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SPHERICAL GABOR TRANSFORMATION OF TYPE δ

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Abstract. The classical Gabor transformation is well known on abelian topological group. In this work, we study the general case of any topological group according to an unitary dual of some compact subgroup.

1. INTRODUCTION

Harmonic Analysis was originally the study of Fourier series with real variables. It has been generalized to commutative locally compact groups. The spectrum of a signal is a representation of the signal called a Fourier transform, which allows the energy of the signal to be obtained by Plancherel's theorem. The Fourier transform is a global transformation that is not suitable for non-stationary signals.To avoid the drawback of the global character, a natural idea is to truncate the signal around a window and perform the Fourier analysis of the truncated signal. This transformation principle is the Gabor transform, which is well suited to non-stationary signals.

In section 2, we present the general theory of the Gabor transform for a continuous signal on an Abelian group. This transform has been studied in the case of commutative groups (see [3]). The Gabor transform for such a signal is defined on $G \times \widehat{G}$ by

$$G_g f(s,\gamma) = \int_G f(t)\overline{\upsilon_g(s,\gamma)(t)}dt =, \forall f \in L^2(G)$$

where $g(t)$ is the window locating the group points and $\gamma(t)$ is the group characters.

In section 3, we generalized this transform to noncommutative groups by assuming that the pair (G,K) is Gelfand with K a compact subgroup of G. We define the spherical Gabor transform for a signal $f(t)$ by:

$$G_g f(s,\varphi) = \int_G f(x)\overline{g(xs^{-1})\varphi(x)}dx$$

where $g(t)$ is the window and $\varphi(t)$ is the spherical function. The signal of a positive type function is reconstructed using the inversion theorem, and the signal energy is obtained from Plancherel's theorem.

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Next in section 4, we define the delta-type spherical Gabor transform, which generalizes the spherical Gabor transform. The spherical delta functions are defined by:

$$\phi(kxk^{'}) = U_\delta(k)\phi(x)U_\delta(k^{'}), \forall k, k^{'} \in K \text{ and } \chi_\delta * \phi(x) = \phi * \chi_\delta(x) = \phi(x).$$

and the delta Gabor transform is defined by:

$$G_g f(s, \phi_\delta^U) = \int_G f(x)\overline{g(xs^{-1})}\left(\phi_\delta^U(x^{-1})\right)dx$$

where $g(t)$ is the window and ϕ_δ^U is the spherical delta function.

Finally, we will give an application of the spherical Gabor transform to groups $SU(2) \times H_2$ and $K = SU(2)$. Let $f \in L^2(SU(2) \times H_2, End_{\mathbb{C}}(V_n))$. The Fourier transform is defined by:

$$\mathcal{F}(f)(\phi) = \int_{SU(2)\times H_2} \chi_w((z,t))\pi_n(U_{a,b})f(U_{a,b}^{-1}; U_{a,b}^{-1}.(z,t))dzdtdadb.$$

2. PRELIMINARIES

Let G be a locally compact commutative group, dx the Haar measure on G and γ a character on G. Let $\widehat{G}$ be the spectrum of G and $d\gamma$ the Plancherel measure of $\widehat{G}$ on which the Fourier transform denoted by $\widehat{f}$ is defined (see [1] or [8]) by

$$\widehat{f}(\gamma) = \int_G f(x)\gamma(x^{-1})dx, \forall \gamma \in \widehat{G}, f \in L^1(G).$$

A function ϕ defined on G, is said to be positive-type (see [1] or [8]) if the inequality

$$\int_G\int_G f(x)\overline{f(y)}\phi(x^{-1}y)dxdy \geq 0, \forall f \in L^1(G)$$

is satisfied. If f is an integrable function and of positive-type on G, then $\widehat{f}$ is integrable (see [1] or [8]) with respect to the Plancherel measure $d\gamma$ and

$$f(x) = \int_{\widehat{G}} \gamma(x)\widehat{f}(\gamma)d\gamma.$$

Let f be is an integrable and square integrable function. The Fourier transform $\widehat{f}$ is square integrable (see [1] or [8]) with respect to the Plancherel measure $d\gamma$ and

$$\int_G |f(x)|^2\,dx = \int_{\widehat{G}} \left|\widehat{f}(\gamma)\right|^2 d\gamma.$$

Let $g \in L^2(G)$, the mapp G_g defined on $G \times \widehat{G}$ by

$$G_g f(s,\gamma) = \int_G f(t)\overline{\upsilon_g(s,\gamma)(t)}dt = \langle f, \upsilon_g(s,\gamma)\rangle = \widehat{f_g^s}(\gamma), \forall f \in L^2(G)$$

where

$$\upsilon_g(s,\gamma)(t) = g(ts^{-1})\gamma(t) = (g_s \cdot \gamma)(t) \text{ and } f_g^s(x) = (f \cdot \overline{g_s})(x)$$

is called Gabor transform (see[3]). The function $\upsilon_g(s,\gamma)$ is called Gabor atoms associated to the window g. We have:

$$\|G_g f\|^2 = \|g\|^2\,\|f\|^2$$

and

$$f(x) = \|g\|^{-2} \int_G \int_{\widehat{G}} G_g f(s,\gamma)\upsilon(s,\gamma)(x) ds d\gamma$$

for all positive-type function f.

3. Spherical Gabor transformation

Let G be a locally compact group, K a compact subgroup of G and dx the Haar measure on G. A function f defined on G, is said to be biinvariant by K if the functional equation

$$f(kxk^{'}) = f(x), \forall x \in G, \forall k, k^{'} \in K$$

is satisfied. We denote by $C_c(G)$, the space of continuous complex values functions on G with compact support and by $C_c^{\#}(G)$ the subspace of $C_c(G)$ containing bi-invariant functions by K. The space $C_c^{\#}(G)$ is a convolution subalgebra of $C_c(G)$ and the map $f \longmapsto f_K$ is a projection of $C_c(G)$ onto $C_c^{\#}(G)$, where f_K is defined by:

$$f_K(x) = \int_K \int_K f(kxk^{'}) dk dk^{'}, (x \in G)$$

Let assume that (G,K) is a Gelfand pair. A bounded function φ which is biinvariant by K is said to be spherical (see [4] or [7]) if the map

$$f \longmapsto \int_G f(x)\varphi(x^{-1})dx$$

is a character of $C_c^{\#}(G)$. Let denote by $S(G)$, the space of spherical functions and by Δ the subspace of $S(G)$ which contains positive-type functions. The space Δ is a locally compact topological space (see [4] or [7]) and we will denote by $d\varphi$ the Plancherel measure on Δ.
For all $f \in C_c^{\#}(G)$, the function $\widehat{f}$ defined on $S(G)$ by

$$\widehat{f}(\varphi) = \int_G f(x)\varphi(x^{-1})dx$$

is called the spherical Fourier transform of f.(see [9]). If f is an integrable function,biinvariant by K and of positive-type on G, then $\widehat{f}$ is integrable (see [4]) with respect to the PLancherel measure $d\varphi$ and

$$f(x) = \int_\Delta \varphi(x)\widehat{f}(\varphi)d\varphi.$$

Let denote by $L^2_\sharp(G)$, the convolution algebra of square integrable biinvariant by K. If f be is an integrable and square integrable function, then the Fourier transform $\widehat{f}$ is square integrable with respect to the Plancherel measure $d\varphi$ on Δ (see [1] or [8]) and

$$\int_G \left|f(x)^2\right| dx = \int_\Delta \left|\widehat{f}(\varphi)\right|^2 d\varphi.$$

Definition 1. *Let $g \in L^2_\sharp(G)$. The mapp G_g defined on $G \times \Delta$ by:*

$$G_g f(s,\varphi) = \int_G f(x)\overline{g(xs^{-1})\varphi(x)}dx, \forall f \in L^2_\sharp(G)$$

is called spherical Gabor transform.

Remark. *Let put* $\upsilon_g(s,\varphi)(t) = g(ts^{-1})\varphi(t) = (g_s \cdot \varphi)(t)$ *and* $f_g^s(x) = (f \cdot \overline{g_s})(x)$, *we have*

$$\upsilon_g(s,\varphi) \in L^2_\sharp(G) \quad \text{and } G_g f(s,\varphi) = \langle f, \upsilon_g(s,\varphi)\rangle .$$

$$f_g^s \in L^1_\sharp(G) \text{ and } G_g f(s,\varphi) = \widehat{f_g^s}(\varphi).$$

The function $\upsilon_g(s,\varphi)$ *is called spherical Gabor atoms associated with* g *and* G_g *is called Gabor transform.*

Theorem 3.1. *Let* $g \in L^2_\sharp(G)$ *be a window.*
i) For any positive-type function $f \in L^2_\sharp(G)$, *we have:*

$$f(x) = \|g\|^{-2} \int_G \int_\Delta G_g f(s,\varphi)\upsilon(s,\varphi)(x) ds d\varphi.$$

ii) Let $g_1, g_2 \in L^2_\sharp(G)$ *be two windows and* $f_1, f_2 \in L^2(G)$, *we have:*

$$\langle G_{g_1} f_1, G_{g_2} f_2\rangle = \langle f_1, f_2\rangle \langle \overline{g_1, g_2}\rangle$$

Proof. i) Let us note that $G_g f$ is a square integrable function of $G \times \Delta$. In fact using Plancherel formula, we have:

$$\begin{aligned}
\int_\Delta \int_G |G_g f(s,\varphi)|^2 ds d\varphi &= \int_\Delta \int_G \left|\widehat{f \cdot \overline{g_s}}(\varphi)\right|^2 ds d\varphi \\
&= \int_G \int_G |f \cdot \overline{g_s}(t)|^2 dt ds \\
&= \int_G \int_G |f(t)|^2 |g(s)|^2 dt ds \\
&= \|f\|^2 \|g\|^2 .
\end{aligned}$$

Using Fubini's theorem, we have:

$$\begin{aligned}
\|g\|^{-2} \int_G \int_\Delta G_g f(s,\varphi)\upsilon(s,\varphi)(x) ds d\varphi &= \|g\|^{-2} \int_G \int_\Delta \widehat{f \cdot \overline{g_s}}(\varphi) g_s(x)\varphi(x^{-1}) ds d\varphi \\
&= \|g\|^{-2} \int_G \int_\Delta \int_G (f \cdot \overline{g_s})(t)\varphi(t^{-1}) g_s(x)\varphi(x^{-1}) dt d\varphi ds \\
&= \|g\|^{-2} \int_G \int_\Delta \int_G f(t)\overline{g_s(t)} g_s(x)\varphi(x^{-1})\varphi(t) dt d\varphi ds \\
&= \int_\Delta \int_G f(t)\varphi(t^{-1}) dt \varphi(x^{-1}) d\varphi \\
&= \int_\Delta \widehat{f}(\varphi)\varphi(x^{-1}) d\varphi = f(x)
\end{aligned}$$

and

$$\begin{aligned}
\langle G_{g_1}f_1, G_{g_2}f_2\rangle &= \int_G\int_\Delta G_{g_1}f_1(s,\varphi)\overline{G_{g_2}f_2(s,\varphi)}dsd\varphi \\
&= \int_G\int_\Delta \widehat{f^s_{1g_1}}(\varphi)\overline{\widehat{f^s_{2g_2}}(\varphi)}dsd\varphi \\
&= \int_G\int_\Delta f^s_{1g_1}(x)\overline{f^s_{2g_2}(x)}dsdx \\
&= \int_G\int_\Delta (f_1\cdot\overline{g_{1s}})(x)\overline{(f_2\cdot\overline{g_{2s}})(x)}dsdx \\
&= \int_G\int_\Delta f_1(x).\overline{g_1}(xs^{-1})\overline{f_2(x).\overline{g_2}(xs^{-1})}dsdx \\
&= \langle f_1, f_2\rangle\,\langle\overline{g_1,g_2}\rangle\,.
\end{aligned}$$

□

Remark. *If the group G is commutative, then the spherical function are characters of G.*

4. Spherical Gabor transformation of type δ

Let G be a locally compact group and K a compact subgoup of G. We denote by $\hat{K}$ the unitary dual of K, for all class δ of $\hat{K}$, let ξ_δ be the character of δ, $d(\delta)$ the degree of δ and $\chi_\delta = d(\delta)\,\xi_\delta$. If $\check{\delta}$ is the class of contragrediant representation of δ in $\hat{K}$, we have $\bar{\chi}_\delta = \chi_\delta^\vee$ and we can verify easily, thanks to the Schur orthogonality relationships, that $\chi_\delta^\vee * \chi_\delta^\vee = \chi_\delta^\vee$. Let $\mathcal{K}(G)$ be the subspace of continuous complex functions on G with compact support . Identifying χ_δ to a bounded measure on G, we put , for any function $f\in\mathcal{K}(G)$,

$${}_\delta f(x) = \bar{\chi}_\delta * f(x)\,,\ f_\delta(x) = f*\chi_\delta(x) \text{ and } I_\delta(G) = \left\{f\in\mathcal{K}(G)\,, f = {}_\delta f = f_{\check{\delta}}, \forall\delta\in\widehat{K}\right\}$$

Let $\mathcal{J}_c(G) = \left\{f\in\mathcal{K}(G)\,, f(kxk^{-1}) = f(x), \forall x\in G, \forall k\in K\right\}$ be the set of the $K-$central functions of $\mathcal{K}(G)$. Assume

$$I_\delta(G)\cap\mathcal{J}_c(G) = I_{c,\delta}(G)\ \ ,$$

and the elements of $I_{c,\delta}(G)$ are said (see[5] or [10]) to be $(K-\delta)-$invariant and K-central functions of $C_{c,\delta}(G)$.. In this work, we give a generalization of Gabor transform on the convolution algebra $I_{c,\delta}(\vartheta)$ of all continuous functions f with compact support on ϑ such that

$$\chi_\delta * f =_\delta f = f, \forall\delta\in\widehat{K}$$

here χ_δ denotes the character of a unitary irreducible representation of K times its dimension. We obtain an inversion formula for the spherical transform by using the Fourier inversion formula in G. ϕ is a continuous functions on G with values in $End_{\mathbb{C}}(E)$ such that

$$\int_K \chi_\delta(k^{-1})\phi(kxk^{-1}y)d(k) = \phi(x)\phi(y), \forall(x,y)\in G^2.$$

Let $\delta \in \widehat{K}$ and $U_\delta : K \to End(E)$ the unitary continuous representation. Then

$$P(\delta) = \int_K \chi_\delta(k^{-1}) U_\delta(k) dk = U_\delta(\chi_\delta)$$

is a continuous projection of E in $E(\delta) = P(\delta)E$ (see [6] or [10]).

Definition 2. *1) Let E be a finite dimensional Hilbert space. The spherical function of type δ (see [4] or [10]) is a quasi-bounded continuous function on G with values in $End_{\mathbb{C}}(E)$ such that:*
(i) $\phi\left(kxk^{-1}\right) = \phi(x)$
(ii) $\chi_\delta * \phi = \phi\,(= \phi * \chi_\delta)$
(iii) The map $u_\phi : f \longrightarrow \phi(f) = \int_G f(x)\,\phi\left(x^{-1}\right) dx$ *is a irreducible representation of the algebra* $I_{c,\delta}(G)$.

2) Let $\delta \in \widehat{K}$ and $U_\delta : K \to End(E)$ the unitary continuous representation. The unitary continuous representation U on G values in $End_{\mathbb{C}}(E)$ is called (see [6]) a dual representation of U_δ if

$$U(kxk^{'}) = U_\delta(k)U(x)U_\delta(k^{'}), \forall k, k^{'} \in K, \forall x \in G.$$

3) Let $\delta \in \widehat{K}$ and U_δ a unitary representation of K in $E(\delta)$. The function $\phi : G \to End_{\mathbb{C}}(E)$ is called (see [4]) $U_\delta-$spherical if:

$$\phi(kxk^{'}) = U_\delta(k)\phi(x)U_\delta(k^{'}), \forall k, k^{'} \in K \text{ and } \chi_\delta * \phi(x) = \phi * \chi_\delta(x) = \phi(x).$$

4) For any function $f \in I_{c,\delta}(G)$, the function $\widehat{f}$ defined on Δ by

$$\widehat{f}(\phi) = \int_G f(x)\phi(x^{-1})dx$$

is called (see [5]) the spherical Fourier transform of type δ. If $f \in I_{c,\delta}(G)$ and of positive-type, then $\widehat{f}$ is integrable (see [10]) with respect to the Plancherel measure $d\phi$ and

$$f(x) = \int_\Delta tr(\phi(x)\widehat{f}(\phi))d\phi.$$

Let denote by $I^2_{c,\delta}(G)$, the convolution algebra of square integrable K-$\delta-$invariant and K-central function with values in $End(E)$. Note by $S_\delta(G)$ the space of $U_\delta-$spherical functions and by Δ_δ the subspace of $S_\delta(G)$ of positive-type functions. Δ_δ is a locally compact topological space and dU designates the Plancherel measure on Δ_δ. If f is a positive-type function of $I^2_{c,\delta}(G)$, then $\widehat{f}$ is square integrable with respect to the Plancherel measure $d\phi$ and

$$\int_G \left|f(x)^2\right| dx = \int_\Delta \left\|\widehat{f}(\phi)\right\|^2 d\phi = \int_\Delta tr((\widehat{f}(\phi)\left(\widehat{f}(\phi)\right)^*)d\phi.$$

Theorem 4.1. *Let $\delta \in \widehat{K}$ and U_δ a unitary representation of K in $E(\delta)$. Let U be a dual unitary representation of G on $End_{\mathbb{C}}(E)$ of U_δ. Then*

i) The function ϕ^U_δ defined by:

$$\phi^U_\delta(x) = P(\delta)U(x)P(\delta)$$

is positive $U_\delta-$spherical.

ii) For any function $f \in I_{c,\delta}(G)$, we have

$$\mathcal{F}f(\phi_\delta^U) = \mathcal{F}f(U) = P(\delta)\mathcal{F}f(U)P(\delta).$$

Proof. i) Let $c_1, c_2, ...c_n \in \mathbb{C}, x_1, x_2, ..., x_n \in G$ et $v \in E$. We have:

$$\begin{aligned}\sum_{i,j} c_i \bar{c}_j \left\langle \phi^U(x_j^{-1}x_i)v, v\right\rangle &= \sum_{i,j} c_i \bar{c}_j \left\langle P(\delta)U(x_j^{-1}x_i)P(\delta)v, v\right\rangle \\ &= \sum_{i,j} c_i \bar{c}_j \left\langle U(x_j^{-1}x_i)P(\delta)v, P(\delta)v\right\rangle \\ &= \left\langle \sum_i c_i U(x_i)P(\delta)v, \sum_j c_j U(x_j)P(\delta)v \right\rangle \geq 0\end{aligned}$$

and

$$(\phi_\delta^U(x))^* = (P(\delta)U(x)P(\delta))^* = P(\delta)(U(x))^*P(\delta) = P(\delta)U(x^{-1})P(\delta) = \phi_\delta^U(x^{-1}).$$

Let U be a dual unitary representation of G on $End_{\mathbb{C}}(E)$ of U_δ. We have:

$$\begin{aligned}\phi^U(kxk') &= P(\delta)U(kxk')P(\delta) = P(\delta)U_\delta(k)U(x)U_\delta(k')P(\delta) \\ &= U_\delta(k)P(\delta)U(x)P(\delta)U_\delta(k') = U_\delta(k)\phi_\delta^U(x)U_\delta(k')\end{aligned}$$

and

$$\begin{aligned}\chi_\delta * \phi_\delta^U(x) &= \int_K \chi_\delta(k)P(\delta)U(k^{-1}x)P(\delta)dk = P(\delta)\int_K \chi_\delta(k)U(k^{-1}x)dkP(\delta) \\ &= P(\delta)(\chi_\delta * U)(x)P(\delta) = P(\delta)U_\delta(x)P(\delta) = \phi_\delta^U(x).\end{aligned}$$

ii) Let $f \in C_{c,\delta}(G)$. We have:

$$\begin{aligned}\mathcal{F}f(\phi_\delta^U) &= \int_G f(x)\phi_\delta^U(x^{-1})dx = \int_G f(x)P(\delta)U(x^{-1})P(\delta)dx = P(\delta)\mathcal{F}f(U)P(\delta) \\ &= \int_K \chi_\delta(k)U(k^{-1})dk \int_G f(x)U(x^{-1})dx \int_K \chi_\delta(k_1)U(k_1^{-1})dk_1 \\ &= \int_G \int_K f(xk_1^{-1})\chi_\delta(k_1)dk_1 U(x)dx \\ &= \int_G f(x)U(x^{-1})dx = \mathcal{F}f(U).\end{aligned}$$

□

Definition 3. *i)Let $\delta \in \widehat{K}$ and U_δ a unitary representation of K in $E(\delta)$. Let U be a dual unitary representation of G on $End_{\mathbb{C}}(E)$ of U_δ.*

Let g, $f \in I_{c,\delta}^2(G)$. The map $G_g f$ from $G \times \Delta_\delta$ to E defined by:

$$G_g f(s, \phi_\delta^U) = \int_G f(x)\overline{g(xs^{-1})}\left(\phi_\delta^U(x^{-1})\right)dx$$

is called $U_\delta-$spherical Gabor transform.

ii) Let $\varepsilon \geq 0$, a function $F \in I_{c,\delta}^2(G \times \Delta_\delta)$, is said $(\varepsilon, \delta)-$concentration on a mesurable set $D_\delta \subseteq G \times \Delta_\delta$, if

$$\left\|\chi_{D_\delta^C} F\right\| \leq \varepsilon \left\|F\right\|.$$

Remark. *Let $g, f \in L^2_{c,\delta}(G)$, the sphérical Gabor transform of type δ verifies the following relation:*

$$\|G_g f\| = \|g\| \, \|f\| \,.$$

In particular, if $\|g\|^2 = 1$, then $\|G_g f\| = \left\|\widehat{f}\right\|$. Thus, in this case the spherical Gabor transform of type δ is an isometry from $L^2_{c,\delta}(G)$ into $L^2_{c,\delta}(\Delta_\delta)$.

For all $f \in L^2_{c,\delta}(G)$ which are positive type, we have:

$$f(x) = \|g\|^{-2} \int_G \int_\Delta tr(G_g f(s, \phi^U_\delta) \upsilon(s, \phi^U_\delta)(x)) ds dU$$

where $\upsilon(s, \phi^U_\delta)$ is defined by $\upsilon(s, \phi^U_\delta)(x) = g(xs^{-1})\phi^U_\delta(x)$.

Theorem 4.2. *Let $\delta \in \widehat{K}$, δ be a unitary representation of K in $E(\delta)$ and U be the dual unitary representation of G. Let $g, f \in L^2_{c,\delta}(G)$, such that $f \neq 0$, and $\varepsilon > 0$. If $G_g f$ is $(\varepsilon, \delta)-$concentration on a mesurable set $D_\delta \subseteq G \times \Delta_\delta$, then for all $p > 2$, we have $|D_\delta| \geq (1-\varepsilon^2)^{\frac{p}{p-2}}$.*

Proof. Let $g, f \in L^2_{c,\delta}(G)$, suppose that $\|f\| = \|g\| = 1$, $D_\delta \subseteq G \times \Delta_\delta$, and $\varepsilon > 0$ with $\int\int_{G\times\Delta_i} \left\|\chi_{D_\delta} G_g f(s, \phi^U_\delta)\right\| ds dU \geq 1 - \varepsilon$. By the Cauchy-Schwartz inegality, we have: $\left\|G_g f(s, \phi^U_\delta)\right\| \leq \|f\| \, \|g\| = 1$, for all $(s, \phi^U_\delta) \in G \times \Delta_\delta$, thus $1 - \varepsilon \leq \int\int_{G\times\Delta_i} \left\|\chi_{D_\delta} G_g f(s, \phi^U_\delta)\right\|^2 ds d\varphi \leq \|G_g f\|^2_\infty |D_\delta| \leq |D_\delta|$. We have

$$\|G_g f\|^2 = \|\chi_{D_\delta} G_g f\|^2 + \left\|\chi_{D^C_\delta} G_g f\right\|^2.$$

$G_g f$ is $(\varepsilon, \delta)-$concentration on a mesurable set $D_\delta \subseteq G \times \Delta_\delta$, then

$$\left\|\chi_{D^C_\delta} G_g f\right\|^2 \leq \varepsilon^2 \|G_g f\|^2 \text{ and } (1-\varepsilon^2) \|G_g f\|^2 \leq \|\chi_{D_\delta} G_g f\|^2 \leq \|f\|^2 \|g\|^2 |D_\delta|$$

and by Plancherel formula we have

$$|D_\delta| \geq 1 - \varepsilon^2.$$

Using the Hôlder inequality, we have

$$\|\chi_{D_\delta} G_g f\|^2 \leq |D_\delta|^{1-\frac{2}{p}} \|G_g f\|^2 = |D_\delta|^{1-\frac{2}{p}} \|f\|^2 \|g\|^2 .$$

Thus

$$(1-\varepsilon^2) \|G_g f\|^2 = (1-\varepsilon^2) \|f\|^2 \|g\|^2 \leq \|\chi_{D_\delta} G_g f\|^2 \leq |D_\delta|^{1-\frac{2}{p}} \|f\|^2 \|g\|^2$$

and we get

$$|D_\delta| \geq (1-\varepsilon^2)^{\frac{p}{p-2}}.$$

□

Remark. *If (G, K) is a Gelfand pair and δ is a trivial representation of 1-dimensional, the algebra $I_{c,\delta}(G)$ is identifed to the algebra of continuous functions with compact support and biinvariant by K and we have the classical Gabor transform.*

Example 1. *Let $H_2 = \mathbb{C}^2 \times \mathbb{R}$ a Heisenberg group. Let $w \in \mathbb{C}^2$, we definit a caractere χ_w in H_2 by: $\chi_w(z, t) = e^{iRe\langle w, z\rangle}$*

Let $SU(2)$ a special unitary group. The action of $U_{a,b}$ in $\mathbb{C}^2$ is:

$$U_{a,b} \cdot z = U_{a,b} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 - \overline{b}z_2 \\ bz_1 + \overline{a}z_2 \end{pmatrix}$$

and the action of $SU(2)$ in H_2 is:

$$U_{a,b}(z,t) = (U_{a,b} \cdot z, t)$$

Let Θ the polynomial space on $\mathbb{C}^2$ and V_n a subspace of Θ of dimension n. We have:

$$V_n = \left\{ P \in \Theta \text{ such as } P(z_1, z_2) = \sum_0^n c_i z_1^i z_2^{n-i} \ ; \ c_i \in \mathbb{C} \right\}.$$

The space V_n is of $n+1$ dimension. Let π a representation of $SU(2)$ defined by:

$$\pi(U_{a,b})P(z_1, z_2) = P(U_{a,b}^{-1} \cdot (z_1, z_2) = P(az_1 - \bar{b}z_2, bz_1 + \bar{a}z_2).$$

Put $\pi_{|V_n} = \pi_n$. The representation π_n is irreductible.

Let $G = SU(2) \times H_2$ and $K = SU(2)$. Let χ_w be a character on H_2 and π_n a representation on $SU(2)$.

Then the function ϕ défined on $SU(2) \times H_2$ by:

$$\phi(U_{a,b}, (z,t)) = \int_{SU(2)} \chi_w(U_{a_1,b_1}(z,t))\pi_n(U_{a_1,b_1}^{-1} U_{a,b} U_{a_1,b_1}) da_1 db_1$$

is a spherical function.

Let $f \in L^2(SU(2) \times H_2, End_{\mathbb{C}}(V_n))$. The Fourier transform is defined by:

$$\mathcal{F}(f)(\phi) = \int_{SU(2)\times H_2} \chi_w((z,t))\pi_n(U_{a,b}) f(U_{a,b}^{-1}; U_{a,b}^{-1}.(z,t)) dz dt da db.$$

For any window ψ on $SU(2) \times H_2$, we define the Gabor transform by:

$$G_\psi f(U_{a,b}, (z,t), \phi) = \int_\Omega \int_{SU(2)} f(U_{a,b}^{-1}; U_{a_1,b_1}^{-1}(z,t))\psi(U_{a,b}^{-1}; U_{a_1,b_1}^{-1}(z,t)) \chi_w(U_{a_1,b_1}(z,t))\pi_n(U_{a_1,b_1}^{-1} U_{a,b} U_{a_1,b_1}) da_1 db_1 d\mu(\phi).$$

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