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ON FACTORIZATION OF TENT SPACES BASED ON LORENTZ CLASSES AND SOME NEW EMBEDDING THEOREMS FOR ANALYTIC SPACES IN THE UNIT DISK

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Abstract. We provide new assertions on factorization of tent spaces and based on them provide new embedding theorems for some analytic spaces in the unit disk.

1. Introduction

In this note, we provide new assertions concerning strong factorization of so called tent spaces. In order to formulate our results we will need some standard definitions ([3, 4, 5]).

Let

$$R^{n+1}_+ = \{(x,t) : x \in R^n, t > 0\},$$
$$\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x-y| < t\}$$

and $B(x,t) = B$ be a ball with center $x \in R^n$.

For $x \in R^n$, let

$$A_\infty(f)(x) = N(f)(x) = \sup_{(y,t)\in\Gamma(x)} |f(y,t)|,$$

$$A_q(f)(x) = \left(\int_{\Gamma(x)} \frac{|f(y,t)|^q}{t^{n+1}} dydt\right)^{1/q}$$

and

$$C_q(f)(x) = \left(\sup_B \frac{1}{|B|}\int_{T(B)} \frac{|f(y,t)|^q}{t} dydt\right)^{1/q},$$

where B contains x and $T(B)$ is the tent over B in R^n (see [3, 4]).

Define spaces T^p_q, T^p_∞ and T^∞_q respectively

$$T^p_q = \left\{f : f \text{ is measurable in } R^{n+1}_+ \text{ satisfying } \|f\|_{T^p_q} = \|A_q(f)(x)\|_{L^p(R^n)} < \infty.\right\},$$

$$T^p_\infty = \{f : f \text{ is measurable in } R^{n+1}_+ \text{ with continuous boundary values on } R^n \text{ such that} \|f\|_{T^p_\infty} = \|A_\infty(f)(x)\|_{L^p(R^n)} < \infty.\}$$

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and

$$T_q^\infty = \{f : f \text{ is measurable in } R_+^{n+1} \text{ satisfying } \|f\|_{T_q^\infty} = \|C_q(f)(x)\|_{L^\infty} < \infty.\}.$$

One of the main results of [3, 4] asserts that

$$T_q^p = T_\infty^p T_q^\infty, \text{for } 0 < p, q < \infty \qquad (A).$$

The mentioned equality was for the first time obtained in [3] for $p > 2$, $q = 2$. Such type strong factorization theorems have numerous applications in the theory of analytic spaces ([2, 4, 6]). We give some results similar in spirit to (A). As we can easily notice mentioned factorization of T_q^p classes were not considered before for $p = \infty$. In this note we, in particular, intend to give an answer to that natural question. On the other hand T_q^p type classes that were defined above are heavily based on classical L^p spaces in R^n. Our next intention is to replace them by their natural extensions the well-known L_q^p Lorentz spaces in R^n and to prove, if possible, a result similar to (A) equality.

2. Main Results

Let $C(n)^{-1}$ be the volume of the unit ball ([4]) so that $\|P_t^0\|_{L^1(R^n)} = 1$, where $P_t^0(x) = C(n)t^{-n}\chi_{B(0,t)}(x)$ and $\chi_{B(0,t)}(x)$ is the characteristic function of the set $B(0,t)$. For $x \in R^n$, define

$$(P_0^*\mu)(x) = C(n)\int_{\Gamma(x)} \frac{d\mu(y,t)}{t^n}$$

where μ is a positive Borel measure in R_+^{n+1}.

Lemma 2.1. *Let $P_0(g)(x,t) = \frac{C(n)}{t^n}\int_{B(x,t)} g(y)dy$, $g \in L^1_{loc}(R^n)$, $S(\mu) = P_0(P_0^*\mu)^{-\tau}$, where $0 < \tau \le 1$ and μ is a positive Borel measure on R^{n+1}. Then*

$$\frac{1}{|B|}\int_{T(B)} S\mu(x,t)d\mu(x,t) \le C\left\|\left(\int_{T(B)\cap\Gamma(y)} \frac{d\mu(x,t)}{t^n}\right)^{1-\tau}\right\|_{L^\infty(B,dy)}.$$

Remark 1: For $\tau = 1$, Lemma 2.1 was proved in [4].

Proof. Let $h(y) = P_0^*\mu(y)$, $y \in R^n$. Modifying proofs in [4] we have

$$\begin{aligned}\int_{T(B)} S\mu(x,t)d\mu(x,t) &\le C(n)\int_{T(B)}\int_{B(x,t)} h(y)^{-\tau}\frac{d\mu(x,t)}{t^n}dy \\ &\le C\int_{R^n} h(y)^{-\tau}\int_{T(B)\cap\Gamma(y)} \frac{d\mu(x,t)}{t^n}dy \\ &\le C|B|\sup_{y\in R^n}\left(\int_{T(B)\cap\Gamma(y)}\frac{d\mu(x,t)}{t^n}\right)^{1-\tau}.\end{aligned}$$

The proof is complete. □

Let X, Y and Z be quasinormed subspaces of a class of all measurable functions in R^n. For $0 < \alpha \le 1$, we say $X \overset{\alpha}{\subset} YZ$, if for any $u \in X$, there exist $w \in Y$, $v \in Z$ such that $u = w \cdot v^\alpha$.

Let $T_q^{\infty,\infty}$ be the class of measurable functions f satisfying

$$\|f\|_{T_q^{\infty,\infty}} = \left\| \left(\int_{\Gamma(y)} \frac{|f|^q}{t^{n+1}} dxdt \right)^{1/q} \right\|_{L^\infty(R^n)} < \infty.$$

Theorem 2.2. *Let $0 < q < \infty$, $p > 0$ and $0 < \alpha = s/q \leq 1$. Then $T_q^{\infty,\infty} \overset{\alpha}{\subset} T_\infty^p T_q^\infty$.*

Remark 2: If we replace $T_q^{\infty\infty}$ classes in Theorem 2.2 with larger T_q^p classes, then for s=q Theorem 2.2 is known (see [3, 4]).

Proof. We will modify the proof of [3, 4]. As the proof in [4] (Page 316), we have

$$\left(\int_X |f|^{-s} d\nu \right)^{-1/s} \leq \left(\int_X |f|^r d\nu \right)^{1/r} \qquad r, s > 0 \tag{$*$}$$

where ν is a measure in R^n. Let us put $d\nu = P_t^0(x)dx$, $f = A_q(u)$ in $(*)$. Then we have $V = (P_0(A_q(u))^r)^{1/r} \geq C(P_0(A_q(u))^{-s})^{-1/s}$, that is, $V^{-s} \leq C(P_0(A_q(u))^{-s})$.

Let $d\mu(x,t) = u(x,t)^q \frac{dxdt}{t}$. Then $(A_q(u))^q = CP_0^*\mu$ and $V^{-s} \leq CP_0(P_0^*\mu)^{-s/q} = P_0(P_0^*\mu)^{-\alpha}$, where $0 < \alpha = s/q \leq 1$. Since $\omega^q = \frac{u^q}{V^s}$ from Lemma 2.1 we have

$$\begin{aligned}
\int_{T(B)} \omega(x,t)^q \frac{dxdt}{t} &\leq C \int_{T(B)} V^{-s} d\mu(x,t) \\
&\leq C \int_{T(B)} S_\alpha(\mu) d\mu \\
&\leq C|B| \left\| \int_{T(B)\cap\Gamma(y)} \frac{|u(x,t)|^q dxdt}{t^{n+1}} \right\|_{L^\infty(B,dy)}^{1-s/q} \\
&\leq C|B| \|u\|_{T_q^{\infty,\infty}}^{q-s},
\end{aligned}$$

which proves that for $u \in T_q^{\infty,\infty}$ and $\omega^q = \frac{u^q}{V^s}$, we have $\omega \in T_q^\infty$.

For $u \in T_q^{\infty,\infty}$, let $V = (P_0(A_q(u))^r)^{1/r}$. Then (see [3, 4]) $NP_0(f) \leq CM(f)$ and hence $N(V) \leq C(M(A_q(u))^r)^{1/r}$, $p > r$, where $M(f)$ is the Hardy-littlewood maximal function. Thus $V \in T_\infty^p$ for every p. Indeed M is a bounded operator from $L^p(R^n)$ into $L^p(R^n)$, $p > 1$. Hence $V \in T_\infty^p$ for every $p > 0$. One the other hand if $\omega = (\frac{u^q}{V^s})^{1/q}$, then we can show that for $u \in T_q^{\infty,\infty}$ and $V \in T_\infty^p$, $p > 0$

$$\left(\frac{1}{|B|} \int_{T(B)} \omega(x,t)^q \frac{dxdt}{t} \right)^{1/q} \leq C\|u\|_{T_q^{\infty,\infty}}^{1-s/q} \quad \text{for } s \leq q.$$

The proof is complete. $\square$

We now turn to another extension of (A). The following facts from the theory of Lorentz classes $L^{p,q}(R^n)$ are needed (see [1, 7]).

For $q, p \in (1, \infty)$, the Hardy-Littlewood maximal operator is extended in $L^{p,q}(R^n)$ (see [5, 1, 7]) and we have

$$\|M(f)\|_{L^{p,q}} \leq C\|f\|_{L^{p,q}}, \tag{1}$$

and

$$\|M(f)\|_{L^{p,\infty}} \leq C\|f\|_{L^{p,\infty}}. \tag{2}$$

For f be a measurable function in R_+^{n+1}, define

$$\|f\|_{LT_q^{p,s}} = \|A_q(f)\|_{L^{p,s}(R^n)},$$

$$\|f\|_{LT_\infty^{p,s}} = \|N(f)\|_{L^{p,s}}.$$

For $0 < p < \infty$, the spaces $LT_q^{p,s}$ and $LT_\infty^{p,s}$ are defined by

$$LT_q^{p,s} = \{f : f \text{ is measurable in } R_+^{n+1} \text{ satisfying } \|f\|_{LT_q^{p,s}} < \infty\}.$$

$$LT_\infty^{p,s} = \{f : f \text{ is measurable in } R_+^{n+1} \text{ satisfying } \|f\|_{LT_\infty^{p,s}} < \infty.\}.$$

Theorem 2.3. *Let $s \le p \le q < \infty$. Then $LT_q^{p,s} = LT_\infty^{p,s} T_q^\infty$.*

Remark 3: For $p = s$, this was obtained in [3, 4] before and it coincides with (A).

Proof. We again use same ideas from [3, 4]. Note first, if $\|A_q(f)\|_{L^{p,s}(R^n)} < \infty$, putting $V = (P_0(A_q(f))^r)^{1/r}$ as in the previous case we have $NV \le C(M(A_q(f)^r))^{1/r}$, $p, s > r$, where M is the Maximal Hardy-Littlewood operator. By (1) we have

$$\|NV\|_{L^{p,s}} \le C\|A_q(f)\|_{L^{p,s}(R^n)} < \infty \quad \text{for } p, s > 0$$

since

$$\||f|^r\|_{L^{p,s}} = \|f\|_{L^{rp,rs}}, p, s, r > 0.$$

The proof of the fact that $\omega = \frac{u}{V} \in T_q^\infty$ follows from the same arguments as in [4]. Let us show the reverse with the same restriction on parameters. Let $\omega \in T_q^\infty$, $V \in LT_\infty^{p,s}$. We will show that

$$\left\| \left(\int_{\Gamma(y)} \frac{|\omega V|^q}{t^{n+s}} dxdt \right)^{1/q} \right\|_{L^{p,s}(R^n)} < \infty.$$

By Holder inequality for Lorentz classes (see [5]), the following estimate holds.

$$D = \left\| \left(\int_{\Gamma(y)} \frac{|\omega(x,t)V(x,t)|^q}{t^{n+s}} dxdt \right)^{1/q} \right\|_{L^{p,s}(R^n)}$$

$$\le C\|NV\|_{L^{\frac{p_1\tau}{q}, \frac{s_1\tau}{q}}} \left\| \int_{\Gamma(y)} \frac{V^{q-\tau}\omega^q}{t^{n+s}} \right\|_{L^{\frac{p_2}{q}, \frac{s_2}{q}}} = AB,$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}$. Choosing τ such that $\frac{\tau p_1}{q} = p$, $\frac{\tau s_1}{q} = s$, then $\frac{p_2}{q} = \frac{s_2}{q} = 1$ and $B \le C\|\omega\|_{T_q^\infty}\|NV\|_{L^{q-\tau}}$ which follows (A). Hence $D \le \|NV\|_{L^{p,s}}\|NV\|_{L^{q-\tau,q-\tau}}$. Note that $\tau = \frac{qs}{s_1} = \frac{pq}{p_1}$, $q - \tau = q(1 - \frac{p}{p_1}) = p = q - qp(\frac{1}{p} - \frac{1}{q})$. Hence using known embeddings for Lorentz classes (see [1, 7]) we have $D \le \|NV\|_{L^{p,s}(R^n)}$ for $s \le p$. The proof is complete.

□

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane and $\mathrm{T} = \{z : |z| = 1\}$. Let $\Gamma_\sigma(\xi)$ be the standard Luzin cone for the unit disk (see [4]), where $\xi \in \mathrm{T}$ and $\sigma > 1$. Based partially on results we provided above we obtained following two assertions (embedding theorems).

Theorem 2.4. *1) Let μ be positive Borel measure on $\mathbb{D}$, $\beta > 0, q > 1, t > \beta + 1$ and let f be analytic on $\mathbb{D}$. Then*

$$\left\| \int_{\Gamma_\sigma(\xi)} \frac{(1-|z|)^t |f(z)| d\mu(z)}{1-|z|} \right\|_{L^{q,\infty}(\mathrm{T})} \leq C \sup_{|z|<1} \left(|f(z)|(1-|z|)^\beta \right)$$

if and only if

$$\left\| \int_{\Gamma_\sigma(\xi)} (1-|z|)^{t-1} \left(\int_{D(z,\rho)} d\mu(w) \right) (1-|z|)^{-\beta-2} \right\|_{L^{q,\infty}} < \infty$$

where $D(z,\rho)$ is Bergman metric ball, $\rho > 0$.

2) Let f be analytic function in $\mathbb{D}$, μ be positive Borel measure on $\mathbb{D}$. Then

$$\left\| \int_{\Gamma_\tau(\xi)} \frac{|f(z)|}{1-|z|} d\mu(z) \right\|_{L^{q,\infty}(\mathrm{T})} \leq C \left\| \int_{\Gamma_\sigma(\xi)} |f(z)|(1-|z|)^{\alpha-1} dm_2(z) \right\|_{L^{q,\infty}}$$

if and only if

$$\sup_{z\in\mathbb{D}} \left[\frac{\mu(D(z,\rho))}{(1-|z|)^{\alpha+2}} \right] < \infty, \rho > 0, q > 1, \alpha > -1.$$

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