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NEW RESULTS ON THE CLOSEDNESS OF THE PRODUCT AND SUM OF CLOSED LINEAR OPERATORS

(COMMUNICATED BY CARLOS KUBRUSLY)

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Abstract. In this paper we give some topological conditions which ensure the closedness of the sum and product of two closed linear operators acting in a Hilbert space. We get also a general result on the formula of the adjoint (well known for bounded operators) of the sum and product in the set of densely defined closed operators. Other related results are also established.

1. Introduction

Let H be a separable Hilbert space. We denote by $\mathcal{C}_1(H)$ (respectively $\mathcal{C}_2(H)$) the set of all closed linear subspaces of H (respectively of $H \oplus H$) and by $C(H)$ the set of densely defined closed linear operators on H. $B(H)$ is the algebra of bounded linear operators on H.

For any unbounded linear operator A on H, the domain of A is denoted by $D(A)$ and $G(A) = \{(x, Ax) \, ; \, x \in D(A)\}$ is its graph, in particular $G(A) \in \mathcal{C}_2(H)$ if $A \in C(H)$. The null space and the range of A will be denoted by $N(A)$ and $R(A)$ respectively. $A^*$is the adjoint of A if A is densely defined on H. The graph $G(A^*)$ of A^* is equal to $V(G(A)^{\perp}) = [V(G(A))]^{\perp}$ where $V(x, y) = (-y, x)$ is an isometric surjection, it is unitary, on $H \oplus H$ satisfying $V^2(x, y) = (-x, -y)$. $\overline{A}$ is the closure of A if A^* is densely defined on H.

We write $A \subset B$ when B is an extension of A, in the sens that $D(A) \subset D(B)$ and the restriction of B to $D(A)$ agrees with A. In particular, $A \subset B$ is equivalent to $G(A) \subset G(B)$. It follows immediately that $\overline{A}$ is the smallest closed extension of A.

The standard and the known definition of the sum and the product of two operators A and B with domains respectively $D(A)$ and $D(B)$ is just :

$$\begin{aligned} (A+B)x &= Ax + Bx \quad \text{for } x \in D(A+B) = D(A) \cap D(B) \\ (AB)x &= A(Bx) \quad \text{for } x \in B^{-1}(D(A)) \end{aligned}$$

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This definition, although being the most natural, has some faults. In particular, if $A, B \in C(H)$ then $A + B$ or AB may just have not any sense or need not be closed (see [4],[7]).

If we talk about the adjoint, then the results are not better. Indeed, the adjoint of $A + B$ and of AB if both operators are unbounded does not equal respectively to $A^* + B^*$ and B^*A^*. We will show that the sum $A+B$ and the product AB can be closed and $(A + B)^* = A^* + B^*$ or $(AB)^* = B^*A^*$ if suitable conditions are imposed on $A, B \in C(H)$.

Concerning the product, J. Dixmier has defined in [1] a new product of two unbounded operators on H, this product is unstable in $C(H)$ but verify the identity of the adjoint of the usual product. Recently, B. Messirdi and M.H. Mortad in [7] have proposed a more general version of the product using the notion of bisecting of an operator in $C(H)$. This new product is stable in $C(H)$ and verify the equality $(A.B)^* = B^*.A^*,\ A, B \in C(H)$.

B. Messirdi, M.H. Mortad, A. Azzouz and G. Djellouli in [6] have already found a topological condition such that the product of two operators of $C(H)$ remains in $C(H)$. Indeed, they show that if $A,\ B \in C(H)$ are such that $d(A, B^*) < 1$ then $D(AB)$ and $D(BA)$ are dense in H and $AB,\ BA \in C(H)$, where d is a metric on $C(H)$ (the gap metric g is strictly finer than d).

Concerning the sum, a natural question that may arise is to find sufficient conditions under different perturbations to ensure that the closedness is preserved in $C(H)$ and that the adjoint of the sum is the sum of adjoints. P. Hess and T. Kato showed in [3] that $(A + B)^* = A^* + B^*$ for perturbations B sufficiently small with respect to A measured in the gap metric g on $C(H)$. On the other hand, the famous theorem of Kato-Rellich and the theorem of Wüst ([4], [9]) are fundamental results of the spectral analysis, they provide some interesting answers to the perturbations problems of selfadjoint and essentially selfadjoint operators especially Schrödinger operators in quantum physics.

Let us first recall an important property of sum of two subspaces of $\mathcal{C}_1(H)$ (see [5], Corollary 1.3.1) :

Proposition 1.1. *Let $M, N \in \mathcal{C}_1(H)$. Then,*

$$M + N \in \mathcal{C}_1(H) \Leftrightarrow M^\perp + N^\perp = (M \cap N)^\perp \in \mathcal{C}_1(H)$$

Let P_M and P_N denote the orthogonal projections on M and N respectively, where $M, N \in \mathcal{C}_1(H)$. Now set

$$g(M, N) = \|P_M - P_N\|_{B(H)}$$

$g(M, N)$ defines a metric on $\mathcal{C}_1(H)$. Moreover, we recall the following result proven the first time in 1980 by J.Ph. Labrousse [5] and taken in 2008 under more general form by B. Messirdi, M.H. Mortad, A. Azzouz and G. Djellouli in [6] for a topological characterization of the product in $C(H)$.

Proposition 1.2. *Let $M, N \in \mathcal{C}_1(H)$. Then the following conditions are equivalent:*

i) $g(M, N) < 1$

ii) $M \oplus N^\perp = H$

g will be used to introduce a metric on $C(H)$. For this we simply use the graph of a closed operator on H.

Definition 1.1. *If $A, B \in C(H)$, we define*

$$g(A,B) = g(G(A), G(B)) = \left\| P_{G(A)} - P_{G(B)} \right\|_{B(H \oplus H)}$$

where $P_{G(A)}$ and $P_{G(B)}$ denote respectively the orthogonal projection in $H \oplus H$ on the graph $G(A)$ of the operator A and the graph $G(B)$ of the operator B.

$C(H)$ equipped with the distance g called "gap" metric becomes a metric space. The topology induced by g on $B(H)$ is equivalent to the usual uniform topology and $B(H)$ is open in $C(H)$ [4].

Nevertheless, the spectral analysis is incomplete in $C(H)$ with respect to the one in $B(H)$ because of certain insufficiencies of algebraic and topological characters taken in $C(H)$. $C(H)$ is not complete for the metric g because if we consider for example the sequence of operators $(nI)_{n \in \mathbb{N}^*}$ where I is the identity operator on H, then the nI are bounded operators, therefore closed on H, for all $n \in \mathbb{N}^*$.

$$\begin{cases} G(nI) = \left\{ \left(\frac{1}{n}x, x \right) \ ; x \in H \right\}, \text{ for all } n \in \mathbb{N}^* \\ \lim\limits_{n \to +\infty} g(G(nI), \ \{0\} \oplus H) = 0 \end{cases}$$

But $\{0\} \oplus H$ cannot be the graph of a linear operator on H, thus $(nI)_{n \in \mathbb{N}^*}$ is a Cauchy sequence without limit in $C(H)$.

M. Fernandez Miranda and J. Ph. Labrousse [2] also have shown that we can remedy these defects by considering the completed of $C(H)$ for the topology defined by the metric g and injecting $C(H)$ in all the set of closed linear relations on H (ie the set of subspaces of $\mathcal{C}_2(H)$ of infinite dimension and codimension). We can also cite here the works of B. Sz. Nagy about perturbation of closed linear relations [8].

In this work we give a sufficient topological conditions on the graph of two operators of $C(H)$, so that their sum and product remain in $C(H)$. We get also a general result on the adjoint of the sum and product in $C(H)$.

2. Product and Sum of Closed Operators

For the convenience of the reader, we remind some classic properties of the sum of the product of unbounded operators. In what follows we suppose that the operators $(A+B)^*$, $(A^*+B^*)^*$, $\overline{A+B}$, $\overline{A^*+B^*}$, $(AB)^*$, B^*A^*, $(B^*A^*)^*$, $\overline{AB}$,... exist when A, B belong in $C(H)$.

Lemma 2.1. *([4],[9]) Let $A, B \in C(H)$, then*
i) $A \subset B$ implies $B^* \subset A^*$
ii) $G(A^* + B^*) \subset [V(G(A+B))]^{\perp}$
iii) $A^* + B^* \subset \overline{A^* + B^*} \subset (A+B)^*$
iv) $A + B \subset \overline{A+B} \subset (A^* + B^*)^*$
v) If $A \in B(H)$, $A + B, BA \in C(H)$ and $(A+B)^* = A^* + B^*$
vi) $B^*A^* \subset (AB)^*$. If , in addition $A \in B(H)$, then $(AB)^* = B^*A^*$
vii) $AB \subset \overline{AB} \subset (B^*A^*)^*$
viii) $(I + AB) \subset \overline{(I+AB)} \subset (I + B^*A^*)^*$

Let us start by giving the following classical result :

Proposition 2.2. *Let $A \in C(H)$. If $g(A,0) < 1$ then for every B in $C(H)$, $A+B \in C(H)$ and $BA \in C(H)$. Thus, $(A+B)^* = A^* + B^*$ and $(AB)^* = B^* A^*$.*

Proof. $g(A,0) < 1$ implies that $A \in B(H)$. Thus, $D(A+B) = D(B)$, $D(AB) = D(B)$, $D(BA) = A^{-1}(D(B))$ and $(A+B)$, $BA \in C(H)$. We obtain also the relations $(A+B)^* = A^* + B^*$ and $(AB)^* = B^* A^*$ by virtue of Lemma 2.1, properties v) and vi). □

We show here the main results of this paper. We establish a topological characterization on the sum and product and also on the adjoint of the sum and product of two operators of $C(H)$.

We begin to show the following lemma :

Lemma 2.3. *Let $A, B \in C(H)$.*
i) If $G(A) + G(-B) \in \mathcal{C}_2(H)$, then $G(A^*) + G(-B^*) \in \mathcal{C}_2(H)$
ii) If $G(A) + V(G(B)) \in \mathcal{C}_2(H)$, then $G(A^*) + V(G(B^*)) \in \mathcal{C}_2(H)$

Proof. i) If $G(A) + G(-B) \in \mathcal{C}_2(H)$ then $G(A)^\perp + G(-B)^\perp \in \mathcal{C}_2(H)$ by virtue of Proposition 1.1. However, $G(A^*) = V(G(A)^\perp) = (V(G(A)))^\perp$ and $G(-B^*) = V(G(-B)^\perp) = (V(G(-B)))^\perp$. So that, $V(G(A))^\perp + V(G(-B))^\perp \in \mathcal{C}_2(H)$ and then $G(A^*) + G(-B^*) \in \mathcal{C}_2(H)$.

ii) is proved in the same manner. □

Now we are able to establish our first main result about the closedness of the product and the adjoint of the product of closed linear operators.

Theorem 2.4. *Let $A, B \in C(H)$ are such that $G(A) + V(G(B)) \in \mathcal{C}_2(H)$ then*
i) $R(I+BA) \in \mathcal{C}_1(H)$ and $R(I+AB) \in \mathcal{C}_1(H)$
ii) $R(I+A^*B^*) \in \mathcal{C}_1(H)$ and $R(I+B^*A^*) \in \mathcal{C}_1(H)$
iii) If, in addition $N(I+BA) \in \mathcal{C}_1(H)$ (respectively $N(I+AB) \in \mathcal{C}_1(H)$), then BA is closed and thus $BA \in C(H)$ if $D(BA)$ is dense in H (respectively AB is closed and thus $AB \in C(H)$ if $D(AB)$ is dense in H).
iv) $(A^*B^*)^* = \overline{BA}$ if $N(\overline{I+A^*B^*}) = N(I+A^*B^*)$ (respectively, $(B^*A^*)^* = \overline{AB}$ if $N(\overline{I+B^*A^*}) = N(I+B^*A^*)$)

Proof. i) Let $(u,0) \in [G(A)+V(G(B))] \cap (H \oplus \{0\})$. Then, $(u,0) = (x - Bs, s + Ax)$ where $x \in D(A)$ and $s \in D(B)$ and $s + Ax = 0$. Hence, $(-Ax, Bs) \in G(B)$ so that $Ax \in D(B)$ and $(x, -Bs) \in G(BA)$ and therefore $(x, x - Bs) = (x, u) \in G(I+BA)$ which shows that $u \in R(I+BA)$.

Conversely, let $u = (I+BA)v \in R(I+BA)$ where $v \in D(BA)$. Then, $(v,u) \in G(I+BA)$, so that $(v, u-v) \in G(BA)$, either if we pose $r = Av$ we have $(v,r) \in G(A)$ and $(r, u-v) \in G(B)$ thus $(u-v, -r) \in V(G(B))$. Consequently, $(u,0) = (v,r) + (u-v,-r) \in [G(A)+V(G(B))] \cap (H \oplus \{0\})$.

Finally, $[G(A)+V(G(B))] \cap (H \oplus \{0\}) = R(I+BA) \oplus \{0\} \in \mathcal{C}_2(H)$ and then $R(I+BA) \in \mathcal{C}_1(H)$ because $G(A)+V(G(B)) \in \mathcal{C}_2(H)$ and $H \oplus \{0\} \in \mathcal{C}_2(H)$.

$R(I+AB) \in \mathcal{C}_1(H)$ because $[G(A)+V(G(B))] \cap (\{0\} \oplus H) = \{0\} \oplus R(I+AB) \in \mathcal{C}_2(H)$.

ii) is established in the same way.

iii) Since $R(I+BA), N(I+BA) \in \mathcal{C}_1(H)$, it follows that

$$\begin{aligned}(H \oplus \{0\}) + G(I+BA) &= H \oplus R(I+BA) \in \mathcal{C}_2(H)\\ (H \oplus \{0\}) \cap G(I+BA) &= N(I+BA) \oplus \{0\} \in \mathcal{C}_2(H)\end{aligned}$$

Hence from Proposition 2.1.1 of [5] we obtain that $G(I+BA) \in \mathcal{C}_2(H)$ and thus $BA \in C(H)$ if $D(BA)$ is dense in H

iv) We have

$$R((I+A^*B^*)^*) \subseteq [N(\overline{I+A^*B^*})]^\perp = [N(I+A^*B^*)]^\perp$$

and

$$R((I+A^*B^*)^*) = \overline{R(I+BA)} = R(I+BA)$$

Let now $(u,v) \in G((I+A^*B^*)^*)$. By Lemma 2.1, it is enough to take $v \in R(I+BA)$, so that $v = (I+BA)w$ where $w \in D(I+BA)$ and $(w,v) \in G(I+BA)$. Thus, $(u-w) \in N((I+A^*B^*)^*) = [R(I+A^*B^*)]^\perp = \overline{N(I+BA)}$.

Therefore, $u = u - w + w \in D(I+BA)$ from which it follows that $G((I+A^*B^*)^*) \subseteq G(\overline{I+BA})$. Finally using properties v) and viii) of Lemma 2.1 we have $G((I+A^*B^*)^*) = G(\overline{I+BA})$ and thus $(A^*B^*)^* = \overline{BA}$.

Since $G(B) + V(G(A)) = V[G(A) + V(G(B)]$ and V is unitary, then we can interchange A and B and obtain $(B^*A^*)^* = \overline{AB}$ if $N(\overline{I+B^*A^*}) = N(I+B^*A^*)$. □

Remark. *The hypothesis* $N(\overline{I+A^*B^*}) = N(I+A^*B^*)$ *or* $N(\overline{I+B^*A^*}) = N(I+B^*A^*)$ *is automatically satisfied if* $G(A)+V(G(B)) \in \mathcal{C}_2(H)$ *and when* A *and* B *are closed linear relations on* H *(see* [2]*).*

Our second main result concerns the sum and the adjoint of the sum of linear closed operators.

Theorem 2.5. *Let* $A, B \in C(H)$ *such that* $G(A) + G(-B) \in \mathcal{C}_2(H)$ *then :*

i) $R(A+B) \in \mathcal{C}_1(H)$

ii) $R(A^*+B^*) \in \mathcal{C}_1(H)$

iii) $(A+B)^* = \overline{A^*+B^*}$ *if* $N((A+B)^*) = N\left(\overline{A^*+B^*}\right)$

iv) $(A^*+B^*)^* = \overline{A+B}$ *if* $N((A^*+B^*)^*) = N\left(\overline{A+B}\right)$

Proof. i) We show in fact that $R(A+B) = [N(A^*+B^*)]^\perp$. Let $(x,y) \in G(A^*) \cap G(-B^*)$ then $x \in D(A^* + (-B^*))$ and $y = A^*x = -B^*x$ hence $x \in N(A^*+B^*)$.

So that, if $z \in [N(A^*+B^*)]^\perp$ then

$$(z,0) \in [G(A^*) \cap G(-B^*)]^\perp = \left[V(G(A))^\perp \cap V(G(-B))^\perp\right]^\perp$$

Using the fact that, $G(A^*) = V(G(A))^\perp$ and $G(-B^*) = V(G(-B))^\perp$ and that V is an isometry on $H \times H$, then $(0,z) \in \left(G(A)^\perp \cap G(-B)^\perp\right)^\perp = G(A) + G(-B)$.

Consequently, for all $z \in [N(A^*+B^*)]^\perp$ there exists a unique decomposition of $(0,z)$ in $(x,Ax) \in G(A)$, $(y,-By) \in G(-B)$ and $(u,v) \in G(A) \cap G(-B)$ such that

$$(0,z) = (x,Ax) + (y,-By) + (u,v)$$

with (x,Ax), $(y,-By) \in [G(A) \cap G(-B)]^\perp$.

Thus,

$$\begin{cases} x + y + u = 0 & (1) \\ Ax - By + v = z & (2) \end{cases}$$

We deduce that $x \in D(A+B)$.

Let us put $x = Tz$ where $T : [N(A^* + B^*)]^{\perp} \to D(A + B)$ satisfying by virtue of (1) and (2)

$$\begin{aligned} (A+B)x &= Ax - By - Bu \\ &= Ax - By + v \\ &= z\ , \text{ for all } z \in [N(A^* + B^*)]^{\perp} \end{aligned}$$

so that,

$$(Tz, z) = (x, (A + B)x) \in G(A + B)$$

then

$$[N(A^* + B^*)]^{\perp} \subset R(A + B)$$

Conversely, we have using the Lemma 2.1, property iii),

$$R(A + B) \subseteq \overline{R((A^* + B^*)^*)} = [N(A^* + B^*)]^{\perp}$$

iii) According to Lemma 2.3, $G(A^*) + G(-B^*) \in \mathcal{C}_2(H)$, there exist as before an application T^* defined from $[N(A + B)]^{\perp}$ on $D(A^* + B^*)$ such that for all $t \in [N(A + B)]^{\perp}$, $(T^*t, t) \in G(A^* + B^*) \subseteq G(\overline{A^* + B^*})$.

Let $v \in D((A+B)^*)$ and $t = (A+B)^*v \in R((A+B)^*) \subseteq [N(A + B)]^{\perp}$ we then have according to Lemma 2.1, $(T^*t, t) \in G(A^* + B^*) \subset G((A + B)^*)$.

Consequently, $(A + B)^*T^*t = t$ and according to i),

$$\begin{aligned} (v - T^*t) &\in N((A + B)^*) = [R(A + B)]^{\perp} \\ &= \overline{N(A^* + B^*)} \subseteq N(\overline{A^* + B^*}) \end{aligned}$$

Then $v \in D(\overline{A^* + B^*})$ since $(v - T^*t) \in D(\overline{A^* + B^*})$ and $T^*t \in D(A^* + B^*)$.

We have also the inverse inclusion from iii) of Lemma 2.1. Consequently, $D((A + B)^*) = D(\overline{A^* + B^*})$.

Moreover, let $(u, v = (A + B)^*u) \in G((A + B)^*)\, , u \in D((A + B)^*) = D(\overline{A^* + B^*})$. Then, $\left(u, w = \overline{(A^* + B^*)}u\right) \in G(\overline{A^* + B^*}) \subseteq G((A + B)^*)$ according to iii) of Lemma 2.1. We obtain $(v - w) = 0$ and $(u, v) \in G(\overline{A^* + B^*})$, thus, $u \in N((A + B)^*) = N(\overline{A^* + B^*})$. This shows that $G((A + B)^*) \subset G(\overline{A^* + B^*})$.

Since the inverse inclusion is true by virtue of Lemma 2.1, we have then $G((A + B)^*) = G(\overline{A^* + B^*})$ and thus $(A + B)^* = \overline{A^* + B^*}$.

ii) and iv) are proved in the same manner by virtue of symmetry of the hypotheses. ☐

Remark. *The hypothesis* $N((A + B)^*) = N(\overline{A^* + B^*})$ *is automatically satisfied if one of the following statements is satisfied :*

i) $G(A) + G(-B) \in \mathcal{C}_2(H)$ *and when* A, B *are closed linear relations on* H *(*[2]*).*

ii) A, $B \in C(H)$ *when* A, B *and* $(A + B)$ *are selfadjoint.*

Corollary 2.6. *Let* $A, B \in C(H)$ *such that* $G(A) + G(-B) \in \mathcal{C}_2(H)$.

i) If $N(A + B) \in \mathcal{C}_1(H)$ *then* $A + B$ *is closed and thus* $A + B \in C(H)$ *if* $D(A) \cap D(B)$ *is dense in* H.

ii) $(A+B)^* = A^* + B^*$ *if* $N(A^* + B^*) \in \mathcal{C}_1(H)$ *and* $N((A + B)^*) = N(\overline{A^* + B^*})$.

Proof. i) Because $R(A+B) \in \mathcal{C}_1(H)$ and $N(A+B) \in \mathcal{C}_1(H)$, then $(H \oplus \{0\}) + G(A+B) = H \oplus R(A+B) \in \mathcal{C}_2(H)$ and $(H \oplus \{0\}) \cap G(A+B) = N(A+B) \oplus \{0\} \in \mathcal{C}_2(H)$. Hence from Proposition 2.1.1 of [5] it follows that $G(A+B)$ is closed in $H \oplus H$.

ii) is obvious. □

By means of the metrics g we can also characterize the closedness and adjoint of the sum and product of unbounded closed linear operators.

Theorem 2.7. *Let $A, B \in C(H)$ are such that $g(A, B^*) < 1$. Then, $AB \in C(H)$, $BA \in C(H)$ and $(AB)^* = B^* A^*$, $(BA)^* = A^* B^*$.*

Proof. Note first that from Theorem 2 of [6] and Proposition 1.2 it follows that since $g(A, B^*) < 1$, $N(I + AB) = N(I + BA) = \{0\}$ and $AB \in C(H)$, $BA \in C(H)$. Now, if $u \in H$ there exists $x \in D(A)$ and $y \in D(B)$ such that

$$(0, u) = (x - By, y + Ax)$$

Thus, $By = x \in D(A)$ and $u = y + ABy = (I + AB)y$. Consequently,

$$\begin{aligned} R(I + AB) &= R(I + BA) = H \\ N((I + AB)^*) &= N((I + BA)^*) = \{0\} \end{aligned}$$

By virtue of Lemma 2.1, properties iii) and iv), we deduce that $N(\overline{I + B^* A^*}) = N(\overline{I + A^* B^*}) = \{0\}$. Finally, it follows from Theorem 2.4 that $(A^* B^*)^* = BA$ and $(B^* A^*)^* = AB$. □

Theorem 2.8. *Let $A, B \in C(H)$ such that $g\left(G(A), G(-B)^{\perp}\right) < 1$, then :*

i) $(A + B)^* = \overline{A^* + B^*}$

ii) $(A + B) \in C(H)$

Proof. Note that from Proposition 1.2, the condition $g\left(G(A), G(-B)^{\perp}\right) < 1$ implies that $G(A) \oplus G(-B) = H \oplus H \in \mathcal{C}_2(H)$.

We have on one hand, $G(A) \cap G(-B) = \{0\}$ and then $N(A + B) = \{0\}$. On the other hand, if $a \in H$ there exists $x \in D(A)$ and $y \in D(B)$ such that

$$(0, a) = (x + y, Ax - By)$$

Thus, $x = -y \in D(A) \cap D(B)$ and $a = (A + B)x$.

Consequently,

$$\begin{aligned} R(A + B) &= H \\ N((A + B)^*) &= \{0\} \end{aligned}$$

By virtue of ii) Lemma 2.1, we have also $N(\overline{A^* + B^*}) = \{0\}$. It follows from Theorem 2.5 that $(A + B)^* = \overline{A^* + B^*}$.

Since 0 is outside the spectrum of $A + B$ we conclude that $(A + B)^{-1} \in B(H)$, and thus $G(A + B) \in \mathcal{C}_2(H)$. In fact, if $(x_n, (A + B)x_n)_n$ converge in $H \oplus H$ to (x, y), then $(x_n)_n \subset D(A) \cap D(B)$ converge to x in H and $((A + B)x_n)_n$ converge to $y = (A + B)z$ in H where $z \in D(A) \cap D(B)$. By continuity of $(A + B)^{-1}$ we deduce that $x = z$ and $(x, y = (A + B)x) \in G(A + B)$. □

Corollary 2.9. *If $A, B \in C(H)$ are such that $g\left(G(A), G(-B)^{\perp}\right) < 1$, then $(A^* + B^*) \in C(H)$ and $(A + B)^* = A^* + B^*$.*

Proof. is an immediate consequence of the previous Theorem since $g\left(G(A), G(-B)^{\perp}\right) = g\left(G(A^*), G(-B^*)^{\perp}\right).$ ☐

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