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SOME PROPERTIES OF QUARTER-SYMMETRIC NON-METRIC CONNECTION ON AN ALMOST CONTACT METRIC MANIFOLD

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Abstract. In the present paper, we have studied the properties of a quarter-symmetric non-metric connection in an almost contact metric manifold.

1. Introduction

The idea of quarter-symmetric linear connection in a differentiable manifold was introduced by Golab [6]. Various properties of quarter-symmetric metric connections have studied by [4], [11], [12], [14], [15], [16], [18], [21], [22] and many others. In 1980, Mishra and Pandey [10] defined and studied the quarter-symmetric metric F-connections in Riemannian, Kahlerian and Sasakian manifolds. In 2003, Sengupta and Biswas [19] defined quarter-symmetric non-metric connection in a Sasakian manifold and studied their properties. Recently present author with Ojha [3] defined a quarter-symmetric non-metric connection on an almost Hermitian manifold and have studied their different geometrical properties. In this series, the properties of quarter-symmetric non-metric connections have been studied by [1], [2], [5] and many others. In the present paper, we studied the properties of a quarter-symmetric non-metric connection in almost contact metric manifold. It has been also proved that the conformal curvature tensor and con-harmonic curvature tensor of the manifold coincide if and only if the scalar curvature with respect to the quarter-symmetric non-metric connection vanishes.

2. Preliminaries

An $n-$dimensional differentiable manifold M_n of differentiability class C^∞ with a $1-$form A, the associated vector field T and a $(1,1)$ tensor field F satisfying

$$(a) \quad \overline{\overline{X}} + X = A(X)T, \quad (b) \quad A(\overline{X}) = 0, \tag{2.1}$$

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for arbitrary vector field X is called an almost contact manifold and the system (M_n, F, A, T) is said to give an almost contact structure to M_n [9], [17]. In consequence of (2.1) (a) and (2.1) (b), we find

$$(a) \quad \overline{T} = 0 \quad (b) \quad A(T) = 1. \tag{2.2}$$

If the associated Riemannian metric g of type $(0, 2)$ in M_n satisfy

$$g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y), \tag{2.3}$$

for arbitrary vector fields X, Y in M_n, then (M_n, g) is said to be an almost contact metric manifold and the system (M_n, g, F, A, T) is said to give an almost contact metric structure to M_n [9], [17]. Putting T for X in (2.3) and using (2.2) (a) and (2.2) (b), we obtain

$$g(T, Y) = A(Y). \tag{2.4}$$

Also,

$$'F(X, Y) \stackrel{\text{def}}{=} g(\overline{X}, Y) \tag{2.5}$$

gives

$$'F(X, Y) +' F(Y, X) = 0. \tag{2.6}$$

If moreover,

$$(D_X F)(Y) = A(Y)X - g(X, Y)T, \tag{2.7}$$

where D denotes the Operator of covariant differentiation with respect to the Riemannian metric g, then the system (M_n, g, F, A, T) is called an almost Sasakian manifold [9], [17]. In almost Sasakian manifold, we also have

$$(a) \qquad 'F(X, Y) = (D_X A)(Y), \tag{2.8}$$

$$(b) \qquad D_X T = \overline{X}$$

and

$$(c) \quad 'K(X, Y, Z, T) = (D_Z{}'F)(X, Y).$$

The conformal curvature tensor V, con-harmonic curvature tensor L and concircular curvature tensor C in the manifold M_n are respectively defined as

$$V(X, Y, Z) = K(X, Y, Z) - \frac{1}{(n-2)}[Ric(Y, Z)X - Ric(X, Z)Y - g(X, Z)RY \tag{2.9}$$
$$+ g(Y, Z)RX] + \tfrac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y],$$

$$L(X, Y, Z) = K(X, Y, Z) - \frac{1}{(n-2)}[Ric(Y, Z)X - Ric(X, Z)Y - g(X, Z)RY + g(Y, Z)RX], \tag{2.10}$$

$$C(X, Y, Z) = K(X, Y, Z) - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \tag{2.11}$$

for arbitrary vector fields X, Y, Z [9]. Here K, Ric and r are respectively defined as curvature tensor, Ricci tensor and scalar curvature with respect to the Riemannian connection D.

The Nijenhuis tensor in an almost contact metric manifold [9] is defined as

$$\begin{aligned} 'N(X, Y, Z) &= (D_{\overline{X}}{}'F)(Y, Z) - (D_{\overline{Y}}{}'F)(X, Z) + (D_X{}'F)(Y, \overline{Z}) \\ &- (D_Y{}'F)(X, \overline{Z}) + A(Z)\{(D_X A)(Y) - (D_Y A)(X)\}, \end{aligned} \tag{2.12}$$

for arbitrary vector fields X, Y and Z.

An almost contact metric manifold (M_n, g) is said to be $\eta-$Einstein if its Ricci-tensor Ric takes the form

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.13}$$

for arbitrary vector fields X, Y; where a and b are functions on (M_n, g). If $b = 0$, then $\eta-$Einstein manifold becomes Einstein manifold. Kenmotsu [8] proved that if (M_n, g) is an $\eta-$Einstein manifold, then $a + b = -(n-1)$.

Definition- A vector field T is said to be a killing vector field [20] if it satisfies

$$(D_X A)(Y) + (D_Y A)(X) = 0 \tag{2.14}$$

and T is said to a harmonic vector field [20] if it satisfies

$$(D_X A)(Y) - (D_Y A)(X) = 0, \tag{2.15}$$

and

$$D_X T = 0, \tag{2.16}$$

for arbitrary vector fields X and Y.

3. Quarter-symmetric non-metric connection

A linear connection B on (M_n, g) defined as

$$B_X Y = D_X Y + A(Y)\overline{X}, \tag{3.1}$$

for arbitrary vector fields X and Y, is said to be a quarter-symmetric non-metric connection [3], [19]. The torsion tensor S of the connection B and the metric tensor g are given by

$$S(X,Y) = A(Y)\overline{X} - A(X)\overline{Y} \tag{3.2}$$

and

$$(B_X g)(Y,Z) = -A(Y)g(\overline{X}, Z) - A(Z)g(\overline{X}, Y), \tag{3.3}$$

for arbitrary vector fields X, Y, Z; where A is $1-$form on M_n with T as associated vector field, i.e., (2.4) is satisfied and D being the Riemannian connection.

Let us put (3.1) as

$$B_X Y = D_X Y + H(X,Y), \tag{3.4}$$

where

$$H(X,Y) = A(Y)\overline{X} \tag{3.5}$$

is a tensor field of type $(1,2)$. If we defined

$$'H(X,Y,Z) \overset{\text{def}}{=} g(H(X,Y),Z), \tag{3.6}$$

then in view of (2.4) and (3.5), (3.6) becomes

$$'H(X,Y,Z) = A(Y)g(\overline{X}, Z). \tag{3.7}$$

On an almost contact metric manifold (M_n, g), the following relations hold

$$(a) \quad H(T,Y) = H(X,T) - \overline{X} = 0, \tag{3.8}$$

$$(b) \quad 'H(X,T,Z) +' F(Z,X) = 0,$$

$$(c) \quad 'H(\overline{X},T,Z) +' H(X,T,\overline{Z}) = 0.$$

Theorem 3.1. *If an almost contact metric manifold admitting a quarter-symmetric non-metric connection B, then the vector field T is said to be a*
(a) killing vector field if

$$(B_X A)(Y) + (B_Y A)(X) = 0,$$

(b) harmonic vector field if

$$(B_X A)(Y) - (B_Y A)(X) = 0$$

and

$$B_X T = \overline{X}.$$

Proof. We have

$$X(A(Y)) = (B_X A)(Y) + A(B_X Y) \\ = (D_X A)(Y) + A(D_X Y),$$

then

$$(B_X A)(Y) - (D_X A)(Y) + A(B_X Y - D_X Y) = 0$$

In view of (2.1) (b) and (3.1), above equation gives

$$(B_X A)(Y) = (D_X A)(Y) \tag{3.9}$$

In consequence of (2.14) and (3.9), we find the first part of the theorem. Also, in view of (2.2) (b), (2.15), (2.16), (3.1) and (3.9), we obtain the second part of the theorem. □

Theorem 3.2. *If an almost contact metric manifold (M_n, g) equipped with a quarter-symmetric non-metric connection B satisfy $B_X{}'F = 0$, then M_n will be normal if and only if T is a harmonic vector field.*

Proof. Taking covariant derivative of $FY = \overline{Y}$ with respect to X, we find

$$(B_X F)(Y) + \overline{B_X Y} = B_X \overline{Y}$$

In consequence of (2.1) (a), (3.1) and $(D_X F)(Y) + \overline{D_X Y} = D_X \overline{Y}$, above equation becomes

$$(B_X F)(Y) = (D_X F)(Y) + A(Y)X - A(X)A(Y)T \tag{3.10}$$

If $B_X{}'F = 0$, then (3.10) gives

$$(D_X F)(Y) = -A(Y)X + A(X)A(Y)T \tag{3.11}$$

In consequence of (2.2) (b), (2.3), (2.4), (3.11) and $g((D_X F)(Y), Z) = (D_X{}'F)(Y, Z)$, we obtain

$$(D_X{}'F)(Y, Z) = -A(Y)g(\overline{X}, \overline{Z}) \tag{3.12}$$

Barring X in (3.12) and then using (2.1) (a), (2.1) (b) and (2.4), we get

$$(D_{\overline{X}}{}'F)(Y, Z) = A(Y)g(X, \overline{Z}) \tag{3.13}$$

Interchanging X and Y in (3.13), we find

$$(D_{\overline{Y}}{}'F)(X, Z) = A(X)g(Y, \overline{Z}) \tag{3.14}$$

Also barring Z in (3.12) and then using (2.1) (a), (2.1) (b) and (2.4), we obtain

$$(D_X{}'F)(Y, \overline{Z}) = A(Y)g(\overline{X}, Z) \tag{3.15}$$

Interchanging X and Y in (3.15), we have

$$(D_Y{}'F)(X, \overline{Z}) = A(X)g(\overline{Y}, Z) \tag{3.16}$$

In view of (2.12), (3.13), (3.14), (3.15) and (3.16), we obtain

$${}'N(X, Y, Z) = A(Z)\{(D_X A)(Y) - (D_Y A)(X)\} \tag{3.17}$$

An almost contact metric manifold is said to be normal if the Nijenhuis tensor ${}'N$ vanishes identically [13]. Hence in consequence of (2.15) and (3.17), we obtain the statement of the theorem. □

Corollary 3.3. *If an almost contact metric manifold (M_n, g) admitting a quarter-symmetric non-metric connection B satisfy $B_X{}'F = 0$, then M_n will be completely integrable.*

Since an almost contact metric manifold M_n will be completely integrable [9] if

$$'N(\overline{X}, \overline{Y}, \overline{Z}) = 0,$$

therefore (2.1)(b) and (3.17) gives the statement of the corollary.

Theorem 3.4. *If a Sasakian manifold admitting a quarter-symmetric non-metric connection B, then we have*

$$(a) \quad (B_{\overline{U}}{}'H)(X, \overline{Y}, Z) + (B_{\overline{Y}}{}'H)(X, \overline{U}, Z) = 0 \tag{3.18}$$

$$(b) \quad (B_T{}'H)(X, \overline{Y}, Z) = 0.$$

Proof. Barring Y in (3.7) and using (2.1) (b), we get

$$'H(X, \overline{Y}, Z) = 0 \tag{3.19}$$

Taking covariant derivative of (3.19) with respect to U and then using (3.19), we find

$$(B_U{}'H)(X, \overline{Y}, Z) +' H(X, (B_U F)(Y), Z) = 0 \tag{3.20}$$

Barring U in (3.20) and using (2.1) (b) and (3.8) (b), we obtain

$$(B_{\overline{U}}{}'H)(X, \overline{Y}, Z) = g(\overline{U}, Y)'H(X, T, Z) \tag{3.21}$$

Interchanging U and Y in (3.21) and then adding with (3.21), we obtain (3.18) (a). Also, replacing U by T in (3.20) and then using (2.2) (b), (2.4), (2.7) and (3.10), we easily find (3.18) (b). □

4. Curvature tensor with respect to the quarter-symmetric non-metric connection

The curvature tensor R of the affine connection B defined as

$$R(X, Y, Z) = B_X B_Y Z - B_Y B_X Z - B_{[X,Y]} Z$$

which satisfies

$$\begin{aligned} R(X, Y, Z) &= K(X, Y, Z) + (D_X A)(Z)\overline{Y} - (D_Y A)(Z)\overline{X} \\ &+ A(Z)\{(D_X F)(Y) - (D_Y F)(X)\}, \end{aligned} \tag{4.1}$$

where

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{4.2}$$

is the Riemannian curvature tensor of the Riemannian connection D.

If we define

$$'R(X, Y, Z, U) \stackrel{\text{def}}{=} g(R(X, Y, Z), U) \tag{4.3}$$

and

$$'K(X, Y, Z, U) \stackrel{\text{def}}{=} g(K(X, Y, Z), U) \tag{4.4}$$

then in view of (4.3) and (4.4), (4.1) becomes

$$\begin{aligned} R(X, Y, Z, U) &= K(X, Y, Z, U) + (D_X A)(Z)g(\overline{Y}, U) - (D_Y A)(Z)g(\overline{X}, U) \\ &+ A(Z)\{(D_X{}'F)(Y, U) - (D_Y{}'F)(X, U)\}, \end{aligned} \tag{4.5}$$

where

$$g((D_X F)(Y), U) = (D_X{}'F)(Y, U) \tag{4.6}$$

Interchanging X and Y in (4.5), we find

$$\begin{aligned} {}'R(Y,X,Z,U) &= {}'K(Y,X,Z,U) + (D_Y A)(Z)g(\overline{X},U) - (D_X A)(Z)g(\overline{Y},U) \\ &+ A(Z)\{(D_Y{}'F)(X,U) - (D_X{}'F)(Y,U)\}, \end{aligned} \tag{4.7}$$

Adding (4.5) and (4.7) and then using

$$'K(X,Y,Z,U) +' K(Y,X,Z,U) = 0,$$

we obtain

$$'R(X,Y,Z,U) +' R(Y,X,Z,U) = 0.$$

Theorem 4.1. *If an almost contact metric manifold M_n admitting a quarter-symmetric non-metric connection B whose curvature tensor vanishes, then the manifold M_n will be flat if and only if*

$$(D_X{}'H)(Y,Z,U) = (D_Y{}'H)(X,Z,U).$$

Proof. From (3.7), we have

$$'H(Y,Z,U) = A(Z)g(\overline{Y},U) \tag{4.8}$$

Taking covariant derivative of (4.8) with respect to X and then using (2.8) (a), (4.6) and (4.8), we find

$$(D_X{}'H)(Y,Z,U) = (D_X A)(Z)g(\overline{Y},U) + A(Z)(D_X{}'F)(Y,U) \tag{4.9}$$

Interchanging X and Y in (4.9), we get

$$(D_Y{}'H)(X,Z,U) = (D_Y A)(Z)g(\overline{X},U) + A(Z)(D_Y{}'F)(X,U) \tag{4.10}$$

Subtracting (4.10) from (4.9), we get

$$\begin{aligned} (D_X{}'H)(Y,Z,U) &- (D_Y{}'H)(X,Z,U) = (D_X A)(Z)g(\overline{Y},U) - (D_Y A)(Z)g(\overline{X},U) \\ &+ A(Z)\{(D_X{}'F)(Y,U) - (D_Y{}'F)(X,U)\}, \end{aligned} \tag{4.11}$$

In consequence of (4.11), (4.5) gives

$$'R(X,Y,Z,U) =' K(X,Y,Z,U) + (D_X{}'H)(Y,Z,U) - (D_Y{}'H)(X,Z,U)$$

Since the curvature tensor with respect to the connection B is zero, therefore above equation becomes

$$'K(X,Y,Z,U) = -(D_X{}'H)(Y,Z,U) + (D_Y{}'H)(X,Z,U) \tag{4.12}$$

From (4.12), it is clear that the manifold will be flat with respect to the Riemannian connection D if and only if

$$(D_X{}'H)(Y,Z,U) = (D_Y{}'H)(X,Z,U).$$

□

Theorem 4.2. *If a Sasakian manifold M_n admits a quarter-symmetric non-metric connection B, then the con-circular curvature tensor C coincide with the Riemannian curvature tensor if and only if the scalar curvature of the connection B vanishes.*

Proof. In view of (2.5), (2.7) and (2.8) (a), (4.1) becomes

$$\begin{aligned} R(X,Y,Z) &= K(X,Y,Z) + g(\overline{X},Z)\overline{Y} - g(\overline{Y},Z)\overline{X} \\ &+ A(Z)\{A(Y)X - A(X)Y\} \end{aligned} \tag{4.13}$$

Contracting (4.13) with respect to X and then using (2.1) (a), we get

$$\widehat{Ric}(Y,Z) = Ric(Y,Z) - g(Y,Z) + nA(Y)A(Z) \tag{4.14}$$

which becomes

$$\widehat{R}Y = RY - Y + nA(Y)T \tag{4.15}$$

and

$$\widehat{r} = r, \tag{4.16}$$

where $\widehat{Ric}$ and Ric are Ricci tensors with respect to the connections ∇ and D respectively and $\widehat{r}$ and r are scalar curvature with respect to ∇ and D respectively. In consequence of (2.11) and (4.16), we find

$$C(X,Y,Z) = K(X,Y,Z) - \frac{\widehat{r}}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{4.17}$$

If $C(X,Y,Z) = K(X,Y,Z)$, then from (4.17) we find $\widehat{r} = 0$. Converse part is obvious. □

Theorem 4.3. *If a Sasakian manifold M_n admitting a quarter-symmetric non-metric connection B whose Ricci tensor vanishes, then the manifold M_n will be η-Einstein manifold.*

Proof. Since the Ricci tensor with respect to the quarter-symmetric non-metric connection vanishes, therefore in view of $\widehat{Ric}(Y,Z) = 0$ and (4.14), we find

$$Ric(Y,Z) = g(Y,Z) - nA(Y)A(Z) \tag{4.18}$$

In consequence of (2.13) and (4.18), we obtain the required result. □

Theorem 4.4. *Let (M_n, g) be a Sasakian manifold equipped with a quarter-symmetric non-metric connection B, then the necessary and sufficient condition for conformal curvature tensor V and con-harmonic curvature tensor L of the manifold coincide is that the scalar curvature with respect to connection B vanishes.*

Proof. In consequence of (2.9), (2.10) and (4.16), we get

$$V(X,Y,Z) = L(X,Y,Z) + \frac{\widehat{r}}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \tag{4.19}$$

If $V(X,Y,Z) = L(X,Y,Z)$, then (4.19) gives $\widehat{r} = 0$, which gives the necessary part. Sufficient part is obvious from (4.19). □

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