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ON LP-SASAKIAN MANIFOLDS

(COMMUNICATED BY UDAY CHAND DE)

ABOLFAZL TALESHIAN, NADER ASGHARI

Abstract. The present paper deals with certain curvature conditions on the projective curvature tensor.

1. Introduction

The notion of a Lorentzian Para Sasakian manifold was introduced by K. Matsumoto [9]. I. Mihai and R. Rosca [11] defined the same notion independently and they obtained several results on this manifold. Also LP-Sasakian manifolds have been studied by K. Matsumoto and I. Mihai [10], and U.C. De [5] and A.A. shaikh [13].
In this paper, we investigate the properties of the LP-Sasakian manifold equipped with projective curvature tensor. we have construct an example of three-dimensional LP-Sasakian manifold.
Next, we study LP-Sasakian manifolds in with $\bar{P}(X,Y).P=0$ and $P(\xi,X).S=0$, where $\bar{P}$ is the Weyl projective curvature tensor. Also, we prove that an LP-Sasakian manifold satisfying $g(P(X,Y)Z,\varphi W)=0$, is an Einstein manifold. Finally, we prove that φ-projectively flat LP-Sasakian manifold is an η-Einstein manifold.

2. Preliminaries

Let M be an n- dimensional Riemanian manifold, φ a $(1,1)$ tensor field, η a 1-form , ξ a contravariant vector field and g a Riemanian metric. Then M is said to admit an almost paracontact Riemanian structure (φ,η,ξ,g) if [9], [11]

$$\varphi^2 = I - \eta\otimes\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

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for all vector fields X, Y. On the other hand, M is said to admit a *Lorentzian* almost *paracontact* structure (φ, η, ξ, g), where φ a $(1,1)$ tensor field, η a 1-form, ξ a *contravariant* vector field and g a *Lorentzian* metric on M, which makes ξ a *timelike* unit vector field such that [5], [11]

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.4}$$

$$\eta(\xi) = -1, \tag{2.5}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.6}$$

$$(a)\nabla_X \xi = \varphi X, (b) g(X,\xi) = \eta(X), \tag{2.7}$$

$$(\nabla_X \varphi)Y = g(X,Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{2.8}$$

for all vector fields X, Y. Where ∇ denotes the operator of covariant differentiation with respect to the *Lorentzian* metric g.
A *Lorentzian* almost para contact manifold is called *Lorentzian* para *sasakian* manifold (briefly, LP-*Sasakian* manifold).
It can be easily seen that in an LP-*Sasakian* manifold, the following relations hold

$$\varphi\xi = 0, \eta(\varphi X) = 0, \tag{2.9}$$

$$rank\varphi = n - 1. \tag{2.10}$$

Again if we put

$$\Omega(X,Y) = g(X, \varphi Y), \tag{2.11}$$

for any vector fields X and Y, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field [5], [9].
Also since the vector field η is closed in an LP-*Sasakian* manifold we have[10], [14]:

$$(\nabla_X \eta)(Y) = \Omega(X,Y), \Omega(X,\xi) = 0, \tag{2.12}$$

for any vector fields X and Y.
Also, an LP-*Sasakian* manifold M is said to be η-Einstein if its *Ricci* tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.13}$$

for any vector fields X, Y where a, b are functions on M.
Further, on such an LP-*Sasakian* manifold the following relations hold [9], [11], [14]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.14}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.15}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.16}$$

$$R(X,\xi)\xi = -X - \eta(X)\xi, \tag{2.17}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.18}$$

$$Q\xi = (n-1)\xi, \tag{2.19}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.20}$$

for any vector fields X,Y,Z, where $R(X,Y)Z$ is the curvature tensor, and S is the *Ricci* tensor.

Definition 2.1. *The projective curvature tensor P on LP-Sasakian manifold M of dimensional n is defined as*

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY], \tag{2.21}$$

for all vector fields X, Y, Z on M. Where Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.
The manifold is said to be projectively flat if P vanishes identically on M.

Definition 2.2. *The Weyl projective curvature tensor $\bar{P}$ of type $(1,3)$ on LP-Sasakian manifold M of dimensional n is defined by*

$$\bar{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{2.22}$$

for all vector fields X, Y, Z on M.

Definition 2.3. *An n-dimensional,$(n > 3)$, LP-Sasakian manifold satisfying the condition*

$$\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0, \tag{2.23}$$

is called φ-projectively flat LP-Sasakian manifold.

3. Main Results

we construct an example of LP-Sasakian manifold.
Example:
We consider 3-dimensional manifold $M = \{(x,y,z) : (x,y,z) \in R^3\}$, where (x,y,z) are standard coordinates in R^3. We choose the vector fields

$$E_1 = -e^x \frac{\partial}{\partial y}, E_2 = e^x(\frac{\partial}{\partial z} - \frac{\partial}{\partial y}), E_3 = \frac{\partial}{\partial x}, \tag{3.1}$$

which are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$g(E_1,E_2) = g(E_1,E_3) = g(E_2,E_3) = 0, \tag{3.2}$$

$$g(E_1,E_1) = g(E_2,E_2) = 1, g(E_3,E_3) = -1. \tag{3.3}$$

Let η be 1-form defined by $\eta(Z) = g(Z,E_3)$ for any vector field Z on M.
We define the $(1,1)$ tensor field φ as $\varphi(E_1) = -E_1$, $\varphi(E_2) = -E_2$, $\varphi(E_3) = 0$. The linearity property of φ and g yields that

$$\eta(E_3) = -1, \varphi^2 U = U + \eta(U)E_3, \tag{3.4}$$

$$g(\varphi U, \varphi W) = g(U,W) + \eta(U)\eta(W), \tag{3.5}$$

for any vector fields Z, U, W on M. Thus for $E_3 = \xi$, (φ, ξ, η, g) defines a Lorentzian paracontact structure on M. Therefore M is a 3-dimensional LP-Sasakian manifold.

Theorem 3.1. *In an n-dimensional LP-Sasakian manifold M if the condition $\bar{P}(X,Y).P = 0$ holds on M, then the equation*
$S(Y,QU) = 2(n-1)S(Y,U) - (n-1)^2 g(Y,U)$ is satisfied on M.

Proof. Since $\bar{P}(X,Y).P = 0$ we have

$$\bar{P}(X,Y).P(Z,U)V = 0, \tag{3.6}$$

this implies that

$$\begin{aligned} 0 &= \bar{P}(X,Y)P(Z,U)V - P(\bar{P}(X,Y)Z,U)V \\ &\quad -P(Z,\bar{P}(X,Y)U)V - P(Z,U)\bar{P}(X,Y)V. \end{aligned} \tag{3.7}$$

Putting $X = \xi$ in (3.7) and using (2.22) we get

$$\begin{aligned} 0 &= g(P(Z,U)V,Y)\xi - g(Y,Z)P(\xi,U)V - g(Y,U)P(Z,\xi)V \\ &\quad -g(Y,V)P(Z,U)\xi + \frac{1}{n-1}[-S(Y,P(Z,U)V)\xi \\ &\quad +S(Y,Z)P(\xi,U)V + S(Y,U)P(Z,\xi)V + S(Y,V)P(Z,U)\xi]. \end{aligned} \tag{3.8}$$

Taking the inner product of the last equation with ξ we obtain

$$\begin{aligned} 0 &= -g(P(Z,U)V,Y) - g(Y,Z)\eta(P(\xi,U)V) \\ &\quad -g(Y,U)\eta(P(Z,\xi)V) - g(Y,V)\eta(P(Z,U)\xi) \\ &\quad +\frac{1}{n-1}[S(Y,P(Z,U)V) + S(Y,Z)\eta(P(\xi,U)V) \\ &\quad +S(Y,U)\eta(P(Z,\xi)V) + S(Y,V)\eta(P(Z,U)\xi)]. \end{aligned} \tag{3.9}$$

With simplify of the above equation we get

$$-g(P(Z,U)V,Y) + \frac{1}{n-1}S(Y,P(Z,U)V) = 0. \tag{3.10}$$

Putting $Z = V = \xi$ in (3.22) we get

$$-g(P(\xi,U)\xi,Y) + \frac{1}{n-1}S(Y,P(\xi,U)\xi) = 0. \tag{3.11}$$

In view of (2.21) we get

$$P(\xi,U)\xi = U - \frac{1}{n-1}QU. \tag{3.12}$$

Using (3.12) in (3.11) we obtain

$$\begin{aligned} 0 &= -g(Y,U) + \frac{1}{n-1}S(Y,U) \\ &\quad +\frac{1}{n-1}S(Y,U) - \frac{1}{(n-1)^2}S(Y,QU). \end{aligned}$$

Finally, with simplify we get

$$S(Y,QU) = 2(n-1)S(Y,U) - (n-1)^2 g(Y,U). \tag{3.13}$$

This the completes the proof of the theorem. □

Theorem 3.2. *In an n-dimensional LP-Sasakian manifold M, if the condition $P(\xi,X).S = 0$ holds on M, then the equation $S(Y,QX) = (n-1)S(Y,X)$ is satisfied on M.*

Proof. If $P(\xi,X).S = 0$ then we have

$$P(\xi,X).S(Y,\xi) = 0, \tag{3.14}$$

this implies that

$$S(P(\xi,X)Y,\xi) + S(Y,P(\xi,X)\xi) = 0. \tag{3.15}$$

In view of (2.21) we get

$$\begin{aligned} 0 &= S(g(X,Y)\xi - \eta(Y)X - \frac{1}{n-1}[(n-1)g(X,Y)\xi - \eta(Y)QX],\xi) \\ &\quad + S(Y, \eta(X)\xi + X - \frac{1}{n-1}[(n-1)\eta(X)\xi + QX]), \end{aligned}$$

with simplify of the above equation we obtain

$$-(n-1)\eta(X)\eta(Y) + \frac{1}{n-1}\eta(Y)S(QX,\xi) + S(Y,X) - \frac{1}{n-1}S(Y,QX) = 0, \quad (3.16)$$

finally we get

$$S(Y,QX) = (n-1)S(Y,X). \quad (3.17)$$

□

Theorem 3.3. *In an n-dimensional LP-Sasakian manifold M if the condition $g(P(X,Y)Z,\varphi W) = 0$ holds on M, then M is an Einstein manifold.*

Proof. If $g(P(X,Y)Z,\varphi W) = 0$ then we get from (2.21)

$$g(R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY], \varphi W) = 0, \quad (3.18)$$

therefore we have

$$g(R(X,Y)Z,\varphi W) - \frac{1}{n-1}[g(Y,Z)S(X,\varphi W) - g(X,Z)S(Y,\varphi W)] = 0. \quad (3.19)$$

Putting $Y = Z = \xi$ we obtain

$$g(-X - \eta(X)\xi, \varphi W) - \frac{1}{n-1}[-S(X,\varphi W) - (n-1)\eta(X)\eta(\varphi W)] = 0. \quad (3.20)$$

With simplify of the above equation we get

$$-g(X,\varphi W) + \frac{1}{n-1}S(X,\varphi W) = 0. \quad (3.21)$$

Taking $X = \varphi X$ we have

$$S(\varphi X,\varphi W) = (n-1)g(\varphi X,\varphi W). \quad (3.22)$$

Using (2.6) and (2.20) we get

$$S(X,W) + (n-1)\eta(X)\eta(W) = (n-1)g(X,W) + (n-1)\eta(X)\eta(W), \quad (3.23)$$

finally we obtain

$$S(X,W) = (n-1)g(X,W). \quad (3.24)$$

The above equation implies that manifold is an Einstein manifold. □

Theorem 3.4. *Let M be an n-dimensional, $(n > 3)$, φ-projectively flat LP-Sasakian manifold then M is an η-Einstein manifold.*

Proof. If M is φ-projectively flat LP-Sasakian manifold then we get from (2.23) that

$$\varphi^2 P(\varphi X,\varphi Y)\varphi Z = 0, \quad (3.25)$$

this implies that

$$g(P(\varphi X,\varphi Y)\varphi Z,\varphi W) = 0, \quad (3.26)$$

for any vector fields X, Y, Z, W on M. Using (2.21) we obtain

$$g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{n-1}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)]. \tag{3.27}$$

Let $\{e_1, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\varphi e_1, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (3.27) and sum up with respect to i, then

$$\sum_{i=1}^{n-1} g(R(\varphi e_i \varphi Y)\varphi Z, \varphi e_i) = \frac{1}{n-1}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]. \tag{3.28}$$

It can be easily verify that

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z) \tag{3.29}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r - (n-1) \tag{3.30}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z). \tag{3.31}$$

So by virtue of (3.29) – (3.31) the equation (3.28) can be written as

$$S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z) = \frac{1}{n-1}[(r-(n-1))g(\varphi Y, \varphi Z) - S(\varphi Y, \varphi Z)], \tag{3.32}$$

this implies that

$$S(\varphi Y, \varphi Z) = \frac{r-2(n-1)}{n} g(\varphi Y, \varphi Z). \tag{3.33}$$

In view of (2.6) and (2.20) we get

$$S(Y, Z) + (n-1)\eta(Y)\eta(Z) = \frac{r-2(n-1)}{n}[g(Y, Z) + \eta(Y)\eta(Z)]. \tag{3.34}$$

Finally, we obtain

$$S(Y, Z) = \frac{r-2(n-1)}{n} g(Y, Z) + [\frac{r-2(n-1)}{n} - (n-1)]\eta(Y)\eta(Z). \tag{3.35}$$

Therefore M is an η -Einstein manifold. The proof is complete. □

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A. Taleshian

Department of Mathematics, Faculty of Sciences, Mazandaran University.P.O.Box 47416-1467,Babolsar, Iran.

E-mail address: taleshian@umz.ac.ir

N. Asghari

Department of Mathematics, Faculty of Sciences, Mazandaran University.P.O.Box 47416-1467,Babolsar, Iran.

E-mail address: nasgharigm2009@yahoo.com