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COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES DEPENDED ON ANOTHER FUNCTION

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Abstract. The purpose of this paper is to prove some coupled fixed point theorems for mappings having the mixed monotone property in partially ordered metric spaces depended on another function. These results are extensions of the main results of Bhaskar and Lakshmikantham [T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal.TMA 65(2006) 1379 - 1393]. We also give some examples to show that our results are effective.

1. Introduction

Existence of fixed points for contraction type mappings in partially ordered metric spaces has been considered recently by many authors (see, for details, [2]- [5], [7]- [24]) .

In [10], Bhaskar and Lakshmikantham introduced notions of a mixed monotone mapping and a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings and discuss the existence and uniqueness of the solutions for periodic boundary value problems. Afterward, some coupled fixed point theorems in partially ordered metric spaces were established by Lakshmikantham and Ciric [14], Luong and Thuan [15], [16], B. Samet [22] and others. The notions of mixed monotone mappings and coupled fixed points state as follows.

Definition 1.1. *([10]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non - decreasing in x and is monotone non - increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

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and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$$

Definition 1.2. *([10]) An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$x = F(x, y) \text{ and } y = F(y, x)$$

The main results of Bhaskar and Lakshmikantham in [10] are the following coupled fixed point theorems

Theorem 1.3. *([10]) Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \text{ for each } x \succeq u \text{ and } y \preceq v. \tag{1.1}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \text{ and } y = F(y, x)$$

Theorem 1.4. *([10]) Let $(X, d, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \text{ for each } x \succeq u \text{ and } y \preceq v.$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \text{ and } y = F(y, x)$$

Definition 1.5. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.*

In this paper we give some coupled fixed point theorems for mappings having the mixed monotone property in partially ordered metric spaces depended on another function which are generalization of the main results of Bhaskar and Lakshmikantham [10].

2. The main results

Let Φ denote all functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy $\varphi(t) < t$ and $\lim_{r\to t+}\varphi(r) < t$ for all $t > 0$.

Theorem 2.1. *Let $(X,\preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space and $T : X \to X$ is an ICS mapping. Let $F : X\times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0,y_0) \quad \text{and} \quad y_0 \succeq F(y_0,x_0)$$

Suppose there exists $\varphi \in \Phi$ such that

$$d(TF(x,y),TF(u,v)) \leq \frac{1}{2}\varphi\left(d(Tx,Tu)+d(Ty,Tv)\right), \tag{2.1}$$

for all $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.
then there exist $x,y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. We construct the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n,y_n) \ \text{ and } \ y_{n+1} = F(y_n,x_n), \ \text{ for all } n \geq 0. \tag{2.2}$$

We shall show that

$$x_n \preceq x_{n+1} \ \text{ for all } n \geq 0 \tag{2.3}$$

and

$$y_n \succeq y_{n+1} \ \text{ for all } n \geq 0 \tag{2.4}$$

Since $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$ and as $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$, we have $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Thus (2.3) and (2.4) hold for $n = 0$.

Suppose now that (2.3) and (2.4) hold for some $n \geq 0$. Then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, and by the mixed monotone property of F, we have

$$x_{n+1} = F(x_n,y_n) \preceq F(x_{n+1},y_n) \ \text{ and } \ F(y_{n+1},x_n) \preceq F(y_n,x_n) = y_{n+1} \tag{2.5}$$

and

$$x_{n+2} = F(x_{n+1},y_{n+1}) \succeq F(x_{n+1},y_n) \ \text{ and } \ F(y_{n+1},x_n) \succeq F(y_{n+1},x_{n+1}) = y_{n+2} \tag{2.6}$$

Now from (2.5) and (2.6), we obtain $x_{n+1} \preceq x_{n+2}$ and $y_{n+1} \succeq y_{n+2}$

Thus by the mathematical induction we conclude that (2.3) and (2.4) hold for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ... \tag{2.7}$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq ... \succeq y_n \succeq y_{n+1} \succeq ... \tag{2.8}$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, from (2.1) and (2.2), we have

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &= d(TF(x_n, y_n), TF(x_{n-1}, y_{n-1})) \\ &\le \frac{1}{2}\varphi\left(d(Tx_n, Tx_{n-1}) + d(Ty_n, Ty_{n-1})\right). \end{aligned} \tag{2.9}$$

Similarly, since $y_{n-1} \succeq y_n$ and $x_{n-1} \preceq x_n$, we have

$$\begin{aligned} d(Ty_n, Ty_{n+1}) &= d(TF(y_{n-1}, x_{n-1}), TF(y_n, x_n)) \\ &\le \frac{1}{2}\varphi\left(d(Ty_{n-1}, Ty_n) + d(Tx_{n-1}, Tx_n)\right). \end{aligned} \tag{2.10}$$

Adding (2.9) and (2.10), we obtain

$$d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1}) \le \varphi\left(d(Tx_{n-1}, Tx_n) + d(Ty_{n-1}, Ty_n)\right). \tag{2.11}$$

Set $d_n := d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1})$, we have

$$d_n \le \varphi\left(d_{n-1}\right) \tag{2.12}$$

Since $\varphi(t) < t$ for all $t > 0$, it follows from (2.12) that $\{d_n\}$ is a decreasing sequence of positive real numbers. Therefore, there exists some $d \ge 0$ such that

$$\lim_{n\to\infty}\left(d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1})\right) = \lim_{n\to\infty} d_n = d+$$

Assume that $d > 0$, taking $n \to \infty$ in two sides of (2.12) and using the property of φ, we have

$$d = \lim_{n\to\infty} d_n \le \lim_{n\to\infty} \varphi\left(d_{n-1}\right) = \lim_{d_{n-1}\to d+} \varphi\left(d_{n-1}\right) < d$$

which is a contradiction. Thus $d = 0$, that is,

$$\lim_{n\to\infty}\left(d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1})\right) = \lim_{n\to\infty} d_n = 0 \tag{2.13}$$

In what follows, we shall prove that $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least of $\{Tx_n\}$ or $\{Ty_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\left\{Tx_{n(k)}\right\}, \left\{Tx_{m(k)}\right\}$ of $\{Tx_n\}$ and $\left\{Ty_{n(k)}\right\}, \left\{Ty_{m(k)}\right\}$ of $\{Ty_n\}$ with $n(k) > m(k) \ge k$ such that

$$d\left(Tx_{n(k)}, Tx_{m(k)}\right) + d\left(Ty_{n(k)}, Ty_{m(k)}\right) \ge \varepsilon \tag{2.14}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such way that it is the smallest integer with $n(k) > m(k) \ge k$ satisfying (2.14). Then

$$d\left(Tx_{n(k)-1}, Tx_{m(k)}\right) + d\left(Ty_{n(k)-1}, Ty_{m(k)}\right) < \varepsilon \tag{2.15}$$

Using (2.14), (2.15) and the triangle inequality, we have

$$\begin{aligned} \varepsilon \le r_k &:= d\left(Tx_{n(k)}, Tx_{m(k)}\right) + d\left(Ty_{n(k)}, Ty_{m(k)}\right) \\ &\le d\left(Tx_{n(k)}, Tx_{n(k)-1}\right) + d\left(Tx_{n(k)-1}, Tx_{m(k)}\right) \\ &\quad + d\left(Ty_{n(k)}, Ty_{n(k)-1}\right) + d\left(Ty_{n(k)-1}, Ty_{m(k)}\right) \\ &\le d\left(Tx_{n(k)}, Tx_{n(k)-1}\right) + d\left(Ty_{n(k)}, Ty_{n(k)-1}\right) + \varepsilon \end{aligned}$$

Letting $k \to \infty$ and using (2.13)

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}\left[d\left(Tx_{n(k)}, Tx_{m(k)}\right) + d\left(Ty_{n(k)}, Ty_{m(k)}\right)\right] = \varepsilon+ \tag{2.16}$$

By the triangle inequality

$$\begin{aligned} r_k &= d\left(Tx_{n(k)}, Tx_{m(k)}\right) + d\left(Ty_{n(k)}, Ty_{m(k)}\right) \\ &\leq d\left(Tx_{n(k)}, Tx_{n(k)+1}\right) + d\left(Tx_{n(k)+1}, Tx_{m(k)+1}\right) + d\left(Tx_{m(k)+1}, Tx_{m(k)}\right) \\ &\quad + d\left(Ty_{n(k)}, Ty_{n(k)+1}\right) + d\left(Ty_{n(k)+1}, Ty_{m(k)+1}\right) + d\left(Ty_{m(k)+1}, Ty_{m(k)}\right) \\ &= d_{n(k)} + d_{m(k)} + d\left(Tx_{n(k)+1}, Tx_{m(k)+1}\right) + d\left(Ty_{n(k)+1}, Ty_{m(k)+1}\right) \quad (2.17) \end{aligned}$$

Since $n(k) > m(k)$, $x_{n(k)} \succeq x_{m(k)}$ and $y_{n(k)} \preceq y_{m(k)}$, from (2.1) and (2.2), we have

$$\begin{aligned} d(Tx_{n(k)+1}, Tx_{m(k)+1}) &= d(TF(x_{n(k)}, y_{n(k)}), TF(x_{m(k)}, y_{m(k)})) \\ &\leq \frac{1}{2}\varphi(d(Tx_{n(k)}, Tx_{m(k)}) + d(Ty_{n(k)}, Ty_{m(k)})) \end{aligned} \tag{2.18}$$

Similarly,

$$\begin{aligned} d(Ty_{n(k)+1}, Ty_{m(k)+1}) &= d(TF(y_{n(k)}, x_{n(k)}), TF(y_{m(k)}, x_{m(k)})) \\ &\leq \frac{1}{2}\varphi(d(Ty_{n(k)}, Ty_{m(k)}) + d(Tx_{n(k)}, Tx_{m(k)})) \end{aligned} \tag{2.19}$$

From (2.17), (2.18) and (2.19), we have

$$r_k \leq d_{n(k)} + d_{m(k)} + \varphi\left(d(Tx_{n(k)}, Tx_{m(k)}) + d(Ty_{n(k)}, Ty_{m(k)})\right) = d_{n(k)} + d_{m(k)} + \varphi(r_k)$$

Taking $k \to \infty$, and using (2.13), (2.16) and the property of φ, we have

$$\varepsilon = \lim_{k\to\infty} r_k \leq \lim_{k\to\infty} \left(d_{n(k)} + d_{m(k)} + \varphi(r_k)\right) = \lim_{r_k \to \varepsilon+} \varphi(r_k) < \varepsilon$$

which is a contradiction.
Therefore, $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in X. Since X is a complete metric space, $\{Tx_n\}$ and $\{Ty_n\}$ are convergent sequences.
Since T is an ICS mapping, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \text{ and } \lim_{n\to\infty} y_n = y. \tag{2.20}$$

Since T is continuous, we have

$$\lim_{n\to\infty} Tx_n = Tx, \text{ and } \lim_{n\to\infty} Ty_n = Ty. \tag{2.21}$$

Suppose that the assumption (a) holds. By (2.20), (2.2) and the continuity of F, we get

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y), \tag{2.22}$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x). \tag{2.23}$$

From (2.22) and (2.23), we get $x = F(x, y)$ and $y = F(y, x)$

Suppose that the assumption (b) holds. Since $\{x_n\}$ is non-decreasing and $x_n \to x$ and as $\{y_n\}$ is non-increasing and $y_n \to y$, by the assumption (b), we have $x_n \preceq x$ and $y_n \succeq y$ for all n. We have

$$\begin{aligned} d(Tx, TF(x,y)) &\leq d(Tx, Tx_{n+1}) + d(Tx_{n+1}, TF(x,y)) \\ &= d(Tx, Tx_{n+1}) + d(TF(x,y), TF(x_n, y_n)) \\ &\leq d(Tx, Tx_{n+1}) + \frac{1}{2}\varphi\left(d(Tx, Tx_n) + d(Ty, Ty_n)\right) \end{aligned}$$

So taking $n \to \infty$ yields $d(Tx, TF(x,y)) \leq 0$. Hence $Tx = TF(x,y)$
Similarly, we can show that $Ty = TF(y,x)$
Since T is an injective mapping, we have $x = F(x,y)$ and $y = F(y,x)$.
Thus we proved that F has a coupled fixed point. □

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

Suppose there exists $\varphi \in \Phi$ such that

$$d(F(x,y), F(u,v)) \leq \frac{1}{2}\varphi\left(d(x,u) + d(y,v)\right),$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.
then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point in X.

Proof. In the Theorem 2.1, taking $Tx = x$, for all $x \in X$, we get Corollary 2.2 □

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space and $T : X \to X$ is an ICS mapping. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

Suppose there exists $k \in [0,1)$ such that

$$d(TF(x,y), TF(u,v)) \leq \frac{k}{2}\left(d(Tx,Tu) + d(Ty,Tv)\right), \tag{2.24}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) F is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.
then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point in X.

Proof. In the Theorem 2.1, taking $\varphi(t) = kt$, for all $t \in [0, \infty)$, we get Corollary 2.3 □

Remark 2.4. *In Corollary 2.3, taking $Tx = x$, for all $x \in X$, we get the main results of Bhaskar and Lakshmikantham* [10] *(Theorem 1.3 and Theorem 1.4). Therefore, Corollary 2.3 is a generalization of Theorem 1.3 and Theorem 1.4 and so are Theorem 2.1 and Corollary 2.2.*

Now we shall prove the uniqueness of the coupled fixed point. Note that if $(X, \preceq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation:

$$\text{for } (x,y),(u,v) \in X \times X, \qquad (x,y) \lesssim (u,v) \Longleftrightarrow x \preceq u, y \succeq v.$$

Theorem 2.5. *In addition to the hypotheses of Theorem 2.1, suppose that for every $(x,y),(z,t) \in X \times X$, there exists a $(u,v) \in X \times X$ that is comparable to (x,y) and (z,t), then F has a unique coupled fixed point.*

Proof. From Theorem 2.1 the set of coupled fixed points is non-empty. Suppose (x,y) and (z,t) are coupled points of F, that is $x = F(x,y), y = F(y,x), z = F(z,t)$ and $t = F(t,z)$, we shall show that $x = z$ and $y = t$.
By assumption, there exists $(u,v) \in X \times X$ that is comparable to (x,y) and (z,t). We define the sequences $\{u_n\}$ and $\{v_n\}$ as follows

$$u_0 = u,\ v_0 = v,\ u_{n+1} = F(u_n, v_n) \quad \text{and} \quad v_{n+1} = F(v_n, u_n), \quad \text{for all } n$$

Since (u,v) is comparable with (x,y), we may assume that $(x,y) \gtrsim (u,v) = (u_0, v_0)$. Now we shall prove that

$$(x,y) \gtrsim (u_n, v_n), \quad \text{for all } n \tag{2.25}$$

Suppose that (2.25) holds for some $n \geq 0$. Then by the mixed monotone property of F, we have

$$u_{n+1} = F(u_n, v_n) \preceq F(x,y) = x,$$

and

$$v_{n+1} = F(v_n, u_n) \succeq F(y,x) = y,$$

that is, $(x,y) \gtrsim (u_{n+1}, v_{n+1})$. Therefore (2.25) holds.
From (2.1), we have

$$d(Tx, Tu_n) = d(TF(x,y), TF(u_{n-1}, v_{n-1})) \leq \frac{1}{2}\varphi\left(d(Tx, Tu_{n-1}) + d(Ty, Tv_{n-1})\right)$$

and

$$d(Tv_n, Ty) = d(TF(v_{n-1}, u_{n-1}), TF(y,x)) \leq \frac{1}{2}\varphi\left(d(Tv_{n-1}, Ty) + d(Tu_{n-1}, Tx)\right)$$

Adding to the above inequalites, we get

$$d(Tx, Tu_n) + d(Ty, Tv_n) \leq \varphi\left(d(Tx, Tu_{n-1}) + d(Ty, Tv_{n-1})\right) \tag{2.26}$$

Set $\delta_n = d(Tx, Tu_n) + d(Ty, Tv_n)$. It follows from (2.26) and the property of φ that $\{\delta_n\}$ is a monotone decreasing sequence of positive real numbers. Therefore there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta = \lim_{n\to\infty} [d(Tx, Tu_n) + d(Ty, Tv_n)] = \delta +$$

Suppose that $\delta > 0$, taking $n \to \infty$ in (2.26), we have

$$\begin{aligned} \delta = \lim_{n\to\infty} d(Tx, Tu_n) + d(Ty, Tv_n) \quad &\leq \quad \lim_{n\to\infty} \varphi\left(d(Tx, Tu_{n-1}) + d(Ty, Tv_{n-1})\right) \\ &= \quad \lim_{\delta_{n-1}\to\delta+} \varphi\left(\delta_{n-1}\right) < \delta \end{aligned}$$

which is a contradiction. Thus $\delta = 0$. Therefore,

$$\lim_{n\to\infty} [d(Tx, Tu_n) + d(Ty, Tv_n)] = 0$$

This implies

$$\lim_{n\to\infty} d(Tx, Tu_n) = \lim_{n\to\infty} d(Ty, Tv_n) = 0 \tag{2.27}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(Tz, Tu_n) = \lim_{n\to\infty} d(Tt, Tv_n) = 0 \tag{2.28}$$

From (2.27) and (2.28), we have $Tx = Tz$ and $Ty = Tt$ which imply $x = z$ and $y = t$ (since T is injective) $\square$

Theorem 2.6. *In addition to the hypotheses of Theorem 2.1, suppose that x_0, y_0 are comparable then $x = y$, that is, $x = F(x, x)$ or F has a fixed point.*

Proof. Let us assume that $x_0 \preceq y_0$. We shall show that

$$x_n \preceq y_n, \quad \text{for all } n. \tag{2.29}$$

where $x_n = F(x_{n-1}, y_{n-1}),\ y_n = F(y_{n-1}, x_{n-1}),\ n = 1, 2, 3, ...$
Suppose that (2.29) holds for some $n \geq 0$. Then by the mixed monotone property of F, we have

$$x_{n+1} = F(x_n, y_n) \preceq F(y_n, x_n) = y_{n+1}$$

Thus (2.29) holds. From (2.29) and (2.1), we have

$$d(TF(y_n, x_n), TF(x_n, y_n)) \leq \frac{1}{2}\varphi\left(2d(Ty_n, Tx_n)\right)$$

By the triangle inequality,

$$\begin{aligned} d(Ty, Tx) &\leq d(Ty, Ty_{n+1}) + d(Ty_{n+1}, Tx_{n+1}) + d(Tx_{n+1}, Tx) \\ &= d(TF(y_n, x_n), TF(x_n, y_n)) + d(Ty, Ty_{n+1}) + d(Tx_{n+1}, Tx) \\ &\leq \frac{1}{2}\varphi\left(2d(Ty_n, Tx_n)\right) + d(Ty, Ty_{n+1}) + d(Tx_{n+1}, Tx) \end{aligned} \tag{2.30}$$

Taking $n \to \infty$ in the above inequality, we have

$$d(Ty, Tx) \leq \frac{1}{2}\varphi\left(2d(Ty, Tx)\right)$$

which implies $d(Ty, Tx) = 0$ or $Ty = Tx$. Since T is injective, we get $x = y$. $\square$

The following example shows that Corollary 2.3, and Theorem 2.1 are both indeed proper extensions of Theorem 1.3 and Theorem 1.4.

Example 2.7. *Let $X = \{0, 1, 2, 8\}$ with the metric $d(x, y) = |x-y|$, for all $x, y \in X$. On the set X, we consider the following relation:*

$$for \ \ x, y \in X,\ \ x \preceq y \Leftrightarrow x = y \ or \ (x, y \in \{0, 1, 2\} \ and \ x \leq y),$$

where $\leq$ is the usual ordering.
Clearly, (X, d) is a complete metric space and $(X, \preceq)$ is a partially orderd set. Moreover, X has the property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$,
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$.

Set $A := \{(2,0),(2,1)\}$ *and* $B := \{(8,0),(8,1),(8,2)\}$
Let $F : X \times X \to X$ *be given by*

$$F(x,y) = \begin{cases} 1, & \text{if } (x,y) \in A \\ 8, & \text{if } (x,y) \in B \\ 0, & \text{if } (x,y) \in X^2 - (A \cup B) \end{cases}$$

Obviously, F has the mixed monotone property. Also, we have

$$0 \preceq 0 = F(0,2) \quad \text{and} \quad 2 \succeq 1 = F(2,0),$$

i.e. there exist $x_0 = 0$ and $y_0 = 2$ such that

$$x_0 \preceq F(x_0,y_0) \quad \text{and} \quad y_0 \succeq F(y_0,x_0)$$

The condition

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)], \forall x \succeq u, y \preceq v,$$

in the Theorem 1.3 and Theorem 1.4 is not true for every $k \in [0,1)$. Indeed, for $x = 2, y = 1, u = 1, v = 1$, we have

$$\frac{k}{2}[d(x,u)+d(y,v)] = \frac{k}{2}[d(2,1)+d(1,1)] = \frac{k}{2} < 1 = d(F(x,y),F(u,v)), \forall k \in [0,1).$$

So we can apply neither Theorem 1.3 nor Theorem 1.4 to the mapping F.
If we define the mapping $T : X \to X$ as follows:

$$T0 = 0, T1 = 1, T2 = 8, T8 = 2$$

then T is an ICS mapping.
Now, we show that F and T satisfy the condition (2.24) with $k = 4/7$. We have the following cases:
Case 1. $(x,y),(u,v) \notin \{(2,1),(2,0)\}$ or $(x,y) = (u,v)$ or $(x,y) = (2,0),(u,v) = (2,1)$, we have

$$d(TF(x,y),TF(u,v)) = 0 \leq \frac{4}{14}[d(Tx,Tu)+d(Ty,Tv)]$$

Case 2. $(x,y) = (2,1), (u,v) \in \{(2,2),(1,1),(1,2),(0,1),(0,2)\}$, we have

$$d(TF(x,y),TF(u,v)) = d(T1,T0) = d(1,0) = 1,$$

and

$$\frac{4}{14}[d(Tx,Tu)+d(Ty,Tv)] = \frac{4}{14}[|T2-Tu|+|T1-Tv|] \geq \frac{4}{14}.7 = 2 > 1$$

Case 3. $(x,y) = (2,0), (u,v) \in \{(2,2),(1,0),(1,1),(1,2),(0,0),(0,1),(0,2)\}$, we have

$$d(TF(x,y),TF(u,v)) = d(T1,T0) = d(1,0) = 1,$$

and

$$\frac{4}{14}[d(Tx,Tu)+d(Ty,Tv)] = \frac{2}{7}[|T2-Tu|+|T0-Tv|] = \frac{2}{7}[|8-Tu|+|Tv|] \geq \frac{2}{7}.8 > 1$$

Consequently, for all $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$,

$$d(TF(x,y),TF(u,v)) \leq \frac{4}{14}[d(Tx,Tu)+d(Ty,Tv)],$$

Therefore, applying Corollary 2.3, we can conclude that F has a coupled fixed point in X.
Notice that since there does not exist any $(x,y) \in X \times X$ such that (x,y) is

comparable with $(0,1)$ and $(0,8)$, F has more one coupled fixed points. In fact, F has three coupled fixed points which are $(0,0),(8,0)$, and $(0,8)$.

The following example shows that Theorem 2.5 is effective.

Example 2.8. *Let $X=[1,64]$ with the metric $d(x,y)=|x-y|$, for all $x,y\in X$ and the usual ordering. Then (X,d) is a complete metric space.*
Let $F:X\times X\to X$ be defined by

$$F(x,y)=8\sqrt[3]{\frac{x}{y}},\ \textit{for all } x,y\in X$$

then F has the mixed monotone property and F is continuous.
Let $T:X\to X$ be defined by

$$Tx=\ln x+1,\quad \textit{for all } x\in X$$

then T is an ICS mapping. Let $\varphi:[0,\infty)\to[0,\infty)$ be defined by

$$\varphi(t)=\frac{2}{3}t,\quad \textit{for all } t\in[0,\infty)$$

Then $\varphi\in\Phi$.
Taking $x,y,u,v\in X$ with $x\geq u$ and $y\leq v$, we have

$$\begin{aligned}
d(TF(x,y),TF(u,v)) &= d\left(\ln 8\sqrt[3]{\frac{x}{y}}+1,\ln 8\sqrt[3]{\frac{u}{v}}+1\right)\\
&= \frac{1}{3}\left|\ln x-\ln y-\ln u+\ln v\right|\\
&\leq \frac{1}{2}.\frac{2}{3}\left(|\ln x-\ln u|+|\ln y-\ln v|\right)\\
&= \frac{1}{2}\varphi\left(d(Tx,Tu)+d(Ty,Tv)\right)
\end{aligned}$$

Also, there exist $x_0=1,y_0=64$ such that

$$F(x_0,y_0)=F(1,64)=8\sqrt[3]{1/64}\geq 1=x_0$$

and

$$F(y_0,x_0)=F(64,1)=8\sqrt[3]{64/1}\leq 64=y_0$$

.
Evidently, for every $(x,y),(z,t)\in X\times X$, there always exists a $(u,v)\in X\times X$ that is comparable to (x,y) and (z,t). Therefore, all the conditions of Theorem 2.5 hold and $(8,8)$ is the unique coupled fixed point of F.
Notice that we can apply neither Theorem 1.3 nor Theorem 1.4 to this example because the condition (1.1) does not hold. Indeed, for $x=8,y=u=v=1$, then (1.1) becomes

$$d\left(8\sqrt[3]{\frac{8}{1}},8\sqrt[3]{\frac{1}{1}}\right)\leq\frac{k}{2}\left(d(8,1)+d(1,1)\right)$$

or

$$8\leq\frac{7k}{2}$$

a contradiction (since $k<1$).

Finally, we give a simple example which shows that if T is not an ICS mapping then the conclusion of Theorem 2.1 fails.

Example 2.9. *Let $X = \mathbb{R}$ with the usual metric and the usual ordering. Let $F : X \times X \to X$ be defined by*

$$F(x, y) = 2x - y + 1, \quad \text{for all } x, y \in X$$

then F has the mixed monotone property and F is continuous and there exist $x_0 = 1, y_0 = 0$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$
Let $\varphi : [0, \infty) \to [0, \infty)$ be defined by

$$\varphi(t) = \frac{t}{2}, \quad \text{for all } t \in [0, \infty)$$

then $\varphi \in \Phi$.
Let $T : X \to X$ be defined by

$$T(x) = 1, \quad \text{for all } x \in X$$

then T is not an ICS mapping. Obviously, the condition (2.1) holds. However, F has no coupled fixed point.

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