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SOME REMARKS ON A FIXED POINT THEOREM

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this note we discuss a special case of the Leray-Schauder alternative for condensing multifunctions.

1. Introduction

A basic metric fixed point theorem, *the Banach contraction principle*, and a basic topological fixed point theorem, *the Schauder fixed point theorem* have been used extensively in the literature of the theory of nonlinear differential and integral equations.

A nice combination of the above two fixed point theorems of Banach and Schauder yields the famous *Krasnoselskii's fixed point theorem* for the sum of two operators in Banach spaces. It is known that this result has a number of interesting applications.

In the last century a Leray-Schauder alternative for multivalued condensing maps was developed; see [6] and the references therein. In particular a discussion of a contraction and a compact map in the multivalued situation is discussed in [6]. Recently a modification of this result was presented by Burton and Kirk [1] in the single valued situation and by Dhage [3] in the multivalued situation. However we would like to point out that the second main result (Theorem 3.3) in [3] is not correct.

In particular [3, Theorem 3.3] is based on a result in [5] which is not correct; see [4, pp 113] for the correct formulation. The correction of Theorem 3.3 in [3] is the main motivation of this paper since recently [2] some authors have used this fixed point theorem in applications so unfortunately mistakes are being made.

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2. A fixed Point Theorem

Let X be a metric space and $\mathcal{P}_b(X)$ be the bounded subsets of X. The Kuratowskii measure of noncompactenes is the map $\alpha : \mathcal{P}_b(X) \to [0, \infty)$ defined by

$$\alpha(A) = \inf\{\epsilon > 0 : A \subseteq \cup_{i=1}^n A_i \text{ and } \operatorname{diam}(A_i) \le \epsilon \text{ for } i = 1, \dots, n\}, \quad A \in \mathcal{P}_b(X),$$

and the ball measure of noncompactenes is the map $\chi : \mathcal{P}_b(X) \to [0, \infty)$ defined by

$$\chi(A) = \inf\{\epsilon > 0 : A \subseteq \cup_{i=1}^n B(x_i, \epsilon) \text{ and } \{x_1, \dots, x_n\} \subseteq E\}, \quad A \in \mathcal{P}_b(X);$$

here $B(x_i, \epsilon)$ is the ball with center x_1 and radius ϵ. Let S be a nonempty subset of X, and for each $x \in X$ let $d(x, S) = \inf_{y \in S} d(x, y)$ and $B(S, r) = \{x \in X; d(x, S) < r\}, r > 0$.

Let $F : S \to \mathcal{P}(X)$. Then:

- (i) F is called k-set contractive ($k \ge 0$) w.r.t. α (respectively w.r.t. χ) is $F(S)$ is bounded and $\alpha(F(Y)) \le k\alpha(Y)$ (respectively $\chi(F(Y)) \le k\chi(Y)$) for all bounded sets Y of S.
- (ii) condensing w.r.t. α (respectively w.r.t. χ) if $\alpha(F(Y)) < \alpha(Y)$ (respectively $\chi(F(Y)) < \chi(Y)$) for all bounded sets Y of S with $\alpha(Y) \ne 0$ (respectively $\chi(Y) \ne 0$).

Theorem 2.1. *[Nonlinear alternative for multivalued condensing maps] Let C be a closed convex subset of a Banach space X and U a relatively open subset of C with $0 \in U$. In addition, assume $F : \overline{U} \to \mathcal{P}_{cv,cp}(C)$ (here $\mathcal{P}_{cv,cp}(C)$ denotes the family of all nonempty, convex compact subsets of C) is upper semicontinuous (u.s.c.) condensing (w.r.t. α or χ) multivalued map with $F(\overline{U})$ bounded. Then, either*

- (A1) *F has a fixed point in $\overline{U}$, or*
- (A2) *there exists $y \in \partial U$ and $\lambda \in (0, 1)$ such that $y \in \lambda F(y)$.*

Example 2.2. *Let $X = (X, d)$ be a Fréchet space, $T : X \to \mathcal{P}_{cv,cp}(X)$ and assume there is a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(z) < z$ for $z > 0$ and*

$$H(Tx, Ty) \le \phi(d(x, y)) \text{ for all } x, y \in X;$$

here H denotes the Hausdorff distance. Then T is condensing w.r.t. χ on bounded subsets of X.

The result follows once we show $\chi(T(\Omega)) \le \phi(\chi(\Omega))$ for any bounded subset Ω of X. The argument we give is a modification of the one in [6, pp 19]. Let $\Omega \subseteq X$ be bounded with $\chi(\Omega) = r > 0$. Let $\epsilon > 0$ be given. Choose $\{x_1, \dots, x_n\} \subseteq X$ with $\Omega \subseteq \cup_{i=1}^n B(x_i, r + \epsilon)$. For each $i \in \{1, \dots, n\}$ choose $\{y_j^i\}_{j=1}^{n(i)}$ with $T(x_i) \subseteq \cup_{j=1}^{n(i)} B(y_j^i, \epsilon)$. We claim

$$T(\Omega) \subseteq \cup_{i=1}^n \cup_{j=1}^{n(i)} B(y_j^i, \phi(r + \epsilon) + \epsilon). \tag{2.1}$$

To prove (2.1) let $z \in T(\Omega)$. Then $z \in Tx$ for some $x \in \cup_{i=1}^n B(x_i, r + \epsilon)$. Choose $i \in \{1, \dots, n\}$ with $d(x, x_i) < r + \epsilon$. Now there exists $w \in Tx_i$ with

$$d(z, w) = d(z, Tx_i) \le H(Tx, Tx_i) \le \phi(d(x, x_i)) \le \phi(r + \epsilon).$$

Select $j \in \{1, \dots, n(i)\}$ with $d(w, y_j^i) < \epsilon$. Then

$$d(z, y_j^i) \le d(z, w) + d(w, y_j^i) \le \phi(r + \epsilon) + \epsilon,$$

so $z \in \cup_{i=1}^{n} \cup_{j=1}^{n(i)} B(y_j^i, \phi(r+\epsilon)+\epsilon)$. As a result (2.1) is true so

$$\chi(T(\Omega)) \le \phi(r+\epsilon)+\epsilon \le \phi(\chi(\Omega)+\epsilon)+\epsilon.$$

Since ϵ arbitrary, $\chi(T(\Omega)) \le \phi(\chi(\Omega))$.

Corollary 2.3. *Let C be a closed convex subset of a Banach space X and U a relatively open subset of C with $0 \in U$. Let $F_1 : X \to \mathcal{P}_{cv,cp}(C)$ and $F_2 : \overline{U} \to \mathcal{P}_{cv,cp}(C)$. In addition assume:*

(i) *there is a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(z) < z$ for $z > 0$ and*

$$H(F_1x, F_1y) \le \phi(d(x,y)) \text{ for all } x, y \in X,$$

(ii) *F_2 is u.s.c. and compact.*

Then if $F = F_1 + F_2$, either

(A1) *F has a fixed point in $\overline{U}$, or*

(A2) *there exists $y \in \partial U$ and $\lambda \in (0,1)$ such that $y \in \lambda F(y)$.*

Proof. The result follows immediately from Theorem 2.1 and Example 2.2 (also recall a multivalued map $T : X \to \mathcal{P}_{cp}(Y)$ (here $\mathcal{P}_{cp}(Y)$ denotes the family of nonempty compact subsets of Y) is continuous if and only if it is continuous in the Hausdorff metric). □

Corollary 2.3 corrects (and extends) Theorem 3.3 in [3]. Note in Theorem 3.3 in [3] the crucial condition on the contraction is $F_1 : X \to \mathcal{P}_{cv,cl}(C)$ (here $\mathcal{P}_{cv,cl}(C)$ denotes the family of nonempty closed convex subsets of C). However to use the theory in the literature [4, pp 113] we need $F_1 : X \to \mathcal{P}_{cv,cp}(X)$.

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