue to arise, not to mention, in biology, the growth of population, the metabolism of cells, the spread of diseases; in chemistry, the reaction rates of chemical compounds and the transport of fluids; in economics, the dynamics of financial markets and growth of economies; in engineering, the control of robots, the operation of power systems

2000 *Mathematics Subject Classification.* 93C43, 93D09, 47H07.

Key words and phrases. Time varying delay differential equations, robustness on stability, delay positive systems in Banach lattices, time varying perturbation, exponential stability, stability radius.

Submitted December 25, 2024. Accepted March 16, 2025. Published March 28, 2025.
Communicated by D. R. Anderson.

and the design of communication networks. The reader can view more applications and obtain more details by browsing one of the following references [17, 29].
The robust control of (TDDS (s)) has been a very active field for the last 50 years and has spawned many branches, for example, stability analysis, stabilization design, H^{∞} control, and stochastic control. Regardless of the branch, stability is the foundation. Therefore, important developments in studying (TDDS (s)) have generally been taken stability as a starting point [19].

Concerning the stability analysis, it is believed by many researchers that perturbations in delay systems involve small changes that impact their behaviour, stability and lead to degradation of performance, this shows the extreme importance of study of robustness in stability, that is, the ability of the system to maintain its stability properties despite the presence of external disturbances. A measure for the stability robustness of a stable system is the stability radius which was introduced in 1986 by Hinrichsen and Pritchard for linear systems, it is defined as the smallest real or complex disturbance, in norm, that destabilizes the system [13, 14]. Depending on the type of disturbance, whether complex, real or positive, there are complex, real or positive stability radii respectively, which are in general different. Over the years, this field has witnessed a great development, as it has expanded to include a large number of types of systems disturbances [2, 8, 9, 10, 15, 16, 20, 21, 22, 24, 25, 26]. Some researchers proposed utilising the concept of Banach lattices which is a mathematical framework that enables an analysis of different types of functions and spaces. Accordingly, in the context of stability analysis, the use of Banach lattices allows one to study the behaviour of a system under different types of perturbations which is particularly important in applications where certainties or external disturbances play a significant role. Therefore, this paper aims to investigate the robust stability of positive linear time delay systems under time varying perturbations using the stability radius approach as a tool of analysis.
It is crucial to mention that the generalization to infinite dimensions is extremely complicated, thus, considerable amount of knowledge about the subject was exploited in order to achieve two goals. The first is to obtain a generalization of a result which has been found by P. H. A. Ngoc and C.T. Tinh in [25] concerning the robust stability of positive delay differential systems under time varying perturbations, from the finite dimension space to the infinite dimension, using semigroup method, the Perron-Frobenius theorem and the principle of comparison. The second is to determinate an upper bound of the complex stability radius. Moreover, we obtain coincidence in the radius of stability, complex, real or positive, in some special cases. It should be noted that the work of S. Murakami and P. H. A. Ngoc in infinite dimension space in [20] was the motivation of the the current research.
The organization of this paper is as follows, in section two, and in order to make this work more specific, we give the most important concepts and theories related to the Banach lattice framework, as well as some results about positive operators, Metzler operators and their properties. Then, we present the system under study and some results about the asymptotic behavior of the solutions. In section three, we apply time varying structured perturbations to this system and we study the robustness on stability. An upper bound of the complex stability radius is found and we get also the coincidence of complex, real and positive stability radii in some special cases.

2. Preliminaries

To make this work self-contained, we divide this section of preliminaries into three subsections in order to summarize concepts and results about Banach lattices, positive operators and delay systems. All these results are necessary in latter use. First, we give basic facts on Banach lattices, the reader can refer to [10, 18, 28, 31] for more details.

BANACH LATTICES. Let $X \neq \{0\}$ be a real vector space endowed with an order relation $\leq$ such that the following properties are satisfied

$$\begin{cases} x \leq y \Rightarrow x+z \leq y+z \, for\, all\, x, y, z \in X; \\ x \leq y \Rightarrow \alpha x \leq \alpha y \, for\, all\, x, y \in X, \alpha \in \mathbb{R}, \alpha \geq 0. \end{cases} \tag{2.1}$$

Then X is called an ordered vector space. Denote the positive elements of X by $X^+ := \{x \in X / x \geq 0\}$ where $x \geq 0$ means $0 \leq x$. If furthermore the lattice property holds, that is, if

$$x \bigvee y := sup\{x, y\} \in X \text{ for all } x, y \in X,$$

then X is called a vector lattice. The set X^+ fulfills the following geometric properties

$$X^+ + X^+ \subseteq X^+; \mathbb{R}_+ X^+ \subseteq X^+; X^+ \cap X^+ = \{0\}. \tag{2.2}$$

In particular, X^+ is a convex cone in X. Conversely, every subset C of X satisfying (2.2) determines an order relation on X by

$$x \leq y :\Leftrightarrow y - x \in C \ \text{ such that (2.1) holds.}$$

Moreover, if X is a vector lattice, the set X^+ is generating, that is, it satisfies

$$X = X^+ - X^+.$$

If X is a vector lattice, the modulus of $x \in X$ is defined by

$$|x| := x \bigvee (-x) := \sup\{x, -x\}.$$

If $\|.\|$ is a norm on the vector lattice X satisfying the lattice norm property, that is,

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|, \ for\ every\ x, y \in X, \tag{2.3}$$

then X is called a normed vector lattice. The lattice norm property implies $\| \, |x| \, \| = \|x\|$ for every $x \in X$. If, in addition, X is norm complete with respect to $\|.\|$, then X is called a real Banach lattice, or only Banach lattice.
Since it is often necessary to consider complex vector spaces, we extend the notion of Banach lattice to the complex case. For this extension, all underlying vector lattice X are assumed to be relatively uniformly complete, that is, for every sequence $(\lambda_n)_{n\in\mathbb{N}} \, in \, \mathbb{R}$ satisfying

$$\sum_{n=1}^{\infty} |\lambda_n| < +\infty$$

and for every $x \in X$ and for every sequence $(x_n)_{n\in\mathbb{N}} \in X$ it holds that

$$0 \leq x_n \leq \lambda_n x \Rightarrow \sup_{n\in\mathbb{N}} \left(\sum_{i=1}^{n} x_i \right) \in X.$$

Now, let X be a relatively uniformly complete vector lattice. The complexification of X is defined by $X_{\mathbb{C}} := X \times X$ with the canonical addition and scalar multiplication. It is frequently written as $X_{\mathbb{C}} := X + iX$. The modulus of $z = x + iy \in X_{\mathbb{C}}$ is defined by

$$|z| = \sup_{0 \leq \varphi \leq 2\pi} |(\cos\varphi)\, x + (\sin\varphi)\, y| \in X. \tag{2.4}$$

Moreover, the modulus mapping $z \to |z|$ of $X_{\mathbb{C}}$ into X is homogeneous, that is, it satisfies the following relations

$$\begin{cases} |z| = 0\, if\, and\, only\, if\ z = 0, \\ |\alpha z| = |\alpha|\ |z|\, for\, all\, z \in X_{\mathbb{C}}, \alpha \in \mathbb{C}, \\ |z_1 + z_2| \leq |z_1| + |z_2|\, for\, all\, z_1, z_2 \in X_{\mathbb{C}}. \end{cases} \tag{2.5}$$

A complex vector lattice is defined as the complixification of a relatively uniformly complete vector lattice endowed with the modulus (2.4). If X is normed by $\|.\|$, then

$$\|z\| := \|\ |z|\ \|,\ for\ every\ z\ \in X_{\mathbb{C}}. \tag{2.6}$$

Define a norm on $X_{\mathbb{C}}$ satisfying the lattice norm property (2.3). If X is a Banach lattice, then $X_{\mathbb{C}}$ endowed with the modulus (2.4) and the norm (2.6) is called a complex Banach lattice.

Next, we summarize some concepts and theorems about positive operators, Metzler operators, spectrum, resolvent and stability. We obtained this information from the following references [1, 4, 5, 6, 10, 18, 27, 28].

POSITIVE OPERATORS. Let X and Y be real Banach lattices and $T \in \mathcal{L}(X,Y)$. If $TX^+ \subseteq Y^+$, then T is called positive ($T \succeq 0$). By $S \preceq T$ we mean that $T - S \succeq 0$ for $S \in \mathcal{L}(X,Y)$.

Every $\mathbb{R}$ -linear map $T \in \mathcal{L}(X,Y)$ has a unique $\mathbb{C}$ -linear extension $T_{\mathbb{C}} \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ given by

$$T_{\mathbb{C}} z := Tx + iTy,\ for\ every\ z = x + iy \in X_{\mathbb{C}}.$$

An operator $T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is called real if $TX \subset Y$, and we introduce the denotation

$$\mathcal{L}^{\mathbb{R}}(X_{\mathbb{C}}, Y_{\mathbb{C}}) = \{T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})\, /T\, is\, real\}.$$

An operator $T \in \mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is called positive ($T \succeq 0$) if T is real and $TX^+ \subset Y^+$, and we introduce the notation

$$\mathcal{L}^{+}(X_{\mathbb{C}}, Y_{\mathbb{C}}) = \left\{T \in \mathcal{L}^{\mathbb{R}}(X_{\mathbb{C}}, Y_{\mathbb{C}})\, /T \succeq 0\right\}.$$

Any positive linear operator T on $X_{\mathbb{C}}$ is a real operator, that is, $T : X \to X$. The cone $\mathcal{L}^{+}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is closed in $\mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$, however, it is not generating in general. For $T \in \mathcal{L}^{+}(X_{\mathbb{C}}, Y_{\mathbb{C}})$, we note that

$$|Tx| \leq T|x|\ for\ every\ x \in \mathbb{C},$$

and we emphasize the simple but important fact

$$\|T\| = \sup_{x \in X^+, \|x\| = 1} \|Tx\|. \tag{2.7}$$

As a consequence, we note that the operator norm is monotone increasing on the positive cone $\mathcal{L}^+(X_\mathbb{C}, Y_\mathbb{C})$, that is

$$0 \preceq S \preceq T \Rightarrow \|S\| \leq \|T\|.$$

An operator $T \in \mathcal{L}(X, Y)$ possesses a modulus if $|T| := \sup\{T, -T\} \in \mathcal{L}(X, Y)$ in the canonical order relation of $\mathcal{L}(X, Y)$. It can be shown that, if $\sup_{|z| \leq x} |Tz| \in Y$ for every $x \in X^+$, then T possesses a modulus $|T|$ and

$$|T|x = \sup_{|z| \leq x} |Tz|, \ \ for\ every\ x \in X^+.$$

Since X^+ is generating, we have that $|T| \in \mathcal{L}^+(X, Y)$.
Let $T \in \mathcal{L}(X_\mathbb{C}, Y_\mathbb{C})$. If $\sup_{|z| \leq x} |Tz| \in Y$ for every $x \in X^+$, then it holds by linear extension that $|T| \in \mathcal{L}^+(X_\mathbb{C}, Y_\mathbb{C})$ and we introduce the denotation

$$\mathcal{L}^{|\cdot|}(X_\mathbb{C}, Y_\mathbb{C}) = \{T \in \mathcal{L}(X_\mathbb{C}, Y_\mathbb{C}) \,/\, |T| \in \mathcal{L}(X_\mathbb{C}, Y_\mathbb{C})\}.$$

We have $|Tx| \leq |T||x|$ for every $T \in \mathcal{L}^{|\cdot|}(X_\mathbb{C}, Y_\mathbb{C})$ and every $x \in X_\mathbb{C}$ as well as $|Tx| \leq T|x|$ for every $T \in \mathcal{L}^+(X_\mathbb{C}, Y_\mathbb{C})$ and every $x \in X_\mathbb{C}$. The lattice norm property (2.3) implies that

$$\|T\| \leq \|\ |T|\ \| \text{ for every } T \in \mathcal{L}^{|\cdot|}(X_\mathbb{C}, Y_\mathbb{C}).$$

Definition 1. *Let $X_\mathbb{C}$ a complex Banach lattice. For a closed linear operator T on $X_\mathbb{C}$, the resolvent set of T is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C}; \lambda I - T : X_\mathbb{C} \to X_\mathbb{C}\, is\, invertible\}.$$

We call $(\lambda I - T)^{-1}$ the resolvent of T and denoted by

$$R(\lambda, T) := (\lambda I - T)^{-1}\, for\, \lambda \in \rho(T).$$

The complement of $\rho(T)$ in $X_\mathbb{C}$ is called the spectrum of T and denoted by $\sigma(T)$. In general, it may be possible that either $\rho(T)$ or $\sigma(T)$ is empty. In what follows, we always assume that the resolvent set is non-empty.

For the C_0-semigroup $(S(t))_{t \geq 0}$ generated by the operator $(T, D(T))$ on the Banach space X, we associate the following quantities (see [30]):
The spectral radius

$$r(T) := \sup\{|\lambda|, \lambda \in \sigma(T)\}.$$

The spectral bound

$$s(T) := \sup\{\Re e\lambda, \lambda \in \sigma(T)\}.$$

The abscissa of uniform boundedness of the resolvent of T

$$s_0(T) := \inf\left\{\omega \in \mathbb{R} : \{\Re\lambda > \omega\} \subset \rho(T)\, and\, \sup_{\Re e\lambda > \omega} \|R(\lambda, T)\|_{\mathcal{L}(X_\mathbb{C})} < \infty\right\}.$$

The growth bound

$$\omega_1(T) := \inf\left\{\omega \in \mathbb{R} : \exists M > 0 : \|S(t)x\|_{\mathcal{L}(X_\mathbb{C})} < Me^{\omega t}\|x\|_{D(T)}, for\, all\, t \geq 0, x \in D(T)\right\}.$$

The uniform growth bound

$$\omega_0(T) := \inf\left\{\omega \in \mathbb{R} : \exists M > 0 : \|S(t)\|_{\mathcal{L}(X_\mathbb{C})} < Me^{\omega t}, for\, all\, t \geq 0\right\}.$$

It is well-known, see for example in [30], that

$$-\infty \leq s(T) \leq \omega_1(T) \leq s_0(T) \leq \omega_0(T) < +\infty.$$

The semigroup $(S(t))_{t\geq 0}$, or the operator T is called

(1) Hurwitz stable if $\sigma(T) \subset \mathbb{C}_- := \{\lambda \in \mathbb{C} : \Re e\lambda < 0\}$,
(2) Strictly Hurwitz stable if $s(T) < 0$,
(3) Exponentially stable if $\omega_1(T) < 0$,
(4) Uniformly exponentially stable if $\omega_0(T) < 0$.

Remark. *The inequality $s(T) \leq \omega_1(T) \leq s_0(T) \leq \omega_0(T)$ might be strict, that is, the exponential stability of a C_0-semigroup, in general, is not controlled by the location of the spectrum of its generator.*

Theorem 2.1. [4] *If the operator T generates a positive C_0-semigroup, that is, $S(t) \succeq 0$, for all $t \geq 0$, then*

$$s(T) = \omega_1(T) \ = s_0(T).$$

Corollary 2.2. *Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice $X_{\mathbb{C}}$ and assume that $B \in \mathcal{L}^+(X)$, then the following holds*

(1) *$A+B$ generates a positive semigroup $(S(t))_{t\geq 0}$ satisfying $0 \preceq T(t) \preceq S(t)$ for all $t \geq 0$,*
(2) *$s(A) \leq s(A+B)$ and $R(\lambda, A) \succeq R(\lambda, A+B)$ for all $\lambda > s(A)$.*

Definition 2. [10] *A closed operator T is said to be Metzler operator if there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(T)$ and $R(t,T)$ is positive for all $t \in (\omega, +\infty)$. Metzler operators are also called resolvent positive operators in the literature. If $T \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{\mathbb{C}})$, then T is a Metzler operator.*

For positive operators, we have the well-known Perron-Frobenius Theorem of bounded operators.

Theorem 2.3. [18] *Suppose $T \in \mathcal{L}^+(X_{\mathbb{C}})$, then*

(1) *$r(T) \in \sigma(T)$,*
(2) *$R(\lambda, T) \succeq 0$ if and only if $\lambda \in \mathbb{R}$ and $\lambda > r(T)$,*
(3) *If $\lambda > r(T)$, then $|R(\lambda, T)x| \leq R(|\lambda|, T)|x|$, for any $x \in X$.*

Theorem 2.4. [10] *Let T a Metzler operator on $X_{\mathbb{C}}$, then*

(1) *$s(T) \in \sigma(T)$ and $s(T) = t - [r(R(t,T))]^{-1}, t > s(T)$,*
(2) *The function $R(., T)$ is positive and decreasing for $t > s(T)$, that is,*

$$s(T) < t_1 \leq t_2 \Rightarrow R(t_2, T) \preceq R(t_1, T),$$

(3) *If T generates a positive C_0-semigroup, then we have $R(t,T)$ is positive if and only if $t > s(T)$.*

Lemma 2.5. [10] *Let T be a Metzler operator on X and $E \in \mathcal{L}^+(X, Y)$, then*

$$|ER(\lambda, T)x| \leq ER(\Re e\lambda, T)|x|,\ \Re e\lambda > s(T),\ x \in X.$$

Last, we give a description of the delay differential system and some results about positivity and exponential stability.

DELAY DIFFERENTIAL SYSTEMS. We consider a time-invariant delay differential system of the form

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x\left(t - h_k\right), \, t \geq 0, \\ x(0) = x_0, \\ x(t) = f(t), t \in [-h, 0). \end{cases} \tag{2.8}$$

Where A_0 is the generator of a C_0-semigroup on a complex Banach lattice $X_{\mathbb{C}}$ with the norm $|.|_{\mathbb{C}}$, A_1, A_2, ..., A_m are bounded linear operators on $X_{\mathbb{C}}$, i.e. $A_k \in \mathcal{L}(X_{\mathbb{C}})$, with the operator norm $\|.\|$ and h_i for every $i = 1, ..., m$ are non-negative real numbers such that $0 \leq h_1 < h_2 < ... \leq h_m := h$.

Here $x_0 \in X_{\mathbb{C}}$ is the initial value and $f \in L^p\left([-h, 0]; X_{\mathbb{C}}\right)$, $p \in [1; \infty)$ is the history function. A mild solution of the delay system (2.8) is the function $x\left(.; x_0, f\right) \in L^p_{loc}\left([-h, \infty); X_{\mathbb{C}}\right)$ satisfying

$$x\left(t; x_0, f\right) = \begin{cases} T\left(t\right) x_0 + \int_0^t T\left(t - s\right) \sum_{k=1}^{m} A_k x\left(s - h_k; x_0, f\right) ds, t \geq 0, \\ f(t), t \in [-h, 0). \end{cases}$$

It follows from [21, Theorem 2.1] that for all $x_0 \in X_{\mathbb{C}}$ and $f \in L^p\left([-h, 0]; X_{\mathbb{C}}\right)$ a unique mild solution $x\left(.; x_0, f\right)$ exists. This solution is continuous on $[0, \infty)$ and exponentially bounded.

Definition 3. *The delay system* (2.8) *is a positive system if for any history function* $f \in L^p\left([-h, 0]; X^+_{\mathbb{C}}\right)$, *and initial value* $x_0 \in X^+_{\mathbb{C}}$, *the corresponding solution satisfies*

$$x\left(t; x_0, f\right) \in X^+_{\mathbb{C}} \text{ for all } t \geq 0.$$

Definition 4. *The delay system* (2.8) *is called exponentially stable if there exist* $M > 0$ *and* $\omega > 0$ *such that for any* $f \in L^p\left([-h, 0]; X^+_{\mathbb{C}}\right)$ *and any* $x_0 \in X^+_{\mathbb{C}}$, *the unique solution of the delay system* (2.8) *satisfies*

$$\left|x\left(t; x_0, f\right)\right|_{X_{\mathbb{C}}} < M e^{-\omega t} \left\|\left(x_0, f\right)\right\|_{X_{\mathbb{C}} \times L^p([-h,0]; X_{\mathbb{C}})}, \text{ for all } t \geq 0,$$

where

$$\left\|\left(x_0, f\right)\right\|_{X_{\mathbb{C}} \times L^p([-h,0]; X_{\mathbb{C}})} = \left(|x_0|^p_{X_{\mathbb{C}}} + \|f\|^p_{L^p([-h,0]; X_{\mathbb{C}})}\right)^{\frac{1}{p}}.$$

Lemma 2.6. *Assume that the delay system* (2.8) *is positive and let* $x_1, x_2 \in X^+_{\mathbb{C}}$, $f_1, f_2 \in L^p\left([-h, 0]; X^+_{\mathbb{C}}\right)$, $x_a\left(t, x_1, f_1\right)$; $x_b\left(t, x_2, f_2\right)$ *are solutions of the delay system* (2.8) *under initial conditions* $x\left(0\right) = x_1, x\left(t\right) = f_1\left(t\right)$ *and* $x\left(0\right) = x_2, x\left(t\right) = f_2\left(t\right)$ *for all* $t \in [-h, 0)$ *respectively, then*

$$\begin{gathered} \text{If } x_1 \leq x_2 \text{ and } f_1\left(t\right) \leq f_2\left(t\right), \text{ for all } t \in [-h, 0) \text{ then} \\ x_a\left(t, x_1, f_1\right) \leq x_b\left(t, x_2, f_2\right), \text{ for all } t \geq 0. \end{gathered}$$

Abstract formulation. In order to study of the solution's asymptotic behaviour of the delay system (2.8) by semigroup method, we introduce the product space $\mathcal{X} := X_{\mathbb{C}} \times L^p([-h,0];X_{\mathbb{C}})$ and define bounded linear operators $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$ as follows: Given a function $x \in L^p_{loc}([-h,\infty);X_{\mathbb{C}})$, for each $t \geq 0$, we define $x_t \in L^p([-h,0];X_{\mathbb{C}})$ by $x_t(s) := x(t+s), s \in [-h,0]$. Denoting the unique mild solution of the delay system (2.8) by $x(.;x_0,f)$, we now define

$$\mathcal{T}(t)(x_0,f) := (x(t;x_0,f), x_t(.;x_0,f)), t \geq 0.$$

Proposition 2.7. [11] *The family $(\mathcal{T}(t))_{t\geq 0}$ defines a C_0-semigroup of linear operators on $\mathcal{X}$. Its generator $\mathcal{A}$ is given by*

$$\mathcal{A}(x_0,f) := \left(A_0x_0 + \sum_{k=1}^{m} A_k f(.-h_k), f'\right),$$

with the domain

$$D(\mathcal{A}) := \left\{(x_0,f) \in \mathcal{X} : f \in W^{1,p}_{loc}([-h,0];X_{\mathbb{C}}), f(0) = x_0 \in D(A_0)\right\}.$$

Here $W^{1,p}_{loc}([-h,0];X_{\mathbb{C}})$ denotes the space of absolutely continuous $X_{\mathbb{C}}$-valued functions f on $[-h,0]$ which are strongly differentiable, i.e. with $f' \in L^p([-h,0];X_{\mathbb{C}})$.

Moreover, the delay system (2.8) is exponentially stable if and only if the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ is exponentially stable, i.e. $\omega_1(\mathcal{A}) < 0$.

Definition 5. [2] *The quasi-polynomial operator associated with the delay system (2.8) is defined by*

$$P(\lambda) := A_0 + \sum_{k=1}^{m} e^{-\lambda h_k} A_k. \tag{2.9}$$

In case where X is a finite dimensional space, the delay system (2.8) is exponentially stable if and only if characteristic roots of equation $det(P(\lambda)) = 0$ lie in the open left half of complex plane, i.e. the exponential stability of the delay system (2.8) is controlled by the location of the spectrum of its quasi-polynomial matrix. In general, this is not the case if the finite dimension assumption is dropped. The spectrum and the resolvent of $\mathcal{A}$ are described by

Proposition 2.8. [11] *We have $\lambda \in \rho(\mathcal{A})$ if and only if $\lambda \in \rho(P(\lambda))$. In this case, the resolvent of $\mathcal{A}$ is given by*

$$R(\lambda;\mathcal{A}) = E_\lambda R(\lambda;P(\lambda)) H_\lambda F + T_\lambda,$$

where $E_\lambda \in \mathcal{L}(X,\mathcal{X}), H_\lambda \in \mathcal{L}(\mathcal{X},X), F \in \mathcal{L}(\mathcal{X},\mathcal{X}), T_\lambda \in \mathcal{L}(\mathcal{X},\mathcal{X})$ are defined by

$$\begin{gathered} E_\lambda x := \left(x, e^{\lambda\cdot}x\right), \\ H_\lambda(x,f) := x + \textstyle\int_{-h}^{0} e^{\lambda s} f(s)\,ds, \\ F(x,f) := \left(x, \displaystyle\sum_{k=1}^{m} \mathcal{X}_{[-h_k,0]}(.)A_k f(-h_k - .)\right), \\ T_\lambda(x,f) := \left(0, \textstyle\int_{.}^{0} e^{\lambda(.-s)} f(s)\,ds\right). \end{gathered}$$

Definition 6. [11] *The spectral set, the resolvent set, and the spectral bound of quasi-polynomial operator $P(.)$ are respectively defined by*

$$\sigma(P(.)) := \{\lambda : \lambda \in \sigma(P(\lambda))\},$$
$$\rho(P(.)) := \mathbb{C} \diagdown \sigma(P(\lambda)),$$
$$s(P(.)) := \sup\{\Re e(\lambda) : \lambda \in \sigma(P(.))\}.$$

Remark. *From the above proposition, we have $\rho(\mathcal{A}) = \rho(P(.))$, hence $s(\mathcal{A}) = s(P(.))$. So, if A generates a positive C_0-semigroup, then the delay system* (2.8) *is exponentially stable if and only if $s(P(.)) < 0$.*

Next, we present some results on an extension of Perron-Frobenius Theorem to quasi-polynomial operator (2.9).

Definition 7. [3] *The quasi-polynomial operator (2.9) is called positive if A_0 generates a positive C_0-semigroup and $A_k \in \mathcal{L}^+(X)$ for all $k \in \{1, 2, ..., m\}$. And if the quasi-polynomial operator (2.9) is positive, then the delay system* (2.8) *is a positive system.*

By [11, Thoerem 3.3'], if A_0 generates a positive C_0-semigroup on a Banach lattice $X_{\mathbb{C}}$ and the operators $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$ for all $k \in \{1, 2, ..., m\}$, then the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is a positive C_0-semigroup.
Using Proposition 2.8, we obtain the following result:

Proposition 2.9. [3] *Let the quasi-polynomial operator* (2.9) *be positive, then for $\lambda_1, \lambda_2 \in \mathbb{R}$. The following statements are equivalent*

1. $R(\lambda_1, P(\lambda_1)) \succeq R(\lambda_2, P(\lambda_2)) \succeq 0,$
2. $R(\lambda_1, \mathcal{A}) \succeq R(\lambda_2, \mathcal{A}) \succeq 0.$

The next theorem is a generalisation of results in [23] to Perron-Frobenius theorem for positive quasi polynomial operator in Banach spaces.

Theorem 2.10. [3] *Let the quasi-polynomial operator* (2.9) *be positive, then*

1. $s(P(.)) \in \sigma(P(.)),$
2. *For $\lambda \in \mathbb{R}$ we have $R(\lambda, P(\lambda)) \in \mathcal{L}^+(X)$ if and only if $\lambda > s(P(.)),$*
3. $R(\lambda_1, P(\lambda_1)) \succeq R(\lambda_2, P(\lambda_2))$ *for* $\lambda_2 \geq \lambda_1 > s(P(.)).$

Theorem 2.11. [3] *Let the delay system* (2.8) *be positive, then the following statements are equivalent*

1. *The delay system* (2.8) *is exponentially stable,*
2. $s(A_0 + A_1 + ... + A_m) < 0,$
3. $s(A_0) < 0$ *and* $r\left(-A_0^{-1}(A_1 + ... + A_m)\right) < 1,$
4. $(-A_0 - A_1 - ... - A_m)^{-1} \succeq 0.$

Remark. *Suppose that A is the generator of a positive C_0-semi-group $(T(t))_{t\geq 0}$ on a Banach lattice $X_{\mathbb{C}}$, that is, $T(t) \in \mathcal{L}^+(X_{\mathbb{C}})$, for all $t \in \mathbb{R}_+$. Then, $\{\lambda \in \mathbb{C} : \Re e\lambda > s(A)\} \subset \rho(A)$ and $R(t, A)$ is positive if and only if $t > s(A)$. Therefore, A is a Metzler operator on $X_{\mathbb{C}}$. Conversely, if A is a Metzler operator and $IntX_{\mathbb{C}}^+ \neq \phi$, then A is the generator of a positive C_0-semigroup on $X_{\mathbb{C}}$ and $s(A) = \omega_0(A)$.*

Lemma 2.12. [20] *Assume that A is the generator of a positive compact C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice $X_{\mathbb{C}}$, and $P \in \mathcal{L}(X_{\mathbb{C}}), Q \in \mathcal{L}^+(X_{\mathbb{C}})$. If $|Px| \preceq Q|x|$ for all $x \in X_{\mathbb{C}}$, then*

$$\omega(A+P) = s(A+P) \leq s(A+Q) = \omega(A+Q).$$

Theorem 2.13. [7] *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, satisfying $\|S(t)\|_{\mathcal{L}(X)} < Me^{\omega t}$, for all $t \geq 0$ and some $\omega \in \mathbb{R}, M > 1$. If $B \in \mathcal{L}(X)$, then $C := A + B$ with $D(C) = D(A)$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ satisfying*

$$\|S(t)\|_{\mathcal{L}(X)} < Me^{(\omega + M\|B\|)t}, \text{ for all } t \geq 0 \ .$$

3. Robust stability

3.1. **Perturbed Delay System.** Assume that the delay system (2.8) is subjected to time varying structured perturbations as follows

$$A_0 \rightsquigarrow A_0 + D_0 \Delta_0 E_0,$$
$$A_k \rightsquigarrow A_k + D_k(t)\Delta_k(t) E_k(t), k \in \{1, 2, ..., m\}.$$

Hence, the perturbed delay system takes the form

$$\dot{x}(t) = (A_0 + D_0\Delta_0 E_0)\, x(t) + \sum_{k=1}^{m} (A_k + D_k(t)\Delta_k(t) E_k(t))\, x(t - h_k), t \geq \sigma, \tag{3.1}$$

where A_0 is the infinitesimal generator of an exponentially stable semigroup $(S(t))_{t\geq 0}$. $D_0 \in \mathcal{L}^+(U_{0,\mathbb{C}}, X_{\mathbb{C}}), E_0 \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{0,\mathbb{C}})$. For any $k \in \{1, 2, ..., m\}, D_k(.) \in PC_b\left(\mathbb{R}, \mathcal{L}^{|.|}(U_{k,\mathbb{C}}, X_{\mathbb{C}})\right)$ and $E_k(.) \in PC_b\left(\mathbb{R}, \mathcal{L}^{|.|}(X_{\mathbb{C}}, Y_{k,\mathbb{C}})\right)$ are given. $\Delta_0 \in \mathcal{L}^+(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}})$ is unknown bounded time-invariant perturbation operator, $\Delta_k(.) \in PC_b\left(\mathbb{R}, \mathcal{L}^{|.|}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})\right)$ are unknown bounded time-varying perturbation operators, $X_{\mathbb{C}}, Y_{k,\mathbb{C}}$ and $U_{k,\mathbb{C}}$ are complex Banach lattices.
The size of the perturbation operator $\Delta_k(.)$ is measured by

$$\|\Delta_k(.)\|_\infty = ess \sup_{t\geq 0} \|\Delta_k(t)\|_{\mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})}.$$

Theorem 3.1. *Under the above assumptions, for a fixed $\sigma \geq 0$ and a given $f \in L^p([-h, 0]; X_{\mathbb{C}})$, the perturbed delay system (3.1) under the above assumptions has a unique mild solution $x(.;\sigma, f) \in L^p_{loc}([-h, \infty); X_{\mathbb{C}})$ with the initial value condition*

$$x(s + \sigma) = f(s), s \in [-h, 0]. \tag{3.2}$$

Proof. In order to apply [21, Theorem 2.1], we need only to verify that $u \in L^p_{Loc}([\sigma, \infty); X_{\mathbb{C}})$, where u is given by $u(t) = \sum_{k=1}^{m} (D_k(t)\Delta_k(t) E_k(t))\, x(t - h_k)$.
For $T \geq \sigma$, we define the operator

$$K_T x : L^p([\sigma - h, T]; X_{\mathbb{C}}) \to L^p([\sigma, T]; X_{\mathbb{C}}),$$

by

$$K_T x(s) = \sum_{k=1}^{m} (D_k(s)\Delta_k(s) E_k(s))\, x(s - h_k), \text{ a.e., } s \in [\sigma, T].$$

Thus,

$$\|K_T x\|_{L^p([\sigma,T];X_{\mathbb{C}})} = \|\sum_{k=1}^{m}(D_k(s)\,\Delta_k(s)\,E_k(s))\,x(s-h_k)\,\|_{L^p([\sigma,T];X_{\mathbb{C}})},$$

$$\leq \sum_{k=1}^{m}\|D_k(.)\,\|_{\mathcal{L}(U_{k,\mathbb{C}},X_{\mathbb{C}})}\|\Delta_k(.)\,\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})}\|E_k(.)\,\|_{\mathcal{L}(X_{\mathbb{C}},Y_{k,\mathbb{C}})}\|x(s-h_k)\,\|_{L^p([\sigma,T];X_{\mathbb{C}})},$$

$$\leq \sum_{k=1}^{m}\|D_k(.)\,\|_{\mathcal{L}(U_{k,\mathbb{C}},X_{\mathbb{C}})}\|\Delta_k(.)\,\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})}\|E_k(.)\,\|_{\mathcal{L}(X_{\mathbb{C}},Y_{k,\mathbb{C}})}\left(\int_{\sigma}^{T}|x(s-h_k)\,|^p ds\right)^{1/p},$$

$$\leq \sum_{k=1}^{m}\|D_k(.)\,\|_{\mathcal{L}(U_{k,\mathbb{C}},X_{\mathbb{C}})}\|\Delta_k(.)\,\|_{\mathcal{L}(Y_{\mathbb{C}},U_{k,\mathbb{C}})}\|E_k(.)\,\|_{\mathcal{L}(X_{\mathbb{C}},Y_{k,\mathbb{C}})}\left(\int_{\sigma-h}^{T}|x(\tau)\,|^p d\tau\right)^{1/p},$$

$$\leq \sum_{k=1}^{m}\|D_k(.)\,\|_{\mathcal{L}(U_{k,\mathbb{C}},X_{\mathbb{C}})}\|\Delta_k(.)\,\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})}\|E_k(.)\,\|_{\mathcal{L}(X_{\mathbb{C}},Y_{k,\mathbb{C}})}\|x(.)\,\|_{L^p([\sigma-h,T];X_{\mathbb{C}})},$$

for $1 \leq p < \infty$, and the inequality is also true for $p = \infty$, then the operator K_T is bounded.

Thus, $u : s \to \sum_{k=1}^{m}(D_k(s)\,\Delta_k(s)\,E_k(s))\,x(s-h_k)$ is in $L^p([\sigma,T];X_{\mathbb{C}})$ for any $T \geq \sigma$, then $u \in L^p_{loc}([\sigma,\infty);X_{\mathbb{C}})$.

Then, for a fixed $\sigma \geq 0$, given $f \in L^p([-h,0];X_{\mathbb{C}})$ and $u \in L^p_{loc}([\sigma,\infty);X_{\mathbb{C}})$, the perturbed delay system (3.1) has a unique mild solution $x(.;\sigma,f) \in L^p_{loc}([-h,\infty);X_{\mathbb{C}})$. Recall that $x(.;\sigma,f)$ is continuous on $[\sigma,\infty)$ and exponentially bounded and satisfies (3.1) for any $t \in [\sigma,\infty)$. □

Definition 8. *The perturbed delay system* (3.1) *is said to be exponentially stable if there exist $M > 0$ and $\beta > 0$ such that for any $f \in L^p([\,-h,0];X_{\mathbb{C}})$ and for any $t \geq \sigma \geq 0$,*

$$|x(t;\sigma,f)\,|_{X_{\mathbb{C}}} < Me^{-\beta(t-\sigma)}\|\,(x_0,f)\,\|_{X_{\mathbb{C}}\times L^p([h,0];X_{\mathbb{C}})}.$$

The first main result of this paper is presented in the following Theorem.

Theorem 3.2. *Let the delay system* (2.8) *be exponentially stable, where A_0 is a Metzler operator on $X_{\mathbb{C}}$ which generates a compact C_0-semigroup with $IntX_{\mathbb{C}} \neq \phi$. For $k \in \{1,2,...,m\}$, assume that*

1. *$A_k \in \mathcal{L}^+(X_{\mathbb{C}})$,*
2. *There exist $D_k \in \mathcal{L}^+(U_{k,\mathbb{C}},X_{\mathbb{C}})$, $E_k \in \mathcal{L}^+(X_{\mathbb{C}},Y_{k,\mathbb{C}})$ and $\Delta_k \in \mathcal{L}^+(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})$ such that*

$$|D_k(t)\,| \leq D_k\ ,|E_k(t)\,| \leq E_k \text{ and } |\Delta_k(t)\,| \leq \Delta_k \text{ for any } t \in \mathbb{R}^+.$$

Then, the perturbed system (3.1) *remains exponentially stable provided*

$$\sum_{k=0}^{m}\|\Delta_k\| < \frac{1}{\max_{i,j\in\{0,1,2,...,m\}}\|E_i\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_j\|}. \tag{3.3}$$

Proof. The proof is in two steps.
Step 1. Since A_0 is a Metzler operator and $IntX_{\mathbb{C}} \neq \phi$, then it generates a positive C_0-semigroup on $X_{\mathbb{C}}$ and while $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$ for all $k \in \{1,2,...,m\}$, then the delay system (2.8) is positive. Since the delay system (2.8) is exponentially stable by Theorem 2.11 this is equivalent to $s(A_0 + A_1 + ... + A_m) < 0$. Since A_0 generates a positive compact C_0-semigroup on $X_{\mathbb{C}}$ and since $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$ for all $k \in \{1,2,...,m\}$, it follows by Theorem 2.13 that the operator $\sum_{k=0}^{m} A_k$ generates a positive compact C_0-semigroup on $X_{\mathbb{C}}$.
Let $t \geq 0$, for any $k \in \{1,2,...,m\}$, $D_k(t) \in \mathcal{L}^{|\cdot|}(U_{k,\mathbb{C}}, X_{\mathbb{C}}), E_k(t) \in \mathcal{L}^{|\cdot|}(X_{\mathbb{C}}, Y_{k,\mathbb{C}}), \Delta_k(t) \in \mathcal{L}^{|\cdot|}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})$, and $D_k \in \mathcal{L}^+(U_k, \mathbb{C}, X_{\mathbb{C}}), E_k \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{\mathbb{C}}), \Delta_k \in \mathcal{L}^+(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})$ such that

$$|D_k(t)| \leq D_k, |E_k(t)| \leq E_k, | \text{ and } \Delta_k(t)| \leq \Delta_k.$$

We have

$$|\left(D_0\Delta_0E_0 + \sum_{k=1}^{m} D_k(t)\Delta_k(t)E_k(t)\right)x| \leq \sum_{k=0}^{m} D_k\Delta_kE_k|x| \text{ for all } x \in X_{\mathbb{C}}.$$

By Lemma 2.12, it follows that

$$s\left(A_0 + D_0\Delta_0E_0 + \sum_{k=1}^{m} A_k + D_k(t)\Delta_k(t)E_k(t)\right) < s\left(\sum_{k=0}^{m} A_k + D_k\Delta_kE_k\right).$$

Step 2. We prove that $s\left(\sum_{k=0}^{m} A_k + D_k\Delta_kE_k\right) < 0.$

By Theorem 2.13 and Corollary 2.2, the operator $\sum_{k=0}^{m} A_k$ generates a positive C_0-semigroup on $X_{\mathbb{C}}$. By Remark 2, the operator $\sum_{k=0}^{m} A_k$ is a Metzler operator, and since the operators D_k, Δ_k and E_k, for any $k \in \{1,2,...,m\}$, are positives, then the operator $\sum_{k=0}^{m} A_k + D_k\Delta_kE_k$ generates a positive C_0-semigroup on $X_{\mathbb{C}}$. Again by Remark 2, the operator $\sum_{k=0}^{m} A_k + D_k\Delta_kE_k$ is a Metzler operator. Now, by Theorem 2.4 assertion (1), we have

$$s\left(\sum_{k=0}^{m} A_k + D_k\Delta_kE_k\right) \in \sigma\left(\sum_{k=0}^{m} A_k + D_k\Delta_kE_k\right).$$

Set $\mu_0 = s\left(\sum_{k=0}^{m} A_k + D_k\Delta_kE_k\right)$. We prove that $\mu_0 < 0$. Assume the contrary, since $\mu_0 \in \sigma\left(\sum_{k=0}^{m} A_k + D_k\Delta_kE_k\right)$, there exist $U_0 \in X_{\mathbb{C}}$ and $U_0 \neq 0$ such that

$$\left(\sum_{k=0}^{m} A_k + D_k\Delta_k E_k\right) U_0 = \mu_0 U_0. \tag{3.4}$$

Let $Q(t) = tI - \sum_{k=0}^{m} A_k$, $t \in \mathbb{R}$. Since the delay system (2.8) is exponentially stable, we have $s\left(\sum_{k=0}^{m} A_k\right) < 0$. Using the fact that

$$s\left(\sum_{k=0}^{m} A_k\right) = \sup\left\{\Re e\lambda : \lambda \in \sigma\left(\sum_{k=0}^{m} A_k\right)\right\} < 0 \leq \mu_0,$$

it follows that $\mu_0 \in \rho\left(\sum_{k=0}^{m} A_k\right)$ thus, $Q(\mu_0)^{-1}$ exists. Equation (3.4) implies

$$Q(\mu_0)^{-1}\left(\sum_{k=0}^{m} D_k\Delta_k E_k\right) U_0 = U_0. \tag{3.5}$$

Let i_0 such that $\|E_{i_0}U_0\| = \max_{i\in\{0,1,\ldots,m\}} \|E_i U_0\|$. It follows from equation (3.5) that $\|E_{i_0}U_0\| > 0$. Multiply both sides of equation (3.5) from the left by E_{i_0} to get

$$E_{i_0}Q(\mu_0)^{-1}\left(\sum_{k=0}^{m} D_k\Delta_k E_k\right) U_0 = E_{i_0}U_0,$$

which implies that

$$\begin{aligned}
\|E_{i_0}U_0\| &\leq \sum_{k=0}^{m} \|E_{i_0}Q(\mu_0)^{-1} D_k\|\|\Delta_k\|\|E_k U_0\|, \\
&\leq \left(\sum_{k=0}^{m} \|E_{i_0}Q(\mu_0)^{-1} D_k\|\|\Delta_k\|\right)\|E_{i_0}U_0\|, \\
&\leq \left(\max_{0\leq i,j\leq m} \|E_i Q(\mu_0)^{-1} D_j\| \sum_{k=0}^{m}\|\Delta_k\|\right)\|E_{i_0}U_0\|, \\
&\leq \max_{0\leq i,j\leq m} \|E_i Q(\mu_0)^{-1} D_j\| \left(\sum_{k=1}^{m}\|\Delta_k\|\right)\|E_{i_0}U_0\|.
\end{aligned}$$

Therefore,

$$\max_{0\leq i,j\leq m} \|E_i Q(\mu_0)^{-1} D_j\| \left(\sum_{k=0}^{m}\|\Delta_k\|\right) \geq 1. \tag{3.6}$$

The resolvent identity

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu = (\lambda - \mu) R_\mu R_\lambda,$$

gives

$$Q(0)^{-1} - Q(\mu_0)^{-1} = \mu_0 Q(0)^{-1} Q(\mu_0)^{-1}.$$

Since the operator $\sum_{k=0}^{m} A_k$ is a Metzler operator with $s\left(\sum_{k=0}^{m} A_k\right) < 0$, $\mu_0 > 0$, it follows from Theorem 2.4, assertion (3), that

$$Q(0)^{-1} \succeq 0, Q(\mu_0)^{-1} \succeq 0 \text{ if and only if } t > s\left(\sum_{k=0}^{m} A_k\right).$$

Therefore, by Theorem 2.4 assertion (2)

$$Q(0)^{-1} \succeq Q(\mu_0)^{-1} \succeq 0,$$

which yields to

$$E_i Q(0)^{-1} D_j \succeq E_i Q(\mu_0)^{-1} D_j \succeq 0, \text{ for any } 0 \leq i, j \leq m,$$

by lattice norm property (2.3), we get

$$\|E_i Q(0)^{-1} D_j\| \geq \|E_i Q(\mu_0)^{-1} D_j\|, \text{ for any } 0 \leq i, j \leq m,$$

and since

$$\|E_i Q(\mu_0)^{-1} D_j\| \leq \max\nolimits_{0 \leq i,j \leq m} \|E_i Q(0)^{-1} D_j\|,$$

then

$$\|E_i Q(\mu_0)^{-1} D_j\| \left(\sum_{k=0}^{m} \|\Delta_k\|\right) \leq \max\nolimits_{0 \leq i,j \leq m} \|E_i Q(0)^{-1} D_j\| \left(\sum_{k=0}^{m} \|\Delta_k\|\right),$$

from (3.6), we get

$$1 \leq \max\nolimits_{0 \leq i,j \leq m} \|E_i Q(\mu_0)^{-1} D_j\| \left(\sum_{k=0}^{m} \|\Delta_k\|\right) \leq$$
$$\max\nolimits_{0 \leq i,j \leq m} \|E_i Q(0)^{-1} D_j\| \left(\sum_{k=0}^{m} \|\Delta_k\|\right).$$

Thus,

$$\sum_{k=0}^{m} \|\Delta_k\| \geq \frac{1}{\max_{i,j \in \{0,1,2,\ldots,m\}} \|E_i \left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|}.$$

However, this conflicts with (3.3), which conclude the proof. □

Corollary 3.3. *Let the linear delay differential system*

$$\dot{x}(t) = (A + A_0)\, x(t) + \sum_{k=1}^{m} A_k(t)\, x(t - h_k), t \geq 0,$$

where A is the generator of a C_0-semigroup, Hurwitz stable and for all $k \in \{1, 2, ..., m\}$, $A_k(.) \in PC(\mathbb{R}_+, \mathcal{L}(X_{\mathbb{C}}))$. Suppose that there exist $\mathcal{A}_k \in \mathcal{L}^+(X_{\mathbb{C}})$ such that $|A_k(t)| \preceq \mathcal{A}_k$ for any $t \in \mathbb{R}_+$, then the system is exponentially stable provided

$$\sum_{k=1}^{m} \|\mathcal{A}_k\| < \frac{1}{\|A^{-1}\|}.$$

3.2. **Stability radii.** For fixed $\sigma \geq 0$ and given $f \in L^p([-h,0], X_{\mathbb{C}})$, let $x(t) := x(t;\sigma,f), t \in [\sigma - h, \infty)$ be the solution of the delay system (3.1) with the initial value condition (3.2). Denote $y(.) = y(., |f|)$ the solution of

$$\begin{cases} \dot{y}(t) = (A_0 + D_0\Delta_0 E_0) y(t) + \sum_{k=1}^{m} (A_k + D_k\Delta_k E_k) y(t - h_k), t \geq \sigma, \\ y(t) = |f|(t), t \in [-h, 0], \end{cases} \tag{3.7}$$

where A_k, D_k, E_k and Δ_k for any $k \in \{1,2,...,m\}$ are defined as in Theorem 3.2 and $|f|(t) := |f(t)|$, for all $t \in [-h,0]$ with $y(\sigma) = y_\sigma$.
For the positive perturbed delay system (3.7), we introduce definitions of complex, real and positive stability radii.

Definition 9. *For the perturbed delay system* (3.1)*, we define complex, real and positive stability radii as follows*

$$r_{\mathbb{C}} = \inf\{\|\Delta_0\| + \sum_{k=1}^{m} \|\Delta_k(.)\|_\infty : \Delta_0 \in \mathcal{L}(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}}), \Delta_k(.) \in PC\left(\mathbb{R}^+; \mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})\right), k = 0,...,m \text{ and } (3.1) \text{ is not exponentially stable } \}.$$

$$r_{\mathbb{R}} = \inf\{\|\Delta_0\| + \sum_{k=1}^{m} \|\Delta_k(.)\|_\infty : \Delta_0 \in \mathcal{L}^{\mathbb{R}}(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}}), \Delta_k(.) \in PC\left(\mathbb{R}^+; \mathcal{L}^{\mathbb{R}}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})\right), k = 0,...,m \text{ and } (3.1) \text{ is not exponentially stable}\}.$$

$$r_{+} = \inf\{\|\Delta_0\| + \sum_{k=1}^{m} \|\Delta_k(.)\|_\infty : \Delta_0 \in \mathcal{L}^{+}(Y_{0,\mathbb{C}}, U_{0,\mathbb{C}}), \Delta_k(.) \in PC\left(\mathbb{R}^+; \mathcal{L}^{+}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})\right), k = 0,...,m \text{ and } (3.1) \text{ is not exponentially stable } \}.$$

respectively.

Definition 10. *For the perturbed delay system (3.7), we define complex, real and positive stability radii as follows*

$$\tilde{r}_{\mathbb{C}} = \inf\{\sum_{k=0}^{m} \|\Delta_k\| : \Delta_k \in \mathcal{L}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}), k = 0,...,m \text{ and } (3.7) \text{ is not exponentially stable } \}.$$

$$\tilde{r}_{\mathbb{R}} = \inf\{\sum_{k=0}^{m} \|\Delta_k\| : \Delta_k \in \mathcal{L}^{\mathbb{R}}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}), k = 0,...,m \text{ and } (3.7) \text{ is not exponentially stable } \}.$$

$$\tilde{r}_{+} = \inf\{\sum_{k=0}^{m} \|\Delta_k\| : \Delta_k \in \mathcal{L}^{+}(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}), k = 0,...,m \text{ and } (3.7) \text{ is not exponentially stable } \}.$$

By convention, $\inf \phi = \infty$.
Now, we establish an upper bound for the complex stability radius.

Proposition 3.4. *The complex stability radius of the perturbed delay system* (3.1) *has the upper bound*

$$r_{\mathbb{C}} \leq \frac{1}{\max_{i,j\in\{0,1,2,\ldots,m\}} \|E_i \left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|}.$$

Proof. By the stability radii definitions, we have

$$r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_{+}.$$

By [10, Appendix A.3], for $\Delta_k(.) \in PC\left(\mathbb{R}, \mathcal{L}^{|\cdot|}\left(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}}\right)\right)$, we have $0 \preceq \Delta_k(.) \preceq |\Delta_k|$, then for any $t \geq 0$ and $k = 0, ..., m$

$$0 \preceq \Delta_k(t) \preceq |\Delta_k(t)| \preceq \Delta_k,$$

by lattice norm property (2.3), we get

$$\|\Delta_k(t)\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})} \leq \||\Delta_k(t)|\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})} \leq \|\Delta_k\|,$$

with

$$\|\Delta_k(.)\|_\infty = ess\sup_{t\geq 0} \|\Delta_k(t)\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})},$$

thus,

$$\|\Delta_k(.)\|_\infty \leq \||\Delta_k(.)|\|_\infty \leq \|\Delta_k\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})},$$

then

$$\|\Delta_0\| + \sum_{k=1}^{m} \|\Delta_k(.)\|_\infty \leq \|\Delta_0\| + \sum_{k=1}^{m} \||\Delta_k(.)|\|_\infty \leq \sum_{k=0}^{m} \|\Delta_k\|_{\mathcal{L}(Y_{k,\mathbb{C}},U_{k,\mathbb{C}})},$$

and with (3.3), we obtain

$$\|\Delta_0\| + \sum_{k=1}^{m} \|\Delta_k(.)\|_\infty \leq \frac{1}{\max_{i,j\in\{0,1,2,\ldots,m\}} \|E_i \left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|},$$

then

$$r_{\mathbb{C}} \leq \frac{1}{\max_{i,j\in\{0,1,2,\ldots,m\}} \|E_i \left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|}.$$

□

Now, we present the second main result of this paper, we compute the stability radii of the perturbed delay system (3.1) in some special cases of perturbations. We are interested in perturbed systems of the form

$$\dot{x}(t) = (A_0 + D_0\Delta_0 E)\, x(t) + \sum_{k=1}^{m} (A_k + D_k(t)\,\Delta_k E)\, x(t - h_k), t \geq \sigma, \qquad (3.8)$$

with the assumptions

(1) For all $k \in \{0, 1, ..., m\}$, $Y_{k,\mathbb{C}} = Y$,
(2) For all $k \in \{1, ..., m\}$, $D_k(.) \in PC_b\left(\mathbb{R}^+, \mathcal{L}^{|\cdot|}\left(U_{k,\mathbb{C}}, X_{\mathbb{C}}\right)\right) \cap PC_b\left(\mathbb{R}^+, \mathcal{L}^+\left(U_{k,\mathbb{C}}, X_{\mathbb{C}}\right)\right)$.

The second main result of this work is given in the following two Theorems.

Theorem 3.5. *Let A_0 a positive operator on $X_{\mathbb{C}}$ which generates a compact C_0-semigroup with $IntX_{\mathbb{C}} \neq \phi$, we assume that*

1. $A_k \in \mathcal{L}^+(X_{\mathbb{C}})$, *for all* $k \in \{1, 2, ..., m\}$,
2. *There exist* $D_k \in \mathcal{L}^+(U_{k,\mathbb{C}}, X_{\mathbb{C}})$, $E \in \mathcal{L}^+(X_{\mathbb{C}}, Y_{k,\mathbb{C}})$ *and* $\Delta_k \in \mathcal{L}^+(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})$ *such that*

$$|D_k(t)| \leq D_k, \text{ for all } t \in \mathbb{R}^+ \text{ and } k \in \{0, 1, 2, ..., m\}.$$

If the delay system (2.8) *with the initial value condition* (3.2) *is exponentially stable, then we have*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = \frac{1}{\max_{j\in\{0,1,2,\ldots,m\}} \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|}. \tag{3.9}$$

Proof. From definitions of complex, real and positive stability radii $r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_+$, and by Theorem 3.2, we have

$$r_{\mathbb{C}} \geq \frac{1}{\max_{i,j\in\{0,1,2,\ldots,m\}} \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|},$$

we get the inequality

$$r_+ \geq \frac{1}{\max_{i,j\in\{0,1,2,\ldots,m\}} \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|},$$

we have to prove

$$r_+ \leq \frac{1}{\max_{i,j\in\{0,1,2,\ldots,m\}} \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\|}.$$

Assume that there exists $j_0 \in \{0, 1, ..., m\}$ such that

$$\max_{j\in\{0,1,2,\ldots,m\}} \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_j\| = \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0}\|.$$

If $\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0}\| = 0$, then $r_{\mathbb{C}} = r_{\mathbb{R}} = r_+ = +\infty$ and equality (3.9) holds.

Now, assume $\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0}\| > 0$. By Theorem 2.11, the delay system (2.8) be positive and exponentially stable equivalent to $s(A_0 + A_1 + ... + A_m) < 0$, this implies that

$$R\left(0, \sum_{k=0}^{m} A_k\right) = \left(-\sum_{k=0}^{m} A_k\right)^{-1} \succeq 0,$$

therefore,

$$E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_{j_0}\in\mathcal{L}^+\left(U_{j_0,\mathbb{C}},Y_{j_0,\mathbb{C}}\right).$$

Let $0<\epsilon<\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_{j_0}\|$ and by (2.7), we can choose $u_{j_0}\in U^+_{j_0,\mathbb{C}}$, $\|u_{j_0}\|=1$ such that

$$\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_{j_0}u_{j_0}\|>\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_{j_0}\|-\epsilon.$$

We look for the disturbance that destabilise the perturbed delay system (3.8).
The case of $j_0=0$.
Set $y_0=E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_{j_0}u\in Y^+_{j_0,\mathbb{C}}$.
By [31, Theorem 39.3 page 249] there exists a positive $g_0\in Y^*_{0,\mathbb{C}}$, $\|g_0\|=1$ satisfying

$$g_0\left(y_0\right)=\|y_0\|=\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0u_0\|,$$

we consider the operator $\Delta_0:Y_{0,\mathbb{C}}\to U_{0,\mathbb{C}}$ defined by

$$y\mapsto\Delta_0y=\frac{g_0\left(y\right)}{\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0u_0\|}u_0,$$

we have $\Delta_0\in\mathcal{L}^+\left(Y_{0,\mathbb{C}},U_{0,\mathbb{C}}\right)$ and

$$\|\Delta_0\|=\frac{1}{\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0u_0\|},$$

set $x_0=\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0u_0$, then $Ex_0=E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0u_0=y_0$, and hence

$$\Delta_0Ex_0=\frac{g_0\left(Ex_0\right)}{\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0u_0\|}u_0=\frac{g_0\left(y_0\right)}{\|E\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0u_0\|}u_0=u_0,$$

then $x_0\neq0$. Moreover we have $x_0=\left(-\sum_{k=0}^{m}A_k\right)^{-1}D_0\left(\Delta_0Ex_0\right)$, or equivalently

$$\left(\sum_{k=0}^{m}A_k+D_0\Delta_0E\right)x_0=0.$$

i.e. $\Delta_0\in\mathcal{L}^+\left(Y_{0,\mathbb{C}},U_{0,\mathbb{C}}\right)$ and $\left(A_0+D_0\Delta_0E+\sum_{k=1}^{m}A_k\right)x_0=0$ and $x_0\neq0$ implies

$$0 \in \sigma\left(A_0 + D_0\Delta_0 E + \sum_{k=1}^{m} A_k\right),$$

then the perturbed system (3.8) with $\Delta_k(t) \equiv 0$ for all $k \in \{1, 2, ..., m\}$ is not exponentially stable, hence by definition of r_+ we get

$$r_+ \leq \|\Delta_0\| + \sum_{k=1}^{m} \|\Delta_k(.)\| \leq \sum_{k=0}^{m} \|\Delta_k\| = \|\Delta_0\|,$$

then

$$r_+ \leq \|\Delta_0\| = \frac{1}{\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_0 u_0\|} < \frac{1}{\|E\left(-\sum_{k=1}^{m} A_k\right)^{-1} D_0\| - \epsilon}.$$

The case of $j_0 = \{1, 2, ..., m\}$.
We construct one rank positive destabilising perturbation.
Let us consider the Banach spaces $\mathcal{U} = \prod_{k=1}^{m} U_k$ and $\mathcal{Y} = \prod_{k=1}^{m} Y_k$ endowed with the norms

$$\|\mathbf{U}\| = \sum_{k=1}^{m} \|u_k\|, \mathbf{U} = (u_1, u_2, ..., u_m) \in \mathcal{U} \text{ such that } u_k \in U_k \text{ with } k = 1, 2, ..., m.$$

$$\|\mathbf{Y}\| = \sum_{k=1}^{m} \|y_k\|, \mathbf{Y} = (y_1, y_2, ..., y_m) \in \mathcal{Y} \text{ such that } y_k \in Y_k \text{ with } k = 1, 2, ..., m.$$

And let us, for all $t \geq 0, x \in X_{\mathbb{C}}$ and $\mathbf{U} = (u_1, u_2, ..., u_m) \in \mathcal{U}$, define the linear operators $\mathcal{D}(.)$ and $\mathcal{E}$ by

$$\mathcal{E}x = \begin{pmatrix} E_1 x \\ E_2 x \\ \vdots \\ E_m x \end{pmatrix}, \mathcal{D}(.)\,\mathbf{U} = (D_1(t), D_2(t), ..., D_m(t)) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \sum_{k=1}^{m} D_k(t)\, u_k,$$

such that $\mathcal{D}(.) \in PC_b\left(\mathbb{R}_+, \prod_{k=1}^{m} \mathcal{L}(U_k, X_{\mathbb{C}})\right)$ and $\mathcal{E}(.) \in PC_b\left(\mathbb{R}_+, \prod_{k=1}^{m} \mathcal{L}(X_{\mathbb{C}}, Y_k)\right)$, and the block diagonal matrix operator

$$\Delta = diag(\Delta_1, \Delta_2, ..., \Delta_m),$$

for all $\Delta_k \in \mathcal{L}(Y_k, U_k)$ and $\mathbf{Y} = (y_1, y_2, ..., y_m)$, by

$$\Delta : \mathcal{Y} \to \mathcal{U}$$

$$\Delta\mathbf{Y} = \begin{pmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \Delta_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \Delta_1 y_1 \\ \Delta_2 y_2 \\ \vdots \\ \Delta_m y_m \end{pmatrix},$$

endowed with the norm

$$\|\Delta\| = \max_{k=1,\dots,m} \|\Delta_k\|.$$

For $j_0 = \{1, 2, ..., m\}$, we can choose $u_{j_0} \in U^+_{j_0,\mathbb{C}}, \|u_{j_0}\| = 1$ such that

$$\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0}\| > \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0}\| - \epsilon,$$

Set $y_{j_0} = E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0} \in Y^+_{j_0,\mathbb{C}}$,

by [31, Theorem 39.3 page 249], there exists a positive $g_{j_0} \in Y^*_{1,\mathbb{C}}, \|g_{j_0}\| = 1$ satisfying

$$g_{j_0}(y_{j_0}) = \|y_{j_0}\| = \|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0}\|,$$

we consider the operator $\Delta_{j_0} : Y_{j_0,\mathbb{C}} \to U_{j_0,\mathbb{C}}$ defined by

$$y \mapsto \Delta_{j_0} y = \frac{g_{j_0}(y)}{\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_1 u_{j_0}\|} u_{j_0},$$

we have $\Delta_{j_0} \in \mathcal{L}^+(Y_{j_0,\mathbb{C}}, U_{j_0,\mathbb{C}})$ and

$$\|\Delta_{j_0}\| = \frac{1}{\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0}\|},$$

set $x_{j_0} = \left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0}$, then $Ex_{j_0} = E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0} = y_{j_0}$, and hence

$$\Delta_{j_0} Ex_{j_0} = \frac{g_{j_0}(Ex_{j_0})}{\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0}\|} u_{j_0} = \frac{g_1(y_1)}{\|E\left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0} u_{j_0}\|} u_{j_0} = u_{j_0},$$

then $x_{j_0} \neq 0$ because of $u_{j_0} \neq 0$. Moreover, we have $x_{j_0} = \left(-\sum_{k=0}^{m} A_k\right)^{-1} D_{j_0}(\Delta_{j_0} Ex_{j_0})$, or equivalently

$$\left(-\sum_{k=0}^{m} A_k\right) x_{j_0} = D_{j_0} \Delta_{j_0} E x_{j_0}, \tag{3.10}$$

then

$$\left(\sum_{k=0}^{m} A_k + D_{j_0} \Delta_{j_0} E\right) x_{j_0} = 0.$$

We have for any $t \geq 0$

$$D_{j_0}(t) \preceq |D_{j_0}(t)| \preceq D_{j_0} \text{ and } D_{j_0}(t) \in \mathcal{L}^+(U_{j_0,\mathbb{C}}, X_{\mathbb{C}}),$$

then

$$0 \preceq D_{j_0}(t)\,\Delta_1 E x_{j_0} \preceq D_{j_0}\Delta_{j_0} E x_{j_0},$$

by equation (3.10)

$$0 \preceq D_{j_0}(t)\,\Delta_{j_0} E x_{j_0} \preceq \left(-\sum_{k=1}^{m} A_k\right) x_{j_0},$$

hence,

$$0 \preceq \left(\sum_{k=1}^{m} A_k\right) x_{j_0} \preceq \left(\sum_{k=1}^{m} A_k\right) x_{j_0} + D_{j_0}(t)\,\Delta_{j_0} E x_{j_0} \preceq 0,$$

therefore,

$$\left(\sum_{k=1}^{m} A_k + D_{j_0}(t)\,\Delta_{j_0} E\right) x_{j_0} = 0,$$

and $x_{j_0} \neq 0$, then $0 \in \sigma\left(\sum_{k=1}^{m} A_k + D_{j_0}(t)\,\Delta_{j_0} E\right)$, it follows that, the perturbed delay system (3.8) with $\Delta_0 = 0$ and $\Delta = (0, ..., \Delta_{j_0}, 0, ..., 0)$ is not exponentially stable. $\square$

Following similar steps, we obtain the same result for perturbed systems of the form

$$\dot{x}(t) = (A_0 + D\Delta_0 E)\,x(t) + \sum_{k=1}^{m} (A_k + D\Delta_k E_k(t))\,x(t - h_k),\, t \geq \sigma, \tag{3.11}$$

with the assumptions

(1) For all $k \in \{0, 1, ..., m\},\ U_{k,\mathbb{C}} = U$,
(2) For all $k \in \{1, ..., m\},\ E_k(.) \in PC_b\left(\mathbb{R}^+, \mathcal{L}^{|.|}(X_\mathbb{C}, Y_{k,\mathbb{C}})\right) \cap PC_b\left(\mathbb{R}^+, \mathcal{L}^+(X_\mathbb{C}, Y_{k,\mathbb{C}})\right)$.

Theorem 3.6. *Let A_0 a positive operator on $X_\mathbb{C}$ which generates a compact C_0-semigroup with $IntX_\mathbb{C} \neq \phi$, we assume that*

(1) $A_k \in \mathcal{L}^+(X_\mathbb{C})$, *for all* $k \in \{1, 2, ..., m\}$,
(2) *There exist* $D \in \mathcal{L}^+(U, X_\mathbb{C}), E_k \in \mathcal{L}^+(X_\mathbb{C}, Y_{k,\mathbb{C}})$ *and* $\Delta_k \in \mathcal{L}^+(Y_{k,\mathbb{C}}, U_{k,\mathbb{C}})$ *such that*

$$|E_k(t)| \leq E_k,\ \textit{for all}\ t \in \mathbb{R}^+\ \textit{and}\ k \in \{0, 1, 2, ..., m\}.$$

If the perturbed delay system (2.8) *with the initial value condition* (3.2) *is exponentially stable, then*

$$r_\mathbb{C} = r_\mathbb{R} = r_+ = \frac{1}{\max_{i,j \in \{0,1,2,...,m\}} \|E_j \left(-\sum_{k=0}^{m} A_k\right)^{-1} D\|}. \tag{3.12}$$

4. Conclusion

Through this work, we tried to study, at some extent, robustness in stability of positive delay systems under time-varying disturbances in the case of infinite dimensions, with the possibility of finding the largest sharp disturbance amplitude that keeps the system stable. In further research, the results that we have found could be extended to a more general class of disturbances.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] Aliprantis, C. D. and Burkinshaw, O, *Positive Operators*, Springer Dordrecht. (2006).
[2] Anh, B. T. and Son, N.K. and Thanh, D. D. X. , *Robust Stability of Metzler Operators and Delay Equation in $L^p([-h,0];X)$*, Vietnam J. Math. **34** **3** (2006) 357–368.
[3] Anh, B. T. and Khanh, D. C. and Thanh, D. D. X., *A Remark on Stability of class of Positive Linear Delay Systems in Banach Lattices.* Commun. Math. Anal. **5** **2**(2008), 26–37.
[4] Arendt, W. and Grabosch, A. and Greiner, G.and Moustakas, U. and Nagel,R. and Schlotterbeck, U. and Groh, U. and Lotz, H. and NeubranderLabahn, F, *One-parameter semigroups of positive operator*, Springer-Verlag Berlin Heidelberg. (1986).
[5] Arendt, W., *Resolvent Positive Operators*, Proc. London Math. Soc. **3** (1987) 321–349.
[6] Bátkai,A. and Piazzera, S., *Semigroups and linear partial differential equations with delay*, J. Math. Anal. Appl. **264** (2001), 1–20.
[7] Engel, K.J. and Nagel, R., *One-parameter semigroups of linear evolution equations*, Springer Science & Business Media, (2000).
[8] Fischer, A., *Stability radii of infinite-dimensional positive systems*, Mathematics of Control, Signals and Systems. **10** (1997) 223–236.
[9] Fischer, A., *On robust stability of positive infinite-dimensional linear systems*, IFAC Proceedings Volumes. **30** (1997) 467–472.
[10] Fischer, A. and Hinrichsen, D. and Son, N.K., *Stability Radii of Metzler Operators*, Vietnam J. Math. **26** **2** (1997) 147–163.
[11] Fischer, A. and Van Neerven, J. M. A. M., *Robust Stability of C_0 Semigroups and Application to Stability of Delay Equations* , J. Math. Anal. Appl. **226** **2** (1998) 82–100.
[12] Hale,J.K. and Verduyn Lunel, S. M., *Introduction to functional differential equations*, Springer Science & Business Media (2013).
[13] Hinrichsen, D. and Pritchard, A. J. , *Stability radii of linear systems*, Systems & Control Letters. **7** (1986) 1–10.
[14] Hinrichsen, D. and Pritchard, A. J., *Stability radius for structured perturbations and the algebraic Riccati equation.* Systems & Control Letters. **8** (1986) 105–113.
[15] Hinrichsen, D. and Pritchard, A. J., *Robust Exponential Stability of Time-Varying Linear Systems*, International Journal of Robust and Non-linear Control. **3** **1** (1993) 63–83.
[16] Hinrichsen, D. and Pritchard, A. J., *Mathematical systems theory I: modelling, state space analysis, stability and robustness*, Springer. (2005).
[17] Kuang, Y. and Pritchard, A. J., *Delay differential equations: with applications in population dynamics*, Elsevier. (1993).
[18] Meyer-Nieberg, P., *Banach Lattices*, springer-Verlag Berlin, Heidelberg, (1991).
[19] Min, W. and Yong,H. and Jin-Hua, S., *Stability analysis and robust control of time-delay systems*, springer-Verlag Berlin, Heidelberg, (2010).
[20] Murakami, S. and PNgoc, P. H. A., *On Stability and Robust Stability of Positive Linear Volterra Equations in Banach Lattices*, Cent. Eur. J. Math. **8** (2010) 966–984.
[21] Nakagiri, S. I., *Optimal Control of Linear Retarded Systems in Banach Spaces*, . Math. Anal. Appl. **120** (1986) 169–210.
[22] Nakagiri, S. I., *Structural Properties of Functional Differential Equations in Banach Spaces*, Osaka. J. Math. **25** (1988) 353–398.
[23] Ngoc, P. H. A., *A Perron Frobenius Theorem for a class of Positive Quasi-polynomial Matrices*, Applied Mathematics Letters. **19** **8** (2006) 747–751.

[24] Ngoc, P. H. A. and Van Minh, N. and Naito, T., *Stability Radii of Positive Linear Functional Differential Systems in Banach Spaces*, International Journal of Evolution Equations. **2 1** (2007) 75–97.
[25] Ngoc, P. H. A. and Tinh, C.T., *Robust Stability of Positive Linear Time Delay Systems Under Time-varying Perturbations*, IBulletin of the Polish Academy of Sciences. Technical Sciences. **63** (2015) 947–954.
[26] Nguyen, K. S. and Ngoc, P. H. A., *Robust stability of positive linear time delay systems under affine parameter perturbations*, Acta Mathematica Vietnamica. **24 3** (1999) 353–372.
[27] Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Springer Science & Business Media , (2012).
[28] Schaefer, H. H., *Banach Lattices and Positive Operators*, Springer, (1974).
[29] Smith, H. L., *An introduction to delay differential equations with applications to the life sciences*, Springer, (2011).
[30] Van Neerven, J., *The Asymptotic Behaviour of Semigroups of Lineair Operators*, Birkhauser Verlag, (1996).
[31] Zaanen, A. C., *Introduction to Operator Theory in Riesz Spaces*, Springer Science & Business Media, (2012).

Assma DERDOUKH
Department of Mathematics and Informatics, Abbes Laghrour University, P.O. Box 1252, Khenchela, 40004, Khenchela, Algeria.
LTM Laboratory, Department of Mathematics, University of Batna 2, P53, Road of Constantine-Fesdis, 05078, Batna, Algeria.

E-mail address: derdoukh_asma@univ-khenchela.dz

Maissa KADA
LTM Laboratory, Department of Mathematics, University of Batna 2, P53, Road of Constantine-Fesdis, 05078, Batna, Algeria.

E-mail address: m.kada@univ-batna2.dz