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AN APPLICATION OF HYPERGEOMETRIC FUNCTIONS ON HARMONIC UNIVALENT FUNCTIONS

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. The purpose of the present paper is to establish connections between various subclasses of harmonic functions by applying certain convolution operator involving hypergeometric functions. To be more precise, we investigate such connections with Goodman-Rønning-type harmonic univalent functions in the open unit disc U.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc U. Hohlov [8] introduced the convolution operator $H(a,b;c) : A \to A$ defined by

$$H(a,b;c)f(z) = zF(a,b;c;z) * f(z),$$

where the symbol "$*$" stands for the convolution of two power series, which is defined for two functions $f, g \in A$, where $f(z)$ and $g(z)$ are of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ as

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$(f*g)(z) = f(z)*g(z) = z+\sum_{n=2}^{\infty} a_n b_n z^n$ and $F(a,b;c;z)$ is a well-known Gaussian hypergeometric function and given by the series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where a, b, c are complex numbers such that $c \neq 0, -1, -2, ...$ and $(a)_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$= \begin{cases} 1 & \text{if } n = 0 \\ a(a+1).........(a+n-1) & \text{if } n \in N = \{1, 2, 3.....\}. \end{cases}$$

A hypergeometric function $F(a,b;c;z)$ is analytic in U and plays an important role in Geometric Function Theory. See, for example, the works by Ahuja [3], Carleson and Shaffer [5], Miller and Mocanu [9], Owa and Srivastava [10], Ponnusamy and Rønning [11], Ruscheweyh and Singh [13] and Swaminathan [14].

Let H be the family of all harmonic functions of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n,\ g(z) = \sum_{n=1}^{\infty} B_n z^n,\ |B_1| < 1,\ (z \in U), \tag{1.2}$$

are in the class A. For complex parameters $a_1, b_1, c_1, a_2, b_2, c_2$ $(c_1, c_2 \neq 0, -1, -2, ...)$, we define the functions $\Phi_1(z) = zF(a_1, b_1; c_1; z)$ and $\Phi_2(z) = zF(a_2, b_2; c_2; z)$.

Corresponding to these functions, we consider the following convolution operator

$$\Omega \equiv \Omega \begin{pmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{pmatrix} : H \to H$$

defined by

$$\Omega \begin{pmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{pmatrix} f = f * (\Phi_1 + \overline{\Phi_2}) = h * \Phi_1 + \overline{g * \phi_2}$$

for any function $f = h + \overline{g}$ in H.

Letting

$$\Omega \begin{pmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{pmatrix} F(z) = H(z) + \overline{G(z)},$$

we have

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n,\ G(z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n z^n. \tag{1.3}$$

We observe that

$$\Omega \begin{pmatrix} a_1, & 1, & a_1 \\ a_2, & 1, & a_2 \end{pmatrix} f(z) = f(z) = f(z) * \left(\frac{z}{1-z} + \overline{\frac{z}{1-z}} \right)$$

is the identity mapping.

This convolution operator Ω were defined and studied by the author in [2].

Denote by S_H the subclass of H that are univalent and sense-preserving in U. Note that $\frac{f - \overline{B_1 f}}{1-|B_1|^2} \in S_H$ whenever $f \in S_H$. We also let the subclass S_H^0 of S_H

$$S_H^0 = \left\{ f = h + \overline{g} \in S_H : g'(0) = B_1 = 0 \right\}.$$

The classes S_H^0 and S_H were first studied in [6]. Also, we let K_H^0, $S_H^{*,0}$ and C_H^0 denote the subclasses of S_H^0 of harmonic functions which are, respectively, convex, starlike and close-to-convex in U. For definitions and properties of these classes, one may refer to ([1], [6]) or [7].

For $0 \leq \gamma < 1$, let

$$N_H(\gamma) = \left\{ f \in H : \operatorname{Re} \frac{f'(z)}{z'} \geq \gamma, z = re^{i\theta} \in U \right\},$$

$$G_H(\gamma) = \left\{ f \in H : \operatorname{Re} \left\{ \left(1 + e^{i\alpha}\right) \frac{zf'(z)}{f(z)} - e^{i\alpha} \right\} \geq \gamma, \alpha \in R, z \in U \right\},$$

where

$$z' = \frac{\partial}{\partial \theta}\left(z = re^{i\theta}\right), \; f'(z) = \frac{\partial}{\partial \theta} f\left(re^{i\theta}\right).$$

Define

$$TN_H(\gamma) = N_H(\gamma) \cap T \text{ and } TG_H(\gamma) = G_H(\gamma) \cap T,$$

where T consists of the functions $f = h + \overline{g}$ in S_H so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \; g(z) = \sum_{n=1}^{\infty} |B_n| z^n. \tag{1.4}$$

The classes $N_H(\gamma)$ and $G_H(\gamma)$ were initially introduced and studied, respectively, in ([4], [12]). A function in $G_H(\gamma)$ is called Goodman-Rønning-type harmonic univalent function in U.

Throughout this paper, we will frequently use the notations

$$\Omega(f) = \Omega \begin{pmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{pmatrix} f,$$

$$D_{n-1} = \frac{(|a_1|)_{n-1} (|b_1|)_{n-1}}{(c_1)_{n-1} (1)_{n-1}}, \; E_{n-1} = \frac{(|a_2|)_{n-1} (|b_2|)_{n-1}}{(c_2)_{n-1} (1)_{n-1}},$$

and a well-known formula

$$F(a, b; c; 1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \; \operatorname{Re}(c-a-b) > 0.$$

The main object of this paper is to establish some important connections between the classes K_H^0, $S_H^{*,0}$, C_H^0, $N_H(\gamma)$ and $G_H(\gamma)$ by applying the convolution operator Ω.

2. Connections with Goodman-Rønning-type Harmonic Univalent Functions

In order to establish connections between harmonic convex functions, we need following results in Lemma 2.1 [6], Lemma 2.2 [12] and Lemma 2.3 [2].

Lemma 2.1. *If $f = h + \overline{g} \in K_H^0$ where h and g are given by (1.2), then*

$$|A_n| \leq \frac{n+1}{2}, \; |B_n| \leq \frac{n-1}{2}.$$

Lemma 2.2. *Let $f = h + \overline{g}$ be given by (1.2). If*

$$\sum_{n=2}^{\infty} (2n-1-\gamma) |A_n| + \sum_{n=1}^{\infty} (2n+1+\gamma) |B_n| \le 1-\gamma, \tag{2.1}$$

then f is sense-preserving, Goodman-Rønning-type harmonic univalent functions in U and $f \in G_H(\gamma)$.

Remark 1. *In [12], it is also shown that $f = h + \overline{g}$ given by (1.4) is in the family $TG_H(\gamma)$, if and only if the coefficient condition (2.1) holds. Moreover, if $f \in TG_H(\gamma)$, then*

$$|A_n| \le \frac{1-\gamma}{2n-1-\gamma}, \ n \ge 2,$$

$$|B_n| \le \frac{1-\gamma}{2n+1+\gamma}, \ n \ge 1.$$

Lemma 2.3. *If $a, b, c > 0$, then*

$$(i) \ \sum_{n=2}^{\infty} (n-1) \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \frac{ab}{c-a-b-1} F(a,b;c;1), \ if c > a+b+1.$$

$$(ii) \ \sum_{n=2}^{\infty} (n-1)^2 \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \left[\frac{(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right] F(a,b;c;1), \ if c > a+b+2.$$

$$(iii) \ \sum_{n=2}^{\infty} (n-1)^3 \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \left[\frac{(a)_3 (b)_3}{(c-a-b-3)_3} + \frac{3(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right] F(a,b;c;1), \ if c > a+b+3.$$

Theorem 2.1. *Let $a_j, b_j \in C\setminus\{0\}$, $c_j \in R$ and $c_j > |a_j| + |b_j| + 2$ for $j = 1, 2$. If for some $\gamma (0 \le \gamma < 1)$, the inequality*

$$Q_1 F(|a_1|, |b_1|; c_1; 1) + R_1 F(|a_2|, |b_2|; c_2; 1) \le 4(1-\gamma)$$

is satisfied, then

$$\Omega\left(K_H^0\right) \subset G_H(\gamma),$$

where

$$Q_1 = \frac{2(|a_1|)_2 (|b_1|)_2}{(c_1 - |a_1| - |b_1| - 2)_2} + (7-\gamma) \frac{|a_1 b_1|}{(c_1 - |a_1| - |b_1| - 1)} + 2(1-\gamma),$$

$$R_1 = \frac{2(|a_2|)_2 (|b_2|)_2}{(c_2 - |a_2| - |b_2| - 2)_2} + (5+\gamma) \frac{|a_2 b_2|}{(c_2 - |a_2| - |b_2| - 1)}.$$

Proof. Let $f = h + \overline{g} \in K_H^0$ where h and g are of the form (1.2) with $B_1 = 0$. We need to show that $\Omega(f) = H + \overline{G} \in G_H(\gamma)$, where H and G defined by (1.3) are analytic functions in U.

In view of Lemma 2.2, we need to prove that

$$P_1 \le 1-\gamma,$$

where

$$P_1 = \sum_{n=2}^{\infty} (2n-1-\gamma) \left|\frac{(a_1)_{n-1} (b_1)_{n-1}}{(c_1)_{n-1} (1)_{n-1}} A_n\right| + \sum_{n=2}^{\infty} (2n+1+\gamma) \left|\frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} B_n\right|.$$

In view of Lemma 2.1 and 2.3, it follows that

$$P_1 \leq \frac{1}{2}\sum_{n=2}^{\infty}(n+1)(2n-1-\gamma)D_{n-1} + \frac{1}{2}\sum_{n=2}^{\infty}(n-1)(2n+1+\gamma)E_{n-1}$$

$$= \frac{1}{2}\sum_{n=2}^{\infty}\left[2(n-1)^2+(5-\gamma)(n-1)+2(1-\gamma)\right]D_{n-1}+\frac{1}{2}\sum_{n=2}^{\infty}\left[2(n-1)^2+(3+\gamma)(n-1)\right]E_{n-1}$$

$$= \frac{1}{2}Q_1F(|a_1|,|b_1|;c_1;1)+\frac{1}{2}R_1F(|a_2|,|b_2|;c_2;1)-(1-\gamma).$$

Now $P_1 \leq 1-\gamma$ follows from the given condition. □

Analogous to Theorem 2.1, we next find connections of the classes $S_H^{*,0}$, C_H^0 with $G_H(\gamma)$. However, we first need the following result which may be found in ([1], [6]) or [15].

Lemma 2.4. *If $f = h+\overline{g} \in S_H^{*,0}$ or C_H^0 with h and g as given by (1.2) with $B_1 = 0$, then*

$$|A_n| \leq \frac{(2n+1)(n+1)}{6},\ |B_n| \leq \frac{(2n-1)(n-1)}{6}.$$

Theorem 2.2. *Let $a_j, b_j \in C\setminus\{0\}$, $c_j \in R$ and $c_j > |a_j|+|b_j|+3$ for $j = 1,2$. If for some $\gamma(0 \leq \gamma < 1)$, the inequality*

$$Q_2F(|a_1|,|b_1|;c_1;1)+R_2F(|a_2|,|b_2|;c_2;1) \leq 12(1-\gamma)$$

is satisfied, then

$$\Omega\left(S_H^{*,0}\right) \subset G_H(\gamma) \text{ and } \Omega\left(C_H^0\right) \subset G_H(\gamma),$$

where

$$Q_2 = 4\frac{(|a_1|)_3(|b_1|)_3}{(c_1-|a_1|-|b_1|-3)_3}+2(14-\gamma)\frac{(|a_1|)_2(|b_1|)_2}{(c_1-|a_1|-|b_1|-1)_2}$$
$$+3(13-3\gamma)\frac{|a_1b_1|}{(c_1-|a_1|-|b_1|-1)}+6(1-\gamma),$$

$$R_2 = 4\frac{(|a_2|)_3(|b_2|)_3}{(c_2-|a_2|-|b_2|-3)_3}+2(10+\gamma)\frac{(|a_2|)_2(|b_2|)_2}{(c_2-|a_2|-|b_2|-1)_2}+3(5+\gamma)\frac{|a_2b_2|}{(c_2-|a_2|-|b_2|-1)}.$$

Proof. Let $f = h+\overline{g} \in S_H^{*,0}(C_H^0)$ where h and g of the form (1.2) with $B_1 = 0$. We need to show that $\Omega(f) = H+\overline{G} \in G_H(\gamma)$, where H and G defined by (1.3) are analytic functions in U.

In view of Lemma 2.2, we need to prove that

$$P_1 \leq 1-\gamma,$$

where

$$P_1 = \sum_{n=2}^{\infty}(2n-1-\gamma)\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_n\right|+\sum_{n=2}^{\infty}(2n+1+\gamma)\left|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}B_n\right|.$$

In view of Lemma 2.3 and 2.4, it follows that

$$P_1 \leq \frac{1}{6}\sum_{n=2}^{\infty}[(2n+1)(n+1)(2n-1-\gamma)]D_{n-1}+\frac{1}{6}\sum_{n=2}^{\infty}[(2n-1)(n-1)(2n+1+\gamma)]E_{n-1}$$

$$= \frac{1}{6}\sum_{n=2}^{\infty}\left[4(n-1)^3 + 2(8-\gamma)(n-1)^2 + (19-7\gamma)(n-1) + 6(1-\gamma)\right]D_{n-1}$$

$$+\frac{1}{6}\sum_{n=2}^{\infty}\left[4(n-1)^3 + 2(4+\gamma)(n-1)^2 + (3+\gamma)(n-1)\right]E_{n-1}$$

$$= \frac{1}{6}Q_2F(|a_1|, |b_1|; c_1; 1) + \frac{1}{6}R_2F(|a_2|, |b_2|; c_2; 1) - (1-\gamma)$$

Now $P_1 \leq 1-\gamma$ follows from the given condition. □

In order to determine connection between $TN_H(\beta)$ and $G_H(\gamma)$, we need the following results in Lemma 2.5 [4] and Lemma 2.6 [3].

Lemma 2.5. *Let $f = h + \overline{g}$ where h and g as given by (1.4) with $B_1 = 0$, and suppose that $0 \leq \beta < 1$. Then*

$$f \in TN_H(\beta) \Leftrightarrow \sum_{n=2}^{\infty} n|A_n| + \sum_{n=2}^{\infty} n|B_n| \leq 1-\beta.$$

Remark 2. *If $f \in TN_H(\beta)$, then*

$$|A_n| \leq \frac{1-\beta}{n},\ n \geq 2,$$

$$|B_n| \leq \frac{1-\beta}{n},\ n \geq 1.$$

Lemma 2.6. *Let $a, b \in C\backslash\{0\}$, $a \neq 1$, $b \neq 1$, $c \in (0,1)\cup(1,\infty)$ and $c > \max\{0, |a|+|b|-1\}$. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)}F(|a|, |b|; c; 1) - \frac{(c-1)}{(|a|-1)(|b|-1)}.$$

Theorem 2.3. *Let $a_j, b_j \in C\backslash\{0\}$, $a_j \neq 1$, $b_j \neq 1$, $c_j \in R$ and $c_j > \max\{0, |a_j|+|b_j|-1\}$ for $j = 1, 2$. If for some $\beta(0 \leq \beta < 1)$ and $\gamma(0 \leq \gamma < 1)$, the inequality*

$$Q_3F(|a_1|, |b_1|; c_1; 1) + R_3F(|a_2|, |b_2|; c_2; 1) \leq \frac{(1-\gamma)(2-\beta)}{(1-\beta)}$$
$$-(1+\gamma)\left[\frac{(c_1-1)}{(|a_1|-1)(|b_1|-1)} - \frac{(c_2-1)}{(|a_2|-1)(|b_2|-1)}\right]$$

is satisfied, then

$$\Omega(TN_H(\beta)) \subset G_H(\gamma),$$

where

$$Q_3 = 2 - (1+\gamma)\frac{(c_1-|a_1|-|b_1|)}{(|a_1|-1)(|b_1|-1)}$$

and

$$R_3 = 2 + (1+\gamma)\frac{(c_2-|a_2|-|b_2|)}{(|a_2|-1)(|b_2|-1)}.$$

Proof. Let $f = h + \overline{g} \in TN_H(\beta)$ where h and g are given by (1.4). In view of Lemma 2.2, it is enough to show that $P_2 \leq 1 - \gamma$, where

$$P_2 = \sum_{n=2}^{\infty} (2n-1-\gamma) \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n \right| + \sum_{n=1}^{\infty} (2n+1+\gamma) \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n \right|.$$

Using Remark 2 and Lemma 2.6, it follows that

$$P_2 \leq (1-\beta) \left[\sum_{n=2}^{\infty} \left(2 - \frac{(1+\gamma)}{n}\right) D_{n-1} + \sum_{n=1}^{\infty} \left(2 + \frac{(1+\gamma)}{n}\right) E_{n-1} \right]$$

$$= (1-\beta) \left[Q_3 F(|a_1|, |b_1|; c_1; 1) + R_3 F(|a_2|, |b_2|; c_2; 1) - (1-\gamma) + \frac{(1+\gamma)(c_1-1)}{(|a_1|-1)(|b_1|-1)} - \frac{(1+\gamma)(c_2-1)}{(|a_2|-1)(|b_2|-1)} \right]$$

$$\leq 1 - \gamma,$$

by the given hypothesis. □

In next theorem, we establish connections between $TG_H(\gamma)$ and $G_H(\gamma)$.

Theorem 2.4. *Let $a_j, b_j \in C \setminus \{0\}$, $c_j \in R$ and $c_j > |a_j| + |b_j|$ for $j = 1, 2$. If for some $\gamma (0 \leq \gamma < 1)$, the inequality*

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 2$$

is satisfied, then

$$\Omega(TG_H(\gamma)) \subset G_H(\gamma).$$

Proof. Making use of Lemma 2.2 and the definition of P_2 in Theorem 2.3, we only need to prove that $P_2 \leq 1 - \gamma$.

Using Remark 1, it follows that

$$\begin{aligned} P_2 &= \sum_{n=2}^{\infty} (2n-1-\gamma) \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n \right| + \sum_{n=1}^{\infty} (2n+1+\gamma) \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n \right| \\ &\leq (1-\gamma) \left[\sum_{n=2}^{\infty} D_{n-1} + \sum_{n=1}^{\infty} E_{n-1} \right] \\ &= (1-\gamma) \left[F(|a_1|, |b_1|; c_1; 1) - 1 + F(|a_2|, |b_2|; c_2; 1) \right] \\ &\leq 1 - \gamma, \end{aligned}$$

by the given condition and this completes the proof. □

In the next result, we establish connections between $TG_H(\gamma)$ and $G_H(\gamma)$ by diluting the restrictions on the complex coefficients of Theorem 2.4.

Theorem 2.5. *If $a_1, b_1 > -1$, $a_1 b_1 < 0$, $c_1 > \max\{0, a_1 + b_1\}$, $a_2, b_2 \in C \setminus \{0\}$ and $c_2 > |a_2| + |b_2|$, then a sufficient condition for*

$$\Omega(TG_H(\gamma)) \subset G_H(\gamma)$$

is that

$$F(a_1, b_1; c_1; 1) - F(|a_2|, |b_2|; c_2; 1) \geq 0$$

for any $\gamma (0 \leq \gamma < 1)$.

Proof. Let $f = h + \overline{g} \in TG_H(\gamma)$ with h and g in (1.4). Then

$$\Omega(f) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| z^n + \overline{\sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| z^n}.$$

This function can be rewritten as

$$\Omega(f) = z + \frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-1}} |A_n| z^n + \overline{\sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| z^n}.$$

In view of Lemma 2.2, we need to prove that $P_3 \leq 1-\gamma$,
where

$$\begin{aligned} P_3 &= \sum_{n=2}^{\infty} (2n-1-\gamma) \frac{|a_1 b_1|}{c_1} \left| \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-1}} A_n \right| \\ &\qquad + \sum_{n=1}^{\infty} (2n+1+\gamma) \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n \right| \\ &\leq (1-\gamma) \left[\frac{|a_1 b_1|}{a_1 b_1} \sum_{n=2}^{\infty} D_{n-1} + \sum_{n=1}^{\infty} E_{n-1} \right] \\ &\leq (1-\gamma) \left[-F(a_1, b_1; c_1; 1) + 1 + F(|a_2|, |b_2|; c_2; 1) \right] \\ &\leq 1-\gamma, \end{aligned}$$

by the given condition. □

In next theorem, we present conditions on the parameters $a_1, a_2, b_1, b_2, c_1, c_2$ and obtain a characterization for operator Ω which maps $TG_H(\gamma)$ onto itself.

Theorem 2.6. *Let $a_j, b_j > 0$, $c_j > a_j + b_j$ for $(j = 1, 2)$ and $\gamma (0 \leq \gamma < 1)$. Then $\Omega(TG_H(\gamma)) \subset TG_H(\gamma)$, if and only if*

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 2.$$

Proof. Let $f = h + \overline{g} \in TG_H(\gamma)$ with h and g in (1.4). In view of Remark 1, we only need to prove that $\Omega(f)$ given by (1.3) is in $TG_H(\gamma)$, if and only if $P_2 \leq 1-\gamma$,
where

$$P_2 = \sum_{n=2}^{\infty} (2n-1-\gamma) \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n \right| + \sum_{n=1}^{\infty} (2n+1+\gamma) \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n \right|.$$

Using the coefficient estimates stated in Remark 1, we obtain

$$\begin{aligned} P_2 &\leq (1-\gamma) \left[\sum_{n=2}^{\infty} D_{n-1} + \sum_{n=1}^{\infty} E_{n-1} \right] \\ &\leq (1-\gamma) \left[F(a_1, b_1; c_1; 1) - 1 + F(a_2, b_2; c_2; 1) \right] \\ &\leq (1-\gamma), \end{aligned}$$

by the given condition. □

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