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ON HERMITE-TRICOMI FUNCTIONS OF THE SINGLE COMPLEX VARIABLE

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Abstract. In this paper, we define a new kind of the hypergeometric function, say, Hermite-Tricomi functions. An explicit representation, recurrence relations for the Hermite-Tricomi functions are given and differential equations satisfied them is presented. A new expansions of the exponential for a wide class of functions, whose properties, including recurrences, addition theorems, generating functions etc. in terms of Hermite-Tricomi functions are studied in detail.

1. Introduction

Theory and applications of scalar orthogonal polynomials are elegant, extensive, and diverse, with fundamental results dating back to developmental work by Hermite, Jacobi, Laguerre, Legendre, Tchebicheff, and others [12, 13, 14]. In a number of previous papers [3, 4, 5, 8, 9], it has been shown that, many properties of ordinary and generalized special functions are easily derived and framed in a more general context. The aim of this paper is to provide some answers to the problems arising in the study of the development of Hermite-Tricomi functions. This approach will permit the introduction of the Tricomi functions to establish the basis of Hermite-Tricomi functions to third order differential equations. Furthermore, we prove that the Hermite-Tricomi functions satisfy a differential equation and thier extension to the two-variables Hermite-Tricomi functions is given.

In these introductory remarks, we discuss the properties of the Tricomi functions and Hermite polynomials defined in [2, 10] and fix the notation in order to make the paper self-consistent, we recall the following specialized version of the definitions.

1.1. **definition.** The Tricomi Functions have been defined in [1, 6] as follows

$$C_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!(n+k)!} \tag{1.1}$$

and

$$C_n(x)=x^{-\frac{n}{2}}J_n(2\sqrt{x}) \tag{1.2}$$

where $J_n(x)$ is a cylindrical Bessel function of first kind.

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Most of the properties of (1.1) can be derived from the generating function

$$\sum_{n=-\infty}^{\infty} t^n C_n(x) = e^{t-\frac{x}{t}} \tag{1.3}$$

and the following properties of the n^{th}-order Tricomi functions, which yield the recurrences

$$\begin{aligned} &\frac{d^r}{dx^r} C_n(x) = (-1)^r C_{n+r}(x), \\ &C_{-n}(x) = (-x)^n C_n(x), \\ &xC_{n+1}(x) - nC_n(x) + C_{n-1}(x) = 0 \end{aligned} \tag{1.4}$$

which, once combined, provide us with the differential equation

$$\left[x\frac{d^2}{dx^2} + (n+1)\frac{d}{dx} + 1\right] C_n(x) = 0. \tag{1.5}$$

By exploiting the addition formula [11] we have

$$C_n(x+y) = \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} C_{n+k}(x). \tag{1.6}$$

The addition theorem is easily derived from (1.7) which yields

$$C_n(x \pm y) = \sum_{k=0}^{\infty} \frac{(\mp 1)^k y^k}{k!} C_{n+k}(x). \tag{1.7}$$

To this aim, we remind the reader that, Hermite polynomials $H_n(x, y)$ are also defined through the operational identity. It is also easy to realize that, we can define the $H_n(x, y)$ through the operational rule

$$H_n(x, y) = \exp\left(-\frac{y}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^n. \tag{1.8}$$

The use of the inverse of (1.8) allows to conclude that

$$(2x)^n = \exp\left(\frac{y}{4}\frac{\partial^2}{\partial x^2}\right) H_n(x, y) \tag{1.9}$$

which is a quite useful operational identity, which can be exploited to state further relations. Special case: we can write $H_n(x, 1) = H_n(x)$.

2. Hermite-Tricomi functions

We define the new generating function which represents a generalization of the Hermite-Tricomi functions in the form

$$\sum_{n=-\infty}^{\infty} t^n \, {}_H C_n(x) = \exp\left(t - \frac{2x}{t} - \frac{1}{t^2}\right) \tag{2.1}$$

and by the series expansion

$${}_H C_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k H_k(x)}{k!(n+k)!}. \tag{2.2}$$

It is clear that

$$ {}_HC_{-1}(x)=0,\quad {}_HC_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^kH_k(x)}{(k!)^2},\quad {}_HC_1(x)=\sum_{k=0}^{\infty}\frac{(-1)^kH_k(x)}{k!(k+1)!}. $$

In the following theorem, we obtain another representation for the Hermite-Tricomi functions [8, 9].

Theorem 2.1. *The Hermite-Tricomi function of n^{th} order has the following representation*

$$ {}_HC_n(x)=\exp\Big(-\frac{1}{4}\frac{d^2}{dx^2}\Big)C_n(2x). \tag{2.3} $$

Proof: By using (1.1), (1.8) and (2.2), we consider the series in the form

$$ \begin{aligned} {}_HC_n(x)&=\sum_{k=0}^{\infty}\frac{(-1)^kH_k(x)}{k!(n+k)!}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\exp\Big(-\frac{1}{4}\frac{d^2}{dx^2}\Big)\big(2x\big)^k\\ &=\exp\Big(-\frac{1}{4}\frac{d^2}{dx^2}\Big)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\big(2x\big)^k=\exp\Big(-\frac{1}{4}\frac{d^2}{dx^2}\Big)C_n(2x). \end{aligned} $$

Proof of Theorem **2.1** is completed.

Furthermore, in view of (2.3), we can write

$$ C_n(2x)=\exp\Big(\frac{1}{4}\frac{d^2}{dx^2}\Big)\ {}_HC_n(x). \tag{2.4} $$

It is worth noting that, for $\frac{x}{2}$ instead of x, the expression (2.3) gives another representation for the Hermite-Tricomi functions in the form

$$ \begin{aligned} {}_HC_n(\frac{x}{2})&=\exp\Big(-\frac{d^2}{dx^2}\Big)C_n(x),\\ C_n(x)&=\exp\Big(\frac{d^2}{dx^2}\Big)\ {}_HC_n(\frac{x}{2}). \end{aligned} $$

The new properties of the the Hermite-Tricomi functions generated by (2.1) yields as given in the following theorem.

Theorem 2.2. *The Hermite-Tricomi functions satisfy the following relations*

$$ {}_HC_n(x+y)=\sum_{k=0}^{\infty}\frac{(-2y)^k}{k!}\ {}_HC_{n+k}(x). \tag{2.5} $$

Proof: By using (2.1), the series can be given in the form

$$ \begin{aligned} \sum_{n=-\infty}^{\infty}{}_HC_n(x+y)t^n&=\exp\Big(t-2\frac{x+y}{t}-\frac{1}{t^2}\Big)=\exp\big(-\frac{2y}{t}\big)\exp\Big(t-\frac{2x}{t}-\frac{1}{t^2}\Big)\\ &=\sum_{n=-\infty}^{\infty}{}_HC_n(x)t^n\sum_{k=0}^{\infty}\frac{(-2y)^kt^{-k}}{k!}=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-2y)^k}{k!}\ {}_HC_n(x)t^{n-k}. \end{aligned} $$

By comparing the coefficients of t^n, we get (2.5) and the proof is established. In the following corollary, we obtain the properties of Hermite-Tricomi functions as follows

Corollary 2.3. *The addition property of the Hermite-Tricomi functions given in the following relation*

$$ {}_HC_n(x \pm y) = \sum_{k=0}^{\infty} \frac{(\mp 2y)^k}{k!}\ {}_HC_{n+k}(x). \tag{2.6}$$

Proof. By exploiting the addition formula (1.7) we have

$$ {}_HC_n(x - y) = \sum_{k=0}^{\infty} \frac{(2y)^k}{k!}\ {}_HC_{n+k}(x).$$

Hence, the proof of Corollary **2.3** is established.

2.1. **Recurrence relations.** Some recurrence relations as carried out on the Hermite-Tricomi functions. We obtain the following

Theorem 2.4. *The Hermite-Tricomi functions satisfy the following relations*

$$\frac{d^k}{dx^k}\ {}_HC_n(x) = (-2)^k\ {}_HC_{n+k}(x); \quad 0 \le k \le n. \tag{2.7}$$

Proof. Differentiating the identity (2.1) with respect to x yields

$$\sum_{n=-\infty}^{\infty} {}_HC_n'(x)t^n = \frac{-2}{t}\exp\left(t - \frac{2x}{t} - \frac{1}{t^2}\right). \tag{2.8}$$

From (2.1) and (2.8), we have

$$\sum_{n=-\infty}^{\infty} {}_HC_n'(x)t^n = -2\sum_{n=-\infty}^{\infty} {}_HC_n(x)t^{n-1}. \tag{2.9}$$

Hence, identifying coefficients of t^n, it follows that

$$ {}_HC_n'(x) = -2\ {}_HC_{n+1}(x). \tag{2.10}$$

The iteration of (2.10), for $0 \le k \le n$, implies (2.7).

Hence for particular values of k and n, (2.7) yields

$$ {}_HC_n(x) = \frac{1}{(-2)^{n-k}}\frac{d^{n-k}}{dx^{n-k}}\ {}_HC_k(x). \tag{2.11}$$

Therefore, the expression (2.7) is established and the proof of Theorem **2.4** is completed.

Differentiating the identity (2.1) with respect to t yields

$$\sum_{n=-\infty}^{\infty} n\ {}_HC_n(x)t^{n-1} = \left(t^3 + 2xt + 2\right)\sum_{n=-\infty}^{\infty} {}_HC_n(x)t^{n-3}$$

from which by comparing the coefficients of t^n on sides of the identity, we obtain the pure recurrence relation

$$(n+1)\ {}_HC_{n+1}(x) = {}_HC_n(x) + 2x\ {}_HC_{n+2}(x) + 2\ {}_HC_{n+3}(x). \tag{2.12}$$

The above recurrence relation will be used in the following theorem.

Theorem 2.5. *For Hermite-Tricomi functions, have*

$$2\ {}_HC_{n+1}(x) + 2x\ {}_HC_n(x) - (n-1)\ {}_HC_{n-1}(x) + {}_HC_{n-2}(x) = 0. \tag{2.13}$$

Proof. The new generating function which represents the Hermite-Tricomi functions given by the following relation

$$F(x,t) = \sum_{n=-\infty}^{\infty} {}_HC_n(x)t^n = \exp\left(t - \frac{2x}{t} - \frac{1}{t^2}\right). \tag{2.14}$$

Differentiating (2.14) with respect to x and t, we find respectively

$$\frac{\partial F}{\partial x} = \sum_{n=-\infty}^{\infty} {}_HC'_n(x)t^n = -2 \sum_{n=-\infty}^{\infty} {}_HC_n(x)t^{n-1}$$

and

$$\frac{\partial F}{\partial t} = \sum_{n=-\infty}^{\infty} n\ {}_HC_n(x)t^{n-1} = \left(t^3 + 2xt + 2\right) \sum_{n=-\infty}^{\infty} {}_HC_n(x)t^{n-3}.$$

Therefore, $F(x,t)$ satisfies the partial differential equation

$$(t^3 + 2xt + 2)\frac{\partial F}{\partial x} + 2t^2\frac{\partial F}{\partial t} = 0$$

which, by using (2.1), becomes

$$(t^3 + 2xt + 2) \sum_{n=-\infty}^{\infty} {}_HC'_n(x)t^n + 2t^2 \sum_{n=-\infty}^{\infty} n\ {}_HC_n(x)t^{n-1} = 0.$$

It follows that

$${}_HC'_{n-3}(x) + 2x\ {}_HC'_{n-1}(x) + 2\ {}_HC'_n(x) + 2(n-1)\ {}_HC_{n-1}(x) = 0. \tag{2.15}$$

Using (2.10) and (2.15), we get (2.13) and the proof of Theorem **2.5** is completed. The following result shows that the Hermite-Tricomi functions appear as finite series solutions of the third order differential equation.

Corollary 2.6. *The Hermite-Tricomi functions are solution of the differential equation of the third order in the form*

$$\left[\frac{d^3}{dx^3} - 2x\frac{d^2}{dx^2} - 2(n+1)\frac{d}{dx} - 4\right] {}_HC_n(x) = 0. \tag{2.16}$$

Proof. Replacing n by $n-1$ in (2.10) gives

$$\begin{aligned} &\frac{d}{dx}\ {}_HC_{n-1}(x) = -2\ {}_HC_n(x), \\ &\frac{d^2}{dx^2}\ {}_HC_{n-1}(x) = -2\frac{d}{dx}\ {}_HC_n(x), \end{aligned} \tag{2.17}$$

$$\frac{d^2}{dx^2}\ {}_HC_{n-2}(x) = (-2)^2\ {}_HC_n(x) \tag{2.18}$$

and

$$\begin{aligned} \frac{d^2}{dx^2}\ {}_HC_n(x) &= -2\frac{d}{dx}\ {}_HC_{n+1}(x) = (-2)^2\ {}_HC_{n+2}(x) \\ \frac{d^3}{dx^3}\ {}_HC_n(x) &= \frac{d^2}{dx^2}\left(-2\ {}_HC_{n+1}(x)\right) = (-2)^2\frac{d}{dx}\ {}_HC_{n+2}(x) \\ &= (-2)^3\ {}_HC_{n+3}(x). \end{aligned} \tag{2.19}$$

Differentiating the identity (2.13) with respect to x yields

$$\begin{aligned} &2\frac{d^2}{dx^2}\ {}_HC_{n+1}(x)+2x\frac{d^2}{dx^2}\ {}_HC_n(x)+4\frac{d}{dx}\ {}_HC_n(x)\\ &-(n-1)\frac{d^2}{dx^2}\ {}_HC_{n-1}(x)+\frac{d^2}{dx^2}\ {}_HC_{n-2}(x)=0. \end{aligned} \tag{2.20}$$

Substituting (2.17), (2.18), (2.19) into (2.20) we obtain (2.16). Thus the proof of Corollary **2.6** is completed.

In the following, we will apply the above results to operator, whose importance has been recognized within the framework of n^{th}-order Hermite-Tricomi functions, and we will see that the results, summarized in this section, so that Hermite-Tricomi functions can be considered as a generalization of the ordinary exponential function, can be exploited to state quite general results. Further speculations will be discussed in the forthcoming section.

3. Hermite-Tricomi functions of the Two variables

The functions represent the Hermite-Tricomi functions ${}_HC_n(x,y)$ of two variables and the generating function is given as

$$\sum_{n=-\infty}^{\infty} t^n\ {}_HC_n(x,y)=\exp\left(t-\frac{2x}{t}-\frac{y}{t^2}\right) \tag{3.1}$$

and by the series expansion

$${}_HC_n(x,y)=\sum_{k=0}^{\infty}\frac{(-1)^k H_k(x,y)}{k!(n+k)!}. \tag{3.2}$$

Theorem 3.1. *The Hermite-Tricomi functions of two variables of n^{th} order has the following representation*

$${}_HC_n(x,y)=\exp\left(-\frac{y}{4}\frac{\partial^2}{\partial x^2}\right)C_n(2x). \tag{3.3}$$

Proof: Instead of (1.1), (1.8) and (3.2),(3.3).

Furthermore, in view of (3.3), we can write

$$C_n(2x)=\exp\left(\frac{y}{4}\frac{\partial^2}{\partial x^2}\right)\ {}_HC_n(x,y). \tag{3.4}$$

In the following theorems, we obtain the addition formula for Hermite-Tricomi functions of two variables.

Theorem 3.2. *The Hermite-Tricomi functions of two variables satisfy the following relations*

$${}_HC_n(x\pm z,y)=\sum_{k=0}^{\infty}\frac{(\mp 2z)^k}{k!}\ {}_HC_{n+k}(x,y). \tag{3.5}$$

Proof: By using (3.1), the series can be given in the form

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}\ {}_HC_n(x\pm z,y)t^n=\exp\left(t-2\frac{x\pm z}{t}-\frac{y}{t^2}\right)\\ &=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(\mp 2z)^k}{k!}\ {}_HC_n(x,y)t^{n-k}. \end{aligned}$$

By comparing the coefficients of t^n, we get (3.5). The proof of Theorem **3.2** is established.

Theorem 3.3. *The Hermite-Tricomi functions satisfy the following relations*

$$ {}_HC_n(x, y \pm w) = \sum_{k=0}^{\infty} \frac{(\mp w)^k}{k!} \; {}_HC_{n+2k}(x, y). \tag{3.6}$$

Proof: By using (3.1), we consider the series in the form

$$\sum_{n=-\infty}^{\infty} {}_HC_n(x, y \pm w)t^n = \exp\left(t - 2\frac{x}{t} - \frac{y \pm w}{t^2}\right) = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \frac{(\mp w)^k}{k!} \; {}_HC_n(x, y)t^{n-2k}.$$

By comparing the coefficients of t^n, we get (3.6) and the proof is completed.

In the following corollary, we obtain the properties Hermite-Tricomi functions of two variables as follows

Corollary 3.4. *The Hermite-Tricomi functions satisfy the following addition formula, yields*

$$2^n \; {}_HC_n(x + z, y + w) = \sum_{m=-\infty}^{\infty} {}_HC_{n-m}(\frac{x}{2}, \frac{y}{4}) \; {}_HC_m(\frac{z}{2}, \frac{w}{4}). \tag{3.7}$$

Proof: By using (3.1), the series can be given in the form

$$\sum_{n=-\infty}^{\infty} {}_HC_n(x + z, y + w)t^n = \exp\left(t - 2\frac{x + z}{t} - \frac{y + w}{t^2}\right)$$
$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} 2^{-(n+m)} \; {}_HC_n(\frac{x}{2}, \frac{y}{4}) \; {}_HC_m(\frac{z}{2}, \frac{w}{4})t^{n+m}.$$

By comparing the coefficients of t^n, we get (3.7). The proof of Corollary **3.4** is completed.

The following corollary, we obtain another recurrence formula Hermite-Tricomi functions of two variables as follows.

Corollary 3.5. *The Hermite-Tricomi functions holds the following*

$$ {}_HC_n(x, y + w) = \exp\left(-\frac{w}{4}\frac{\partial^2}{\partial x^2}\right) {}_HC_n(x, y). \tag{3.8}$$

Proof. By using the Theorem **3.1**, we get directly the equation (3.8).

3.1. Recurrence relations for Hermite-Tricomi functions of two variables.

Theorem 3.6. *The Hermite-Tricomi functions satisfy the following relations*

$$\frac{\partial^k}{\partial x^k} \; {}_HC_n(x, y) = (-2)^k \; {}_HC_{n+k}(x, y); \quad 0 \le k \le n \tag{3.9}$$

and

$$\frac{\partial^k}{\partial y^k} \; {}_HC_n(x, y) = (-1)^k \; {}_HC_{n+2k}(x, y); \quad 0 \le k \le [\frac{n}{2}]. \tag{3.10}$$

Proof. Differentiating the identity (3.1) with respect to x yields

$$\sum_{n=-\infty}^{\infty} \frac{\partial}{\partial x} \ {}_HC_n(x,y)t^n = \frac{-2}{t} \exp\left(t - \frac{2x}{t} - \frac{y}{t^2}\right). \tag{3.11}$$

From (3.1) and (3.11), we have

$$\sum_{n=-\infty}^{\infty} \frac{\partial}{\partial x} \ {}_HC_n(x,y)t^n = -2 \sum_{n=-\infty}^{\infty} \ {}_HC_n(x,y)t^{n-1}$$

from identifying coefficients in t^n, it follows that

$$\frac{\partial}{\partial x} \ {}_HC_n(x,y) = -2 \ {}_HC_{n+1}(x,y). \tag{3.12}$$

The iteration of (3.12), for $0 \leq k \leq n$, implies (3.9). Differentiating the identity (3.1) with respect to y and identifying coefficients of t^n, we get

$$\frac{\partial}{\partial y} \ {}_HC_n(x,y) = - \ {}_HC_{n+2}(x,y) \tag{3.13}$$

and, in general,

$$\frac{\partial^k}{\partial y^k} \ {}_HC_n(x,y) = (-1)^k \ {}_HC_{n+2k}(x,y).$$

Hence for particular values of k and n, (3.9) and (3.10), yield

$$\begin{aligned} {}_HC_n(x,y) &= \frac{1}{(-2)^{n-k}} \frac{\partial^{n-k}}{\partial x^{n-k}} \ {}_HC_k(x,y) = \frac{1}{(-2)^k} \frac{\partial^k}{\partial x^k} \ {}_HC_{n-k}(x,y) \\ &= (-1)^k \frac{\partial^k}{\partial x^k} \ {}_HC_{n-2k}(x,y). \end{aligned} \tag{3.14}$$

Differentiating the identity (3.1) with respect to t yields

$$\sum_{n=-\infty}^{\infty} n \ {}_HC_n(x,y)t^{n-1} = \left(t^3 + 2xt + 2y\right) \sum_{n=-\infty}^{\infty} \ {}_HC_n(x,y)t^{n-3}$$

from which by comparing the coefficients of t^n on both sides of the identity, we obtain the pure recurrence relation

$$(n+1) \ {}_HC_{n+1}(x,y) = \ {}_HC_n(x,y) + 2x \ {}_HC_{n+2}(x,y) + 2y \ {}_HC_{n+3}(x,y). \tag{3.15}$$

Therefore, the expressions (3.9) and (3.10) are established and the proof of Theorem **3.6** is completed.

The following corollary is a consequence of Theorem **3.6**.

Corollary 3.7. *The Hermite-Tricomi functions satisfy the following relation*

$$\frac{\partial^2}{\partial x^2} \ {}_HC_n(x,y) + 4\frac{\partial}{\partial y} \ {}_HC_n(x,y) = 0. \tag{3.16}$$

Proof. By (3.9) and (3.10) the equation (3.16) follows directly.

According to (3.16), it is clear that the ${}_HC_n(x,y)$ are the natural solutions of the heat partial differential equation. The above recurrence relation will be used in the following theorem.

Theorem 3.8. *The Hermite-Tricomi functions of two variables, have the following*

$$2y \ {}_HC_{n+1}(x,y) + 2x \ {}_HC_n(x,y) - (n-1) \ {}_HC_{n-1}(x,y) + \ {}_HC_{n-2}(x,y) = 0. \tag{3.17}$$

Proof. The generating function gives

$$F(x,y,t)=\sum_{n=-\infty}^{\infty}{}_HC_n(x,y)t^n=\exp\left(t-\frac{2x}{t}-\frac{y}{t^2}\right). \tag{3.18}$$

Differentiating (3.18) with respect to x and t, we find respectively

$$\frac{\partial F}{\partial x}=\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial x}\,{}_HC_n(x,y)t^n=-2\sum_{n=-\infty}^{\infty}{}_HC_n(x,y)t^{n-1}$$

and

$$\frac{\partial F}{\partial t}=\sum_{n=-\infty}^{\infty}n\,{}_HC_n(x,y)t^{n-1}=\Big(t^3+2xt+2y\Big)\sum_{n=-\infty}^{\infty}{}_HC_n(x,y)t^{n-3}.$$

Therefore, $F(x,y,t)$ satisfies the partial differential equation

$$(t^3+2xt+2y)\frac{\partial F}{\partial x}+2t^2\frac{\partial F}{\partial t}=0$$

which, by using (3.1), becomes

$$\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial x}\,{}_HC_n(x,y)t^{n+3}+2x\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial x}\,{}_HC_n(x,y)t^{n+1}$$
$$+\,2y\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial x}\,{}_HC_n(x,y)t^{n}+2\sum_{n=-\infty}^{\infty}n\,{}_HC_n(x,y)t^{n+1}=0.$$

It follows

$$\frac{\partial}{\partial x}\,{}_HC_{n-3}(x,y)+2x\frac{\partial}{\partial x}\,{}_HC_{n-1}(x,y)+2y\frac{\partial}{\partial x}\,{}_HC_n(x)+2(n-1)\,{}_HC_{n-1}(x,y)=0. \tag{3.19}$$

Using (3.10) and (3.19), we get (3.17). Finally, the Hermite-Tricomi functions appear as finite series solutions of the differential equation in the following corollary that is a consequence of Theorem **3.8** .

Corollary 3.9. *The Hermite-Tricomi functions are solution of the differential equation of the third order in the form*

$$\left[y\frac{\partial^3}{\partial x^3}-2x\frac{\partial^2}{\partial x^2}-2(n+1)\frac{\partial}{\partial x}-4\right]{}_HC_n(x,y)=0. \tag{3.20}$$

Proof. Replacing n by $n-1$ in (3.9) gives

$$\begin{aligned}\frac{\partial}{\partial x}\,{}_HC_{n-1}(x,y)&=-2\,{}_HC_n(x,y),\\ \frac{\partial^2}{\partial x^2}\,{}_HC_{n-1}(x,y)&=\frac{\partial}{\partial x}\Big(-2\,{}_HC_n(x,y)\Big)\\ &=-2\frac{\partial}{\partial x}\,{}_HC_n(x,y),\end{aligned} \tag{3.21}$$

$$\frac{\partial^2}{\partial x^2}\,{}_HC_{n-2}(x,y)=\frac{\partial}{\partial x}\Big(-2\,{}_HC_{n-1}(x,y)\Big)=(-2)^2\,{}_HC_n(x,y) \tag{3.22}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\ {}_HC_n(x,y) = -2\frac{\partial}{\partial x}\ {}_HC_{n+1}(x,y) = (-2)^2\ {}_HC_{n+2}(x,y)\\
&\frac{\partial^3}{\partial x^3}\ {}_HC_n(x,y) = (-2)^2\frac{\partial}{\partial x}\ {}_HC_{n+2}(x,y)\\
&\qquad\qquad\qquad = -2\frac{\partial^2}{\partial x^2}\ {}_HC_{n+1}(x,y) = (-2)^3\ {}_HC_{n+3}(x,y).
\end{aligned} \tag{3.23}$$

Differentiating the identity (3.18) with respect to x yields

$$\begin{aligned}
2y\frac{\partial^2}{\partial x^2}\ {}_HC_{n+1}(x,y) + 2x\frac{\partial^2}{\partial x^2}\ {}_HC_n(x,y) + 4\frac{\partial}{\partial x}\ {}_HC_n(x)\\
-(n-1)\frac{\partial^2}{\partial x^2}\ {}_HC_{n-1}(x,y) + \frac{\partial^2}{\partial x^2}\ {}_HC_{n-2}(x,y) = 0.
\end{aligned} \tag{3.24}$$

Substituting (3.21), (3.22), (3.24) into (3.24) we obtain (2.16). Thus the proof of Corollary **3.9** is completed.

Special case: By exploiting the same argument of the previous section, it is evident that ${}_HC_n(x,y)$ satisfies the properties [7]

$$\begin{aligned}
{}_HC_n(x,y)|_{y=0} &= C_n(2x),\\
{}_HC_n(x,y)|_{y=1} &= \ {}_HC_n(x).
\end{aligned} \tag{3.25}$$

Further examples proving the usefulness of the present methods can be easily worked out, but are not reported here for conciseness. These last identities indicate that the method described in this paper can go beyond the specific problem addressed here and can be exploited in a wider context.

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