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A FIXED POINT THEOREM VIA GENERALIZED W-DISTANCE

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Abstract. In this paper we first introduce the concept of generalized w-distance in a metric space and prove a fixed point theorem which generalizes Banach contraction theorem.

1. Introduction

In 1996, W. Takahashi et. al.[5] had introduced the concept of w-distance in a metric space and proved some fixed point theorems in complete metric spaces. In this paper we first introduce the concept of generalized w-distance in a metric space. At the beginning of the paper an example is provided to show that the class of generalized w-distance functions is strictly larger than the class of w-distance functions. Finally we prove a fixed point theorem in a complete metric space by using the concept of generalized w-distance. This theorem is a generalization of Banach contraction theorem.

2. Definitions and Examples

Definition 2.1. [5] *Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a w- distance on X if the following conditions are satisfied :*
(i) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(ii) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semicontinuous ;
(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Clearly every metric is a w-distance but the converse is not true. The following example supports our contention.

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Example 2.1. [5] *Let (X,d) be a metric space. A function $p : X \times X \to [0,\infty)$ defined by $p(x,y) = c$ for every $x, y \in X$ is a w-distance on X, where c is a positive real number. But p is not a metric since $p(x,x) = c \neq 0$ for any $x \in X$.*

Definition 2.2. *Let (X,d) be a metric space and $j \in N$. A function $p : X \times X \to [0,\infty)$ is called a generalized w- distance of order j on X if for all $x, z \in X$ and for all distinct points $x_i \in X,\ i \in \{1,2,3,\cdots,j\}$, each of them different from x and z, the following conditions are satisfied:*

(i) $p(x,z) \leq \sum_{i=0}^{j} p(x_i, x_{i+1}),$ *where* $x_0 = x,\ x_{j+1} = z;$

(ii) for any $x \in X,\ p(x,.) : X \to [0,\infty)$ is lower semicontinuous;

(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

From Definition 2.2 it follows that every w-distance is a generalized w-distance of order 1.

Now we consider the following example to show that a generalized w-distance may not be a w-distance.

Example 2.2. *Let $X = \{1,2,3,4\}$ be a metric space with metric $d(x,y) =| x - y |$ for all $x, y \in X$. Let $p : X \times X \to [0,\infty)$ be defined by*

$$p(1,2) = p(2,1) = 3,\ p(1,3) = p(3,1) = p(2,3) = p(3,2) = 1,$$

$$p(1,4) = p(4,1) = p(2,4) = p(4,2) = p(3,4) = p(4,3) = 2$$

$$\text{and } p(x,x) = 0.6 \text{ for every } x \in X.$$

Then p satisfies condition (i) of Definition 2.2 for $j = 2$. Also, condition (ii) of Definition 2.2 is obvious. To show (iii), for any $\epsilon > 0$, put $\delta = \frac{1}{2}$. Then

$$p(z,x) \leq \delta \text{ and } p(z,y) \leq \delta \text{ imply } d(x,y) \leq \epsilon.$$

Thus p is a generalized w-distance of order 2 on X but it is not a w-distance on X since it lacks the triangular property:

$$p(1,2) = 3 > 1 + 1 = p(1,3) + p(3,2).$$

3. Main Result

In this section we prove a fixed point theorem in a complete metric space by employing notion of generalized w-distance. The following Lemma is crucial in the proof of the theorem.

Lemma 3.1. *Let (X,d) be a metric space and let p be a generalized w-distance of order j on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold :*

(i) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;

(ii) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z;

(iii) if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is a d-Cauchy

sequence;
(iv) if $p(y, x_n) \leq \alpha_n$ *for any* $n \in N$*, then* $\{x_n\}$ *is a d-Cauchy sequence.*

Proof. Proof is similar to that of Lemma 1 [5] and we left it. □

Theorem 3.1. *Let* (X, d) *be a complete metric space, let* p *be a generalized w-distance of order* j *on* X *and let* T *be a mapping from* X *into itself. Suppose that there exists* $r \in [0, 1)$ *such that*

$$p(Tx, Ty) \leq r\, p(x, y) \tag{3.1}$$

for every $x, y \in X$*. Then there exists* $z \in X$ *such that* $z = Tz$*. Moreover, if* $v = Tv$*, then* $p(v, v) = 0$*.*

Proof. Let u be an arbitrary element of X. We consider the sequence $\{u_n\}$ where $u_n = T^n u$ for any $n \in N$. We can suppose that $T^n u \neq T^m u$ for all distinct $n, m \in N$. In fact, if $T^n u = T^m u$ for some $m, n \in N$, $m \neq n$ then assuming $m > n$, we have

$$T^{m-n}(T^n u) = T^n u$$

$$i.e.,\ T^k y = y \text{ where } k = m - n > 0 \text{ and } y = T^n u.$$

If $k = 1$, then $Ty = y$ and y is a fixed point of T.

Again if $k > 1$, then

$$p(y, Ty) = p(T^k y, T^{k+1} y) \leq r^k\, p(y, Ty)$$

and being $r < 1$ one has $p(y, Ty) = 0$.
Also,

$$p(y, y) = p(T^k y, T^k y) \leq r^k\, p(y, y)$$

and being $r < 1$ one has $p(y, y) = 0$.
Since $p(y, Ty) = 0$ and $p(y, y) = 0$, by using Lemma 3.1(i), we get $Ty = y$ i.e., y is a fixed point of T.
Thus in the sequel of the proof we can suppose that $T^n u \neq T^m u$ for all distinct $n, m \in N$.

Let us now prove that for all $n, m \in N$, one has

$$p(T^n u, T^{n+m} u) \leq \frac{r^n}{1-r}\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\}. \tag{3.2}$$

By using (3.1), we have

$$p(T^n u, T^{n+m} u) \leq r^n p(u, T^m u). \tag{3.3}$$

If $m \leq j$, then

$$\begin{aligned} p(u, T^m u) &\leq (1 + r + r^2 + \cdots)\, p(u, T^m u) \\ &\leq \frac{1}{1-r}\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\}. \end{aligned}$$

If $m > j$, then there exists $s \in N$ such that $m = sj + t$, where $0 \leq t < j$.
If $t = 0$, then by using (3.1)

$$\begin{aligned} p(u, T^m u) &\leq p(u, Tu) + p(Tu, T^2 u) + \cdots + p(T^{j-1} u, T^j u) + p(T^j u, T^m u) \\ &\leq p(u, Tu) + rp(u, Tu) + \cdots + r^{j-1} p(u, Tu) + r^j p(u, T^{m-j} u) \\ &= \sum_{q=0}^{j-1} r^q p(u, Tu) + r^j p(u, T^{m-j} u). \end{aligned} \tag{3.4}$$

By repeated application of (3.4), we obtain at $(s-1)$-th step that

$$\begin{aligned} p(u, T^m u) &\leq \sum_{q=0}^{(s-1)j-1} r^q p(u, Tu) + r^{(s-1)j} p(u, T^j u) \\ &\leq (1 + r + r^2 + \cdots + r^{(s-1)j})\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\} \\ &\leq \frac{1}{1-r}\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\}. \end{aligned}$$

If $t \neq 0$, then by (3.1)

$$\begin{aligned} p(u, T^m u) &\leq p(u, Tu) + p(Tu, T^2 u) + \cdots + p(T^{j-1} u, T^j u) + p(T^j u, T^m u) \\ &\leq p(u, Tu) + rp(u, Tu) + \cdots + r^{j-1} p(u, Tu) + r^j p(u, T^{m-j} u) \\ &= \sum_{q=0}^{j-1} r^q p(u, Tu) + r^j p(u, T^{m-j} u). \end{aligned} \tag{3.5}$$

By repeated application of (3.5), we obtain at s-th step that

$$\begin{aligned} p(u, T^m u) &\leq \sum_{q=0}^{sj-1} r^q p(u, Tu) + r^{sj} p(u, T^t u) \\ &\leq (1 + r + r^2 + \cdots + r^{sj})\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\} \\ &\leq \frac{1}{1-r}\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\}. \end{aligned}$$

So, if $m > j$ then it must be the case that

$$p(u, T^m u) \leq \frac{1}{1-r}\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\}.$$

Now, using (3.3) we have for all $n, m \in N$,

$$p(T^n u, T^{n+m} u) \leq \frac{r^n}{1-r}\, max \left\{ p(u, T^i u) : i = 1, 2, \cdots, j \right\}.$$

By Lemma 3.1(iii), $\{u_n\}$ is a Cauchy sequence in (X, d) which is a complete metric space. So there exists a point $z \in X$ such that $z = \lim_n u_n$.
Let $n \in N$ be fixed. Since $\{u_m\}$ converges to z and $p(u_n, .)$ is lower semi continuous, one obtains

$$p(u_n, z) \leq \lim_{m \to \infty} inf\, p(u_n, u_m) \leq \frac{r^n}{1-r}\, max \left\{ p(u, Tu), p(u, T^2 u) \right\},$$

which implies that, $p(u_n, z) \to 0$ as $n \to \infty$.
Again, from (3.1)

$$p(u_{n+1}, Tz) = p(Tu_n, Tz) \leq r\, p(u_n, z) \to \ as\ n \to \infty.$$

Thus, by Lemma 3.1(i), $p(u_{n+1}, Tz) \to 0$ and $p(u_{n+1}, z) \to 0$ imply that $Tz = z$. Therefore, z becomes a fixed point of T.
If $v = Tv$, then

$$p(v, v) = p(Tv, Tv) \leq r\, p(v, v)$$

and hence $p(v, v) = 0$.

□

Corollary 3.1. *(Banach Contraction Theorem) Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{3.6}$$

for all $x, y \in X$ *and* $0 < \alpha < 1$. *Then* T *has a unique fixed point in* X.

Proof. We see that d is a generalized w-distance of order 1. So, by Theorem 3.1 there exists $z \in X$ such that $Tz = z$. Uniqueness follows from condition (3.6).

□

We now furnish an example which shows that the condition (3.1) in Theorem 3.1 can neither be relaxed.

Example 3.1. *Take* $X = [2, \infty) \cup \{0, 1\}$, *which is a complete metric space with usual metric* d *of reals. Define* $T : X \to X$ *where*

$$\begin{aligned} Tx &= 0 \, for \ x \in (X \setminus \{0\}) \\ &= 1 \, for \ x = 0. \end{aligned}$$

Clearly, T *possesses no fixed point in* X.
In fact, for $x = 0$ *and* $y = Tx = T0$ *in* X, *we find that*

$$d(Tx, Ty) = 1 > r\, d(x, Tx)$$

for any $r \in [0, 1)$.
Hence condition (3.1) *fails and Theorem* 3.1 *does not hold.*

Note: *In example above we treat* d *as a generalized* w*-distance of order* 1 *in* X *in reference to Theorem* 3.1.

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