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ON KENMOTSU MANIFOLD

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Abstract. The object of the present paper is to construct an example of a three-dimensional Kenmotsu manifold with η-parallel Ricci tensor. Condition for a vector field to be Killing vector field in Kenmotsu manifold is obtained.

1. Introduction

The notion of Kenmotsu manifolds was defined by K. Kenmotsu [9]. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a non-zero constant. Kenmotsu manifolds were studied by many authors such as G. Pitis [14], De and Pathak [6], Jun, De and Pathak [8], Binh, Tamassy, De and Tarafdar [5], Bagewadi and colaborates [2], [3], [4],Ozgur [12],[13] and many others. In [6], the authors proved that a three-dimensinal Kenmotsu manifold with η-parallel Ricci tensor is of constant scalar curvature. In the present paper we like to verify this theorem by a concrete example. We also like to obtain the condition for a vector field in a Kenmotsu manifold to be Killing vector field. The present paper is organized as follows:
In section 2 we recall some preliminary results. Section 3 contains an example of three-dimensional Kenmotsu manifold satisfying $\eta-$parallel Ricci tensor. In Section 4 we deduce conditions for a vector field in a Kenmotsu manifold to be Killing.

2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold, where ϕ is a (1,1) tensor field, η is a 1-form and g is the Riemannian metric. It is well known that [1], [16]

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

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for any vector fields X, Y on M . If, moreover,

$$(\nabla_X \phi)Y = -\eta(Y)\phi(X) - g(X, \phi Y)\xi, \quad X,\ Y \in \chi(M), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold [9].

In Kenmotsu manifolds the following relations hold[9]:

$$(\nabla_X \eta)Y = g(\phi X, \phi Y), \tag{2.7}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{2.8}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$(a)\ \ R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \ \ (b)\ \ R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \tag{2.11}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.12}$$

$$(\nabla_Z R)(X, Y)\xi = g(Z, X)Y - g(Z, Y)X - R(X, Y)Z, \tag{2.13}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor. In a Riemannian manifold we also have

$$g(R(W, X)Y, Z) + g(R(W, X)Z, Y) = 0, \tag{2.14}$$

for every vector fields X, Y, Z.

3. Example of a three-dimensional Kenmotsu manifold with η-parallel Ricci tensor

Definition 3.1. *The Ricci tensor S of a Kenmotsu manifold is called η-parallel if it satisfies*

$$(\nabla_X S)(\phi Y, \phi Z) = 0. \tag{3.1}$$

The notion of Ricci $\eta-$parallelity for Sasakian manifolds was introduced by M. Kon [11].

In [6] the authors proved that a three-dimensional Kenmotsu manifold has η-parallel Ricci tensor if and only if it is of constant scalar curvature. In this section we verify this theorem by a concrete example.

We consider the three-dimensional manifold $M = \{(x, y, z) \in R^3, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = -z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

The Rimennian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{array}{lll} \nabla_{e_1} e_3 = e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = 0, \\ \nabla_{e_2} e_3 = e_2, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From the above it follows that the manifold satisfies $\nabla_X \xi = X - \eta(X)\xi$, for $\xi = e_3$. Hence the manifold is a Kenmotsu Manifold. With the help of the above results we can verify the following:

$$\begin{array}{lll} R(e_1, e_2)e_2 = 0, & R(e_1, e_3)e_3 = -e_1, & R(e_2, e_1)e_1 = 0, \\ R(e_2, e_3)e_3 = -e_2, & R(e_3, e_1)e_1 = 0, & R(e_3, e_2)e_2 = 0, \\ R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_1 = 0, & R(e_3, e_1)e_2 = 0 \end{array}$$

Now from the definition of the Ricci tensor in three dimensional manifold we get

$$S(X, Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i). \tag{3.2}$$

From the components of the curvature tensor and (3.2) we get the following results.

$$\begin{array}{lll} S(e_1, e_1) = 0, & S(e_2, e_2) = 0, & S(e_3, e_3) = -2, \\ S(e_1, e_2) = 0, & S(e_1, e_3) = 0, & S(e_2, e_3) = 0. \end{array}$$

With the help of the above results we can easily verify the following :

$$\begin{array}{lll} (\nabla_X S)(\phi e_1, \phi e_2) = 0, & (\nabla_X S)(\phi e_2, \phi e_3) = 0, & (\nabla_X S)(\phi e_1, \phi e_1) = 0, \\ (\nabla_X S)(\phi e_1, \phi e_3) = 0, & (\nabla_X S)(\phi e_3, \phi e_1) = 0, & (\nabla_X S)(\phi e_2, \phi e_2) = 0, \\ (\nabla_X S)(\phi e_2, \phi e_1) = 0, & (\nabla_X S)(\phi e_3, \phi e_2) = 0, & (\nabla_X S)(\phi e_3, \phi e_3) = 0. \end{array}$$

Thus we note that

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{3.3}$$

for all $X, Y, Z \in \chi(M)$. Hence the Ricci tensor is η-parallel.

Here we also note that the scalar curvature of the manifold is -2 which is constant.

4. Conditions for a vector field in Kenmotsu manifold to be Killing

Definition 4.1. *A vector field X on a Kenmotsu manifold is said to be conformal Killing vector field*[17] *if*

$$L_X g = \rho g,$$

where ρ is a function on the manifold.

If $\rho = 0$, then the vector field X is said to be a Killing vector field.

Let the vector field X be a conformal Killing vector field on a Kenmotsu manifold M^{2n+1}. Then for a function ρ we get

$$(L_X g)(Y, Z) = \rho g(Y, Z). \tag{4.1}$$

From (2.6) we get $\nabla_\xi \xi = 0$. So the integral curves are geodesics and we have from (4.1), by putting $Y = Z = \xi$

$$\rho = (L_X g)(\xi, \xi).$$

Now

$$(L_X g)(\xi, \xi) = 2g(\nabla_\xi X, \xi).$$

Again

$$2\nabla_\xi (g(X, \xi)) = 2g(\nabla_\xi X, \xi).$$

So, we have

$$\rho = (L_X g)(\xi, \xi) = 2g(\nabla_\xi X, \xi) = 2\nabla_\xi (g(X, \xi)). \tag{4.2}$$

Now if X is orthogonal to ξ, $\rho = 0$ and hence $(L_X g) = 0$; i.e., X is a Killing vector field. Thus we are in a position to state

Theorem 4.1. *If a conformal Killing vector field X on a Kenmotsu manifold is orthogonal to ξ, then X is Killing.*

Let V be a vector field on a Kenmotsu manifold M^{2n+1} such that $L_V R = 0$.

Now from (2.14) we have

$$g(R(W, X)Y, Z) + g(R(W, X)Z, Y) = 0.$$

Taking the Lie derivative of the above identity along V we get

$$(L_V g)(R(W, X)Y, Z) + (L_V g)(R(W, X)Z, Y) = 0. \tag{4.3}$$

Putting $W = Y = Z = \xi$ in (4.3) and using (2.10)(b)we get

$$(L_V g)(X - \eta(X)\xi, \xi) + (L_V g)(X - \eta(X)\xi, \xi) = 0,$$

or,

$$(L_V g)(X, \xi) = \eta(X)(L_V g)(\xi, \xi). \tag{4.4}$$

Again putting $W = Y = \xi$ in (4.3) and using (2.10)(a) we get

$$(L_V g)(X - \eta(X)\xi, Z) + (L_V g)(\eta(Z)X - g(X, Z)\xi, \xi) = 0,$$

or,

$$\begin{aligned} &(L_V g)(X, Z) - \eta(X)(L_V g)(\xi, Z) + \eta(Z)(L_V g)(X, \xi) \\ -\ &(L_V g)(\xi, \xi)g(X, Z) = 0. \end{aligned} \tag{4.5}$$

By (4.4), (4.5) yields

$$(L_V g)(X, Z) = (L_V g)(\xi, \xi)g(X, Z),$$

i.e.,

$$(L_V g) = (L_V g)(\xi, \xi)g. \tag{4.6}$$

From (2.12) we know $S(\xi, \xi) = -2n$. Applying Lie derivative on it and keeping in mind that $L_V R = 0$ implies $L_V S = 0$, we get

$$S(L_V \xi, \xi) = 0.$$

But $S(X, \xi) = -2n$. So, $L_V \xi = 0$. Hence $g(L_V \xi, \xi) = 0$. Thus $(L_V g)(\xi, \xi) = 0$. So, in view of (4.2) we get $\rho = 0$. In other words the vector field V is Killing vector field. Thus we can state

Theorem 4.2. *If a vector field V on a Kenmotsu manifold leaves the curvature tensor invariant, then V is Killing vector field.*

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