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SOME EXTRAGRADIENT METHODS FOR NONCONVEX QUASI VARIATIONAL INEQUALITIES

(COMMUNICATED BY M.S.MOSLEHIAN)

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Abstract. In this paper, we introduce and consider a new class of variational inequalities involving two operators, which is called the general nonconvex quasi variational inequality. Several special cases are discussed. We use the projection technique to establish the equivalence between the general nonconvex quasi variational inequalities and the fixed point problems. This alternative equivalent formulation is used to study the existence of a solution of the general nonconvex quasi variational inequalities. Using these equivalent formulations, we suggest and analyze a wide class of new extragradient methods for solving the general nonconvex quasi variational inequalities. Convergence criteria of these new iterative methods is considered under some suitable conditions.

1. Introduction

Quasi variational inequalities, which were introduced in early 1970's, are being used to model various problems arising in different branches of pure and applied sciences in a unified and general manner. Quasi variational inequalities continuously benefit from cross-fertilization between functional analysis, convex analysis, numerical analysis and physics. This interaction between these fields have played a significant and important role in developing several numerical techniques for solving quasi variational inequalities and related optimization problems, see [1-25] and the references therein.

It is worth mentioning that almost all the results regarding the existence and iterative schemes for quasi variational inequalities have been investigated and considered, if the underlying set is a convex set. This is because all the techniques are based on the properties of the projection operator over convex sets, which may

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not hold for the nonconvex sets. Motivated and inspired by the ongoing research in this area, we introduce and consider a new class of quasi-variational inequalities, which is called the general nonconvex quasi-variational inequality on the uniformly prox-regular sets, It is well-known [3,24] that the prox-regular sets are nonconvex sets and include the convex sets as a special case, see [3,24]. Using the idea and technique of Noor [16,18-21], we show that the projection technique can be extended for the general nonconvex quasi-variational inequalities. We establish the equivalence between the general nonconvex quasi-variational inequalities and fixed point problems using the projection technique. This equivalent alternative formulation is used to discuss the existence of a solution of the general nonconvex quasi variational inequalities, which is Theorem 3.1. We use this alternative equivalent formulation to suggest and analyze some implicit type iterative methods for solving the general nonconvex quasi variational inequalities. In order to implement these new implicit methods, we use the predictor-corrector technique to suggest some two-step methods for solving the general nonconvex variational inequalities, which are Algorithm 3.4 and Algorithm 3.6. We consider the convergence (Theorem 3.2) of the new iterative method (Algorithm 3.1) under some suitable conditions. We have also suggested three-step iterative methods for solving the general nonconvex quasi-variational inequalities. Some special cases are also discussed. We would like to point out that our method of proofs is very simple as compared with other techniques. These results can be viewed as a significant refinement and improvement of the previously known results for different classes of variational inequalities and optimization problems. We hope that the results proved in this paper may lead to novel and innovative applications of the general nonconvex quasi variational inequalities in various branches of pure and applied sciences.

2. Basic Concepts

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [3,24].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$,
if and only if, there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K.$$

Poliquin et al. [24] and Clarke et al [3] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.2. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N^P_{K_r}(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u\rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [3,24]. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N^P_{K_r}(u)$ is closed as a set-valued mapping.

For given nonlinear operators T, h and a point-to-set mapping $K_r : u \longrightarrow K_r(u)$, which associates a closed uniformly prox-regular set $K_r(u)$ of H with any element of H, we consider the problem of finding $u \in H : h(u) \in K_r(u)$ such that

$$\langle \rho Tu + h(u) - u, v - h(u)\rangle \geq 0, \quad \forall v \in K_r(u), \tag{2.1}$$

which is called the *general nonconvex quasi variational inequality.*

If $h \equiv I$, the identity operator and $K_r(u) \equiv K_r$, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$\langle \rho Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.2}$$

which is known as the nonconvex variational inequality, studied and introduced by Noor [18].

We now consider the following simple examples to give an idea of the importance of the nonconvex sets. These examples are due to Noor [21]

Example 2.1[21]. Let $u = (x, y)$ and $v = (t, z)$ belong to the real Euclidean plane and consider $Tu = (2x, 2(y - 1))$. Let $K = \{t^2 + (z - 2)^2 \geq 4, \quad -2 \leq t \leq 2, \quad z \geq -2\}$ be a subset of the Euclidean plane. Then one can easily show that the set K is a prox-regular set K_r. It is clear that nonconvex variational inequality (2.2) has no solution.

Example 2.2 [21]. Let $u = (x, y) \in R^2, \quad v = (t, z) \in R^2$ and let $Tu = (-x, 1 - y)$. Let the set K be the union of 2 disjoint squares, say A and B having respectively, the vertices in the points $(0, 1), (2, 1), (2, 3), (0, 3)$ and in the points $(4, 1), (5, 2), (4, 3), (3, 2)$.
The fact that K can be written in the form:

$$\big\{(t, z) \in R^2 : \max\{|t - 1|, |z - 2|\} \leq 1\} \cup \{|t - 4| + |z - 2| \leq 1\}\big\}$$

shows that it is a prox-regular set in R^2 and the nonconvex variational inequality (2.1) has a solution on the square B. We note that the operator T is the gradient of a strictly concave function. This shows that the square A is redundant.

We note that, if $K_r(u) \equiv K(u)$, the convex set in H, then problem (2.1) is equivalent to finding $u \in H : h(u) \in K(u)$ such that

$$\langle \rho Tu + h(u) - u, v - h(u)\rangle \geq 0, \qquad \forall v \in K(u). \tag{2.3}$$

Inequality of type (2.3) is called the *general quasi variational inequality.*

If $h(u) = u$, then problem (2.1) is equivalent to finding $u \in H : h(u) \in K_r(u)$ such that

$$\langle T(h(u)), v - h(u)\rangle \geq 0, \quad \forall v \in K_r(u), \tag{2.4}$$

which is also called the general nonconvex quasi variational inequality and appears to be a new one.

If $K_r(u) \equiv K(u)$, the convex-valued set in H, then problem (2.4) is equivalent to finding $u \in H : h(u) \in K(u)$ such that

$$\langle T(h(u)), v - h(u)\rangle \geq 0, \forall v \in K(u), \tag{2.5}$$

which was introduced and studied by Noor [9] in 1988. It has been shown that the minimum of a differentiable nonconvex function can be characterized by the general variational inequality (2.5).

If $h \equiv I$, the identity operator, then problem (2.5) is equivalent to finding $u \in K(u)$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad v \in K(u), \tag{2.6}$$

which is known as the classical quasi variational inequality, introduced and studied by Bensoussan and Lions [1].

In brief, for suitable choice of the operators and the spaces, one can obtain several new and previous known classes of (quasi) variational inequalities and related optimization problems. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1-25] and the references therein.

We note that for the nonconvex-valued (uniformly prox-regular) set $K_r(u)$, problem (2.1) is equivalent to finding $u \in K_r(u)$ such that

$$0 \in \rho Tu + h(u) - u + \rho N^P_{K_r(u)}(h(u)), \tag{2.7}$$

where $N^P_{K_r(u)}(u)$ denotes the normal cone of $K_r(u)$ at u in the sense of nonconvex analysis. Problem (2.8) is called the general nonconvex quasi variational inclusion problem associated with nonconvex variational inequality (2.1). This implies that the general nonconvex quasi variational inequality (2.1) is equivalent to finding a zero of the sum of two monotone operators (2.7). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the general nonconvex variational inequality (2.1).

We now recall the well known proposition which summarizes some important properties of the uniform prox-regular sets.

Lemma 2.2. Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d_K(u) < r\}$. If K_r is uniformly prox-regular, then

i. $\forall u \in K_r, P_{K_r}(u) \neq \emptyset$.

ii. $\forall r' \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $K_{r'}$.

Definition 2.3. An operator $T : H \to H$ is said to be:

(i) *strongly monotone*, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2, \quad \forall u, v \in H.$$

(ii) *Lipschitz continuous*, if and only if, there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta ||u - v||, \quad \forall u, v \in H.$$

3. Main Results

In this section, we establish the equivalence between the general nonconvex quasi variational inequality (2.1) and the fixed point problem using the projection operator technique. This alternative formulation is used to discuss the existence of a solution of the problem (2.1) and to suggest some new iterative methods for solving the general nonconvex quasi variational inequality (2.1).

Lemma 3.1. $u \in H : h(u) \in K_r(u)$ is a solution of the general nonconvex quasi variational inequality (2.1), if and only if, $u \in H : h(u) \in K_r(u)$ satisfies the relation

$$h(u) = P_{K_r(u)}[u - \rho Tu], \tag{3.1}$$

where $P_{K_r(u)}$ is the projection of H onto the uniformly prox-regular set $K_r(u)$.

Proof. Let $u \in H : h(u) \in K_r(u)$ be a solution of (2.1). Then, for a constant $\rho > 0$,

$$\begin{aligned} 0 &\in h(u) + \rho N^P_{K_r(u)}(h(u)) - (u - \rho Tu) = (I + \rho N^P_{K_r(u)})(h(u)) - (u - \rho Tu) \\ &\Longleftrightarrow \\ h(u) &= (I + \rho N^P_{K_r(u)})^{-1}[u - \rho Tu] = P_{K_r(u)}[u - \rho Tu], \end{aligned}$$

where we have used the well-known fact that $P_{K_r(u)} \equiv (I + N^P_{K_r(u)})^{-1}$. □

We would like to point out that the implicit projection operator $P_{K_r(u)}$ is not nonexpansive. We shall assume that the implicit projection operator $P_{K_r(u)}$ satisfies the Lipschitz type continuity, which plays an important and fundamental role in the existence theory and in developing numerical methods for solving nonconvex quasi-variational inequalities.

Assumption 3.1 [18]. For all $u, v, w \in H$, the implicit projection operator $P_{K_r(u)}$ satisfies the condition

$$\|P_{K_r(u)}w - P_{K_r(v)}w\|| \leq \nu \|u - v||,$$

where $\nu > 0$ is a positive constant.

Lemma 3.1 implies that the general nonconvex quasi-variational inequality (2.1) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical point of views.

We rewrite the the relation (3.1) in the following form

$$F(u) = u - h(u) + P_{K_r(u)}[u - \rho Tu], \tag{3.2}$$

which is used to study the existence of a solution of the general nonconvex quasi-variational inequality (2.1).

We now study those conditions under which the general nonconvex quasi-variational inequality (2.1) has a solution and this is the main motivation of our next result.

Theorem 3.1. Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let T, h be strongly monotone with constants $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants $\beta > 0, \delta > 0$, respectively. If Assumption 3.1 holds and there exists a constant $\rho > 0$ such that

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\delta^2\alpha^2 - \beta^2(\delta^2 - (1-k)^2))}}{\delta\beta^2}, \quad \delta\alpha > \beta\sqrt{\delta^2 - (1-k)^2}, \quad (3.3)$$

$$k = \nu + \sqrt{1 - 2\sigma + \delta^2} < 1, \quad (3.4)$$

then there exists a solution of the problem (2.1).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2), has a fixed point. For all $u \neq v \in K_r$, we have

$$\begin{aligned} ||F(u) - F(v)|| &= \|u - v - (h(u) - h(v))\| + \|P_{K_r(u)}[u - \rho Tu] - P_{K_r(v)}[v - \rho Tv]|| \\ &\leq \|u - v - (h(u) - h(v))\| + \|P_{K_r(u)}[u - \rho Tu] - P_{K_r(v)}[u - \rho Tu]|| \\ &\quad + \|P_{K_r(v)}[u - \rho Tu] - P_{K_r(v)}[v - \rho Tv]\| \\ &\leq \|u - v - (h(u) - h(v)) + \delta\|u - v - \rho(Tu - Tv)\| + \nu\|u - v\|, \quad (3.5) \end{aligned}$$

where we have used the fact that the operator P_{K_r} is a Lipschitz continuous operator with constant δ and the Assumption 3.1.

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} ||u - v - \rho(Tu - Tv)||^2 &\leq ||u - v||^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2||Tu - Tv||^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)||u - v||^2. \quad (3.6) \end{aligned}$$

In a similar way, we have

$$\|u - v - (h(u) - h(v))\| \leq \sqrt{1 - 2\sigma + \delta^2}\|u - v|, \quad (3.7)$$

where $\sigma > 0$ is the strongly monotonicity constant and $\delta > 0$ is the Lipschitz continuity constant of the operator h respectively.

From (3.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned} ||F(u) - F(v)|| &\leq \left\{k + \delta\sqrt{1 - 2\alpha\rho + \beta^2\rho^2)}\right\}||u - v|| \\ &= \theta||u - v||, \end{aligned}$$

where

$$\theta = \delta\sqrt{1 - 2\alpha\rho + \beta^2\rho^2} + k. \quad (3.8)$$

From (3.3), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (3.2), has a fixed point, which is the unique solution of (2.1). □

This fixed point formulation (3.1) is used to suggest the following iterative method for solving the general nonconvex quasi-variational inequality (2.1).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = (1 - \alpha_n)u_n - \alpha_n\{u_n - h(u_n) + P_{K_r(u_n)}[u_n - \rho Tu_n]\}, \quad n = 0, 1, 2, \ldots, (3.9)$$

where $\alpha_n \in [0, 1], \forall n \geq 0$ is a constant. Algorithm 3.1 is also called the Mann iteration process. For $\alpha_n = 1$, Algorithm 3.1 collapse to:

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$h(u_{n+1}) = P_{K_r(u_n)}[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

We again use the fixed-point formulation (3.1) to suggest and analyze an iterative method for solving the general nonconvex quasi-variational inequalities (2.1) as:

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$h(u_{n+1}) = P_{K_r(u_n)}[u_{n+1} - \rho T u_{n+1}], \quad n = 0, 1, 2, \dots$$

Algorithm 3.3 is an implicit type iterative method, which is difficult to implement. To implement Algorithm 3.3, we use the predictor-corrector technique. Here we use the Algorithm 3.1 as a predictor and Algorithm 3.3 as a corrector. Consequently, we have the following iterative method

Algorithm 3.4. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} h(y_n) &= P_{K_r(u_n)}[u_n - \rho T u_n] \\ h(u_{n+1}) &= P_{K_r(y_n)}[y_n - \rho T y_n], \quad n = 0, 1, 2, \dots \end{aligned}$$

which is called the two-step or splitting type iterative method for solving the general nonconvex quasi-variational inequalities (2.1). Algorithm 3.4 is also known as the modified extragradient method. It is worth mentioning that Algorithm 3.4 can be suggested by using the updating the technique of the solution.

Using the fixed point formulation (3.1), one can suggest and analyze the following iterative method for solving the general nonconvex quasi-variational inequality (2.1) as;

Algorithm 3.5. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K_r(u_n)}[u_n - \rho T u_{n+1}], \quad n = 0, 1, 2, \dots,$$

which is an implicit iterative method. To implement Algorithm 3.5, we use the predictor-corrector technique to suggest the following iterative method for solving the general nonconvex quasi-variational inequality (2.1) as:

Algorithm 3.6. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= P_{K_r(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= P_{K_r(y_n)}[u_n - \rho T y_n], \quad n = 0, 1, 2, \dots, \end{aligned}$$

which is called the extragradient method. We would like to mention that Algorithm 3.4 and Algorithm 3.6 are remarkably different from each other.

Algorithm 3.4 can be used suggest and analyze the following two-step iterative method for solving the general nonconvex quasi-variational inequality (2.1).

Algorithm 3.7. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= (1 - \beta_n) u_n + \beta_n \{y_n - h(y_n) + P_{K_r(u_n)}[u_n - \rho T u_n]\} \\ u_{n+1} &= (1 - \alpha_n) u_n + \alpha_n \{u_n - h(u_n) + P_{K_r(y_n)}[y_n - \rho T y_n]\}, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where $\alpha_n, \beta_n \in [0, 1], \quad \forall n \geq 0$.

Clearly for $\alpha_n = \beta_n = 1$, Algorithm 3.7 reduces to Algorithm 3.4. It is worth mentioning that, if $r = \infty$, then the nonconvex set $K_r(u)$ reduces to a convex-valued set $K(u)$. Consequently Algorithms 3.1- 3.7 collapse to the algorithms for solving the classical quasi variational inequalities (2.6). We would like to point that Algorithm 3.4 appears to be a new one for solving the quasi variational inequalities.

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result. In a similar way, one can consider the convergence criteria of other Algorithms.

Theorem 3.2. Let $P_{K_r(u)}$ be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let the operators $T, h : H \longrightarrow H$ be strongly monotone with constants $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants with $\beta > 0, \delta > 0$, respectively. If Assumption 3.1 and condition (3.3) hold with, $\alpha_n, \in [0,1], \quad \forall n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm 3.1 converges to a solution $u \in H$ satisfying the general nonconvex quasi variational inequality (2.1).

Proof. Let $u \in H : h(u) \in K_r(u)$ be a solution of the general nonconvex quasi variational inequality (2.1). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n\{u - h(u) + P_{K_r(u)}[u - \rho Tu]\}, \tag{3.10}$$

where $0 \leq \alpha_n \leq 1$ is a constant.

From (3.3) and (3.5)-(3.10) and using the Lipschitz continuity of the projection $P_{K_r(u)}$ with constant δ and Assumption 3.1, we have

$$\begin{aligned}
||u_{n+1} - u|| &= ||(1-\alpha_n)(u_n - u) + \alpha_n\{P_{K_r(u_n)}[u_n - \rho Tu_n] - P_{K_r(u)}[u - \rho Tu]\}|| \\
&\quad + \alpha_n\|u_n - u - (h(u_n) - h(u))\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\|P_{K_r(u_n)}[u_n - \rho Tu_n] - P_{K_r(u_n)}[u - \rho Tu]\| \\
&\quad + \alpha_n k\|u_n - u\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\delta\|u_n - u + \rho(Tu_n - Tu)\| + \alpha_n k\|u_n - u\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\{k + \delta\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u_n - u\| \\
&= [1 - \alpha_n(1-\theta)]\,\|u_n - u\| \\
&\leq \prod_{i=0}^{n}[1 - \alpha_i(1-\theta)]\,\|u_0 - u\|,
\end{aligned}$$

where,

$$\theta = k + \delta\sqrt{1 - 2\rho\alpha + \beta^2\rho^2} < 1.$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\lim_{n\to\infty}\{\prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\} = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof. □

Using the technique of the updating the solution, one can rewrite the equation (3.1) in the following form:

$$\begin{aligned}
h(y) &= P_{K_r(u)}[u - \rho Tu] \\
h(w) &= P_{K_r(y)}[y - \rho Ty] \\
h(u) &= P_{K_r(w)}[w - \rho Tw],
\end{aligned}$$

which is another fixed point formulation. This fixed-point formulation is used to suggest the following three-step iterative method for solving the general nonconvex quasi variational inequality (2.1).

Algorithm 3.8. For a given $u_n \in H$, find the approximate solution u_{n+1} by the iterative schemes:

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n\{y_n - h(y_n) + P_{K_r(u_n)}[u_n - \rho T u_n]\} \\ w_n &= (1-\beta_n)u_n + \beta_n\{w_n - h(w_n) + P_{K_r(y_n)}[y_n - \rho T y_n]\} \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\{u_n - h(u_n) + P_{K_r(w_n)}[w_n - \rho T w_n]\}, \quad n = 0, 1, \ldots, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ are some constants.

We would like to mention that three-step iterative methods are also known as Noor iteration for solving the variational inequalities and equilibrium problems, see the references. Note that for different and suitable choice of the constants α_n, β_n and γ_n, one can easily show that the Noor iterations include the Mann and Ishikawa iterations as special cases. One can consider the convergence criteria of Algorithm 3.8 using the technique of this paper.

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