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SUBORDINATION RESULTS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION WITH COMPLEX ORDER

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Abstract. In this paper, we drive several interesting subordination results of certain classes of analytic functions defined by convolution with complex order.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We also denote by K the class of functions $f(z) \in A$ which are convex in U.

For functions f given by (1.1) and $g \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \qquad (c_n \geq 0), \tag{1.2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (g * f)(z).$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [5] and [14]):

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$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

(Subordinating Factor Sequence) [21]. A sequence $\{b_n\}_{n=1}^{\infty}$ *of complex numbers is said to be a subordinating factor sequence if, whenever* f *of the form (1.1) is analytic, univalent and convex in* U*, we have the subordination given by*

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad (z \in U;\ a_1 = 1). \tag{1.3}$$

For $\lambda \geq 0, 0 \leq \alpha < 1, b \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and for all $z \in U$, let $S(f, g; \lambda, \alpha, b)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and $g(z)$ of the form (1.2) and satisfying the analytic criterion:

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{(f*g)(z)}{z} + \lambda\,(f*g)'(z) - 1\right]\right\} > \alpha, \tag{1.4}$$

and for $\lambda \geq 0, \beta > 1, b \in \mathbb{C}^*$ and for all $z \in U$, let $M(f, g; \lambda, \beta, b)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and $g(z)$ of the form (1.2) and satisfying the analytic criterion:

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{(f*g)(z)}{z} + \lambda\,(f*g)'(z) - 1\right]\right\} < \beta. \tag{1.5}$$

We note that for suitable choices of g, λ, α and β, we obtain the following subclasses.

(1) If $g(z) = z + \sum\limits_{n=2}^{\infty} \Psi_n(\alpha_1) z^n$ *(or* $c_n = \Psi_n(\alpha_1)$*)*, where

$$\Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1} \ldots\ldots\ldots\ldots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \ldots\ldots (\beta_s)_{n-1}\,(n-1)!} \tag{1.6}$$

$(\alpha_i > 0,\ i = 1, \ldots.., q; \beta_j > 0,\ j = 1, \ldots., s;\ q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$

$\mathbb{N} = \{1, 2, ...\})$, then the class $S(f, z + \sum\limits_{n=2}^{\infty} \Psi_n(\alpha_1) z^n; \lambda, \alpha, b)$ reduces to the class $S_{q,s}([\alpha_1]; \lambda, \alpha, b)$

$$= \left\{ f \in A : \operatorname{Re}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{H_{q,s}(\alpha_1)f(z)}{z} + \lambda\,(H_{q,s}(\alpha_1)f(z))' - 1\right]\right\} > \alpha, \right.$$
$$\left. \lambda \geq 0,\ 0 \leq \alpha < 1, b \in \mathbb{C}^*, z \in U \right\},$$

and the class $M(f, z + \sum\limits_{n=2}^{\infty} \Psi_n(\alpha_1) z^n; \lambda, \beta, b)$ reduces to the class $M_{q,s}([\alpha_1]; \lambda, \beta, b)$

$$= \left\{ f \in A : \operatorname{Re}\left\{1 + \frac{1}{b}\left[(1-\lambda)\frac{H_{q,s}(\alpha_1)f(z)}{z} + \lambda\,(H_{q,s}(\alpha_1)f(z))' - 1\right]\right\} < \beta, \right.$$
$$\left. \lambda \geq 0,\ \beta > 1, b \in \mathbb{C}^*, z \in U \right\},$$

where $H_{q,s}(\alpha_1)$ is the Dziok-Srivastava operator (see [10] and [11]) which contains well known operators such as Carlson-Shaffer linear operator (see [6]), the Bernardi-Libera-Livingston operator (see [4], [12] and [13]), Srivastava - Owa fractional derivative operator (see [16]), the Choi-Saigo-Srivastava operator (see [9]), the Cho-Kwon-Srivastava operator (see [8]), the Ruscheweyh derivative operator (see [17]) and the Noor integral operator (see [15]);

(2) If $g(z)=z+\sum\limits_{n=2}^{\infty}\left(\frac{1+\gamma(n-1)+l}{1+l}\right)^{m}z^{n}$ $\left(or\ c_n=\left(\frac{1+\gamma(n-1)+l}{1+l}\right)^{m}, \gamma\geq 0,\ l\geq 0,\ m\in\mathbb{N}_0\right)$, then the class $S(f,z+\sum\limits_{n=2}^{\infty}\left(\frac{1+\gamma(n-1)+l}{1+l}\right)^{m}z^{n};\lambda,\alpha,b)$ reduces to the class $S(\gamma,l,m;\lambda,\alpha,b)$

$$=\left\{f\in A:\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{I^{m}(\gamma,l)f(z)}{z}+\lambda\left(I^{m}(\gamma,l)f(z)\right)^{'}-1\right]\right\}>\alpha,\right.$$
$$\left.\lambda\geq 0,\ \ 0\leq\alpha<1,\gamma\geq 0,l\geq 0,m\in\mathbb{N}_0,b\in\mathbb{C}^{*},z\in U\right\},$$

the class $S(\gamma,l,m;\lambda,0,b)$ reduces to the class $G^{m}(\gamma,l;\lambda,b)$

$$=\left\{f\in A:\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{(I^{m}(\gamma,l)f(z))}{z}+\lambda\left(I^{m}(\gamma,l)f(z)\right)^{'}-1\right]\right\}>0,\right.$$
$$\left.\lambda\geq 0,\ \ ,\gamma\geq 0,l\geq 0,m\in\mathbb{N}_0,b\in\mathbb{C}^{*},z\in U\right\}\ \text{(see [2])},$$

and the class $M(f,z+\sum\limits_{n=2}^{\infty}\left(\frac{1+\gamma(n-1)+l}{1+l}\right)^{m}z^{n};\lambda,\beta,b)$ reduces to the class $M(\gamma,l,m;\lambda,\beta,b)$

$$=\left\{f\in A:\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{I^{m}(\gamma,l)f(z)}{z}+\lambda\left(I^{m}(\gamma,l)f(z)\right)^{'}-1\right]\right\}<\beta,\right.$$
$$\left.\lambda\geq 0,\ \ \beta>1,\gamma\geq 0,l\geq 0,m\in\mathbb{N}_0,b\in\mathbb{C}^{*},z\in U\right\},$$

where $I^{m}(\gamma,l)f(z)$ is the extended multiplier transformation (see [7]);

(3) If $g(z)=z+\sum\limits_{n=2}^{\infty}n^{k}z^{n}$ $\left(or\ c_n=n^{k},\ k\in\mathbb{N}_0\right)$, then the class $S(f,z+\sum\limits_{n=2}^{\infty}n^{k}z^{n};\lambda,\beta,b)$ reduces to the class $S\Im(k;\lambda,\alpha,b)$

$$=\left\{f\in A:\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{D^{k}f(z)}{z}+\lambda\left(D^{k}f(z)\right)^{'}-1\right]\right\}>\alpha,\ \lambda\geq 0,\right.$$
$$\left.0\leq\alpha<1,k\in\mathbb{N}_0,b\in\mathbb{C}^{*},z\in U\right\},$$

the class $S\Im(k;\lambda,0)=G_k(\lambda,b)$

$$=\left\{f\in A:\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{D^{k}f(z)}{z}+\lambda\left(D^{k}f(z)\right)^{'}-1\right]\right\}>0,\ \lambda\geq 0,\right.$$
$$\left.k\in\mathbb{N}_0,b\in\mathbb{C}^{*},z\in U\right\}\ \text{(see [1])},$$

and the class $M(f,z+\sum\limits_{n=2}^{\infty}n^{k}z^{n};\lambda,\beta,b)$ reduces to the class $M\Im(k;\lambda,\beta,b)$

$$=\left\{f\in A:\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\lambda)\frac{D^{k}f(z)}{z}+\lambda\left(D^{k}f(z)\right)^{'}-1\right]\right\}<\beta,\right.$$
$$\left.\lambda\geq 0,\beta>1,k\in\mathbb{N}_0,b\in\mathbb{C}^{*},z\in U\right\},$$

where D^{k} is the Sălăgean differential operator (see [18]);

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\lambda\geq 0,0\leq\alpha<1,\ \beta>1,\ n\geq 2,\ z\in U\ ,b\in\mathbb{C}^{*}$ *and* $g(z)$ is defined by (1.2).
To prove our main results we need the following lemmas.

Lemma 2.1. *[20]. The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \ , \quad (z \in U). \tag{2.1}$$

Lemma 2.2. *Let the function f defined by (1.1) satisfy the following condition:*

$$\sum_{n=2}^{\infty} [1+\lambda(n-1)]\, c_n \left|a_n\right| \leq (1-\alpha)\left|b\right|. \tag{2.2}$$

Then $f \in S(f, g; \lambda, \alpha, b)$.

Proof. Assume that the inequality (2.2) holds true. Then we find that

$$\left|(1-\lambda)\frac{(f*g)(z)}{z} + \lambda (f*g)^{'}(z) - 1\right|$$
$$-\left|(1-\lambda)\frac{(f*g)(z)}{z} + \lambda (f*g)^{'}(z) + 2(1-\alpha)b - 1\right|$$

$$= \left|\sum_{n=2}^{\infty} [1+\lambda(n-1)]\, c_n a_n z^{n-1}\right| - \left|2(1-\alpha)b + \sum_{n=2}^{\infty} [1+\lambda(n-1)]\, c_n a_n z^{n-1}\right|$$
$$\leq \sum_{n=2}^{\infty} [1+\lambda(n-1)]\, c_n \left|a_n\right| \left|z^{n-1}\right| - \left\{2(1-\alpha)\left|b\right| - \sum_{n=2}^{\infty} [1+\lambda(n-1)]\, c_n \left|a_n\right| \left|z^{n-1}\right|\right\}$$
$$\leq \sum_{n=2}^{\infty} [1+\lambda(n-1)]\, c_n \left|a_n\right| \leq (1-\alpha)\left|b\right|. \tag{2.3}$$

This completes the proof of Lemma 2.2. □

Let the function $f(z)$ defined by (1.1) be in the class $S(f, g; \lambda, \alpha, b)$, *then*

$$\left|a_n\right| \leq \frac{(1-\alpha)\left|b\right|}{[1+\lambda(n-1)]\, c_n} \ (n \geq 2). \tag{2.4}$$

The result is sharp for the function

$$f(z) = z + \frac{(1-\alpha)\left|b\right|}{[1+\lambda(n-1)]\, c_n} z^n \ (n \geq 2). \tag{2.5}$$

Lemma 2.3. *Let the function f defined by (1.1) satisfy the following condition:*

$$\sum_{n=2}^{\infty} [1+\lambda(n-1)] c_n \left|a_n\right| \leq (\beta-1)\left|b\right|. \tag{2.6}$$

Then $f \in M(f, g; \lambda, \beta, b)$.

Proof. Assume that the inequality (2.6) holds true. Then we find that

$$\left|(1-\lambda)\frac{(f*g)(z)}{z} + \lambda (f*g)^{'}(z) - 1\right|$$
$$\leq \left|(1-\lambda)\frac{(f*g)(z)}{z} + \lambda (f*g)^{'}(z) - [2(\beta-1)b+1]\right|,$$

that is, that

$$\left|\sum_{n=2}^{\infty}[1+\lambda(n-1)]\,c_n a_n z^{n-1}\right| \le \left|2(\beta-1)b+\sum_{n=2}^{\infty}[1+\lambda(n-1)]\,c_n a_n z^{n-1}\right|.$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}[1+\lambda(n-1)]\,c_n\,|a_n| \le 2(\beta-1)\,|b| - \sum_{n=2}^{\infty}[1+\lambda(n-1)]\,c_n\,|a_n|\,.$$

Then $f \in M(f,g;\lambda,\beta,b)$. This completes the proof of Lemma 2.3.

□

Corollary 2.4. *Let the function $f(z)$ defined by (1.1) be in the class $M(f,g;\lambda,\beta,b)$, then*

$$|a_n| \le \frac{(\beta-1)\,|b|}{[1+\lambda(n-1)]\,c_n}\ (n\ge 2). \tag{2.7}$$

The result is sharp for the function

$$f(z) = z+\frac{(\beta-1)\,|b|}{[1+\lambda(n-1)]\,c_n}z^n\ (n\ge 2). \tag{2.8}$$

Let $S^*(f,g;\lambda,\alpha,b)$ denote the class of functions $f(z)\in A$ whose coefficients satisfy the condition (2.2). We note that $S^*(f,g;\lambda,\alpha,b)\subseteq S(f,g;\lambda,\alpha,b)$ and let $M^*(f,g;\lambda,\beta,b)$ denote the class of functions $f(z)\in A$ whose coefficients satisfy the condition (2.6). We note that $M^*(f,g;\lambda,\beta,b)\subseteq M(f,g;\lambda,\beta,b)$.

Employing the technique used earlier by Attiya [3] and Srivastava and Attiya [19], we prove:

Theorem 2.5. *Let $f\in S^*(f,g;\lambda,\alpha,b)$, $c_n\ge c_2>0\ (n\ge 2)$. Then for every function $\psi\in K$, we have*

$$\frac{(1+\lambda)\,c_2}{2[(1+\lambda)\,c_2+(1-\alpha)\,|b|]}(f*\psi)(z)\prec\psi(z), \tag{2.9}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{(1+\lambda)\,c_2+(1-\alpha)\,|b|}{(1+\lambda)\,c_2}. \tag{2.10}$$

The constant $\dfrac{(1+\lambda)\,c_2}{2[(1+\lambda)\,c_2+(1-\alpha)\,|b|]}$ *is the best estimate.*

Proof. Let $f\in S^*(f,g;\lambda,\alpha,b)$ and let $\psi(z)=z+\sum\limits_{n=2}^{\infty}d_n z^n\in K$. Then we have

$$\begin{aligned}&\frac{(1+\lambda)\,c_2}{2[(1+\lambda)\,c_2+(1-\alpha)\,|b|]}(f*\psi)(z)\\ &=\frac{(1+\lambda)\,c_2}{2[(1+\lambda)\,c_2+(1-\alpha)\,|b|]}\left(z+\sum_{n=2}^{\infty}a_n d_n z^n\right).\end{aligned} \tag{2.11}$$

Thus, by Definition 1, the subordination result (2.9) will hold true if the sequence

$$\left\{\frac{(1+\lambda)\,c_2}{2[(1+\lambda)\,c_2+(1-\alpha)\,|b|]}a_n\right\}_{n=1}^{\infty}, \tag{2.12}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 2.1, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+\sum_{n=1}^{\infty}\frac{(1+\lambda)\,c_2}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}a_n z^n\right\}>0. \tag{2.13}$$

Now, since

$$\{[1+\lambda(n-1)]c_n\},$$

is an increasing function of n $(n \geq 2)$, we have

$$\operatorname{Re}\left\{1+\sum_{n=1}^{\infty}\frac{(1+\lambda)\,c_2}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}a_n z^n\right\}$$

$$=\operatorname{Re}\left\{1+\frac{(1+\lambda)\,c_2}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}z+\frac{1}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}\sum_{n=2}^{\infty}(1+\lambda)\,c_2 a_n z^n\right\}$$

$$\geq 1-\frac{(1+\lambda)\,c_2}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}r-\left(\frac{1}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}\sum_{n=2}^{\infty}[1+\lambda(n-1)]c_n\,|a_n|\,r^n\right)$$

$$>1-\frac{(1+\lambda)\,c_2}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}r-\frac{(1-\alpha)\,|b|}{(1+\lambda)\,c_2+(1-\alpha)\,|b|}r$$
$$=1-r>0\ (|z|=r<1),$$

where we have used assertion (2.2) of Lemma 2.2. Thus (2.13) holds true in U. This proves the inequality (2.9). The inequality (2.10) follows from (2.9) by taking the convex function $\psi(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n \in K$.

To prove the sharpness of the constant $\dfrac{(1+\lambda)\,c_2}{2\left[(1+\lambda)\,c_2+(1-\alpha)\,|b|\right]}$, we consider the function $f_0(z) \in S^*(f,g;\lambda,\alpha,b)$ given by

$$f_0(z)=z-\frac{(1-\alpha)\,|b|}{(1+\lambda)\,c_2}z^2. \tag{2.14}$$

Thus from (2.9), we have

$$\frac{(1+\lambda)\,c_2}{2\left[(1+\lambda)\,c_2+(1-\alpha)\,|b|\right]}f_0(z)\prec\frac{z}{1-z}. \tag{2.15}$$

Moreover, it can be verified for the function $f_0(z)$ given by (2.14) that

$$\min_{|z|\leq r}\left\{\operatorname{Re}\frac{(1+\lambda)\,c_2}{2\left[(1+\lambda)\,c_2+(1-\alpha)\,|b|\right]}f_0(z)\right\}=-\frac{1}{2}. \tag{2.16}$$

This show that the constant $\dfrac{(1+\lambda)\,c_2}{2\left[(1+\lambda)\,c_2+(1-\alpha)\,|b|\right]}$ is the best possible. This completes the proof of Theorem 2.5.

□

Putting $g(z)=z+\sum_{n=2}^{\infty}\Psi_n z^n$ $(or\ c_n=\Psi_n)$, where Ψ_n is defined by (1.6) in Lemma 2.2 and Theorem 2.5, we obtain the following corollary:

Corollary 2.6. *Let f defined by (1.1) be in the class $S^*_{q,s}([\alpha_1];\lambda,\alpha,b)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}[1+\lambda(n-1)]\Psi_n(\alpha_1)|a_n| \leq (1-\alpha)|b|.$$

Then for every function $\psi \in K$, we have

$$\frac{(1+\lambda)\Psi_2(\alpha_1)}{2[(1+\lambda)\Psi_2(\alpha_1)+(1-\alpha)|b|]}(f*\psi)(z) \prec \psi(z),$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{(1+\lambda)\Psi_2+(1-\alpha)|b|}{(1+\lambda)\Psi_2}.$$

The constant $\dfrac{(1+\lambda)\Psi_2(\alpha_1)}{2[(1+\lambda)\Psi_2(\alpha_1)+(1-\alpha)|b|]}$ *is the best estimate.*

Remark. (1) *Putting $c_n = n^k$ $(k \in \mathbb{N}_0)$ and $\alpha = 0$ in Lemma 2.2 and Theorem 2.5, we obtain the result obtained by Aouf [1, Theorem 1];*

(2) Putting $c_n = \left(\frac{1+\gamma(n-1)+l}{1+l}\right)^m$ $(\gamma \geq 0, l \geq 0, m \in \mathbb{N}_0)$ *and $\alpha = 0$ in* Lemma *2.2 and Theorem 2.5, we obtain the result obtained by Aouf and Hidan [2, Theorem 3].*

Similarly, we can prove the following theorem.

Theorem 2.7. *Let $f \in M^*(f,g;\lambda,\beta,b)$, $c_n \geq c_2 > 0$ $(n \geq 2)$. Then for every function $\psi \in K$, we have*

$$\frac{(1+\lambda)c_2}{2[(1+\lambda)c_2+(\beta-1)|b|]}(f*\psi)(z) \prec \psi(z) \tag{2.17}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{(1+\lambda)c_2+(\beta-1)|b|}{(1+\lambda)c_2}. \tag{2.18}$$

The constant $\dfrac{(1+\lambda)c_2}{2[(1+\lambda)c_2+(\beta-1)|b|]}$ *is the best estimate.*

Putting $g(z) = z + \sum_{n=2}^{\infty}\Psi_n(\alpha_1)z^n$ *(or $c_n = \Psi_n(\alpha_1)$)*, where $\Psi_n(\alpha_1)$ is defined by(1.6) in Lemma 2.3 and Theorem 2.7, we obtain the following corollary:

Corollary 2.8. *Let f defined by (1.1) be in the class $M^*_{q,s}([\alpha_1];\lambda,\beta,b)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}[1+\lambda(n-1)]\Psi_n(\alpha_1)|a_n| \leq (\beta-1)|b|.$$

Then for every function $\psi \in K$, we have

$$\frac{(1+\lambda)\Psi_2(\alpha_1)}{2[(1+\lambda)\Psi_2(\alpha_1)+(\beta-1)|b|]}(f*\psi)(z) \prec \psi(z),$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{(1+\lambda)\Psi_2(\alpha_1)+(\beta-1)|b|}{(1+\lambda)\Psi_2(\alpha_1)}.$$

The constant $\dfrac{(1+\lambda)\Psi_2(\alpha_1)}{2[(1+\lambda)\Psi_2(\alpha_1)+(\beta-1)|b|]}$ *is the best estimate.*

Remark. *Specializing g, λ and β, in Lemma 2.3 and Theorem 2.7, we obtain the corresponding results for the corresponding operators (1-3) defined in the introduction.*

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