

**A UNIQUE COMMON FIXED POINT THEOREM FOR FOUR
MAPS UNDER $\psi - \phi$ CONTRACTIVE CONDITION IN
PARTIAL METRIC SPACES**

(COMMUNICATED BY SIMEON REICH)

K.P.R.RAO AND G.N.V.KISHORE

ABSTRACT. In this paper, we obtain a unique common fixed point theorem for four self maps satisfying $\psi - \phi$ contractive condition in partial metric spaces. Our result generalizes and improves a theorem of Altun et. al. in partial metric spaces.

1. INTRODUCTION

The notion of partial metric space was introduced by S.G.Matthews [1] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation ([2 - 9], etc).

S.G.Matthews [1], Sandra Oltra and Oscar Valero[10] and Salvador Romaguera [11] and I.Altun, Ferhan Sola, Hakan Simsek [12] prove fixed point theorems in partial metric spaces for a single map.

In this paper, we obtain a unique common fixed point theorem for four self mappings satisfying a generalized $\psi - \phi$ contractive condition in partial metric spaces. Our result generalizes and improves a theorem of Altun et. al.[12] and some known theorems in partial metric spaces.

First we recall some definitions and lemmas of partial metric spaces.

2. BASIC FACTS AND DEFINITIONS

Definition 2.1. [1]. *A partial metric on a nonempty set X is a function $p : X \times X \rightarrow R^+$ such that for all $x, y, z \in X$:*

$$(p_1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(p_2) \quad p(x, x) \leq p(x, y), p(y, y) \leq p(x, y),$$

$$(p_3) \quad p(x, y) = p(y, x),$$

$$(p_4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

(X, p) is called a partial metric space.

2000 *Mathematics Subject Classification.* 54H25, 47H10.

Key words and phrases. partial metric, weakly compatible maps, complete space.

©2008 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted February 14, 2011. Accepted June 7, 2011.

It is clear that $|p(x, y) - p(y, z)| \leq p(x, z) \forall x, y, z \in X$.
 Also clear that $p(x, y) = 0$ implies $x = y$ from (p_1) and (p_2) .
 But if $x = y$, $p(x, y)$ may not be zero. A basic example of a partial metric space is the pair (R^+, p) , where $p(x, y) = \max\{x, y\}$ for all $x, y \in R^+$.
 Each partial metric p on X generates τ_0 topology τ_p on X which has a base the family of open p - balls $\{B_p(x, \epsilon)/x \in X, \epsilon > 0\}$ for all $x \in X$ and $\epsilon > 0$, where $B_p(x, \epsilon) = \{y \in X/p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.
 If p is a partial metric on X , then the function $p^s : X \times X \rightarrow R^+$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X .

Definition 2.2. [1]. Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) is said to converge to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in (X, p) is said to be Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and is finite .

(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, w.r.to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

Lemma 2.3. [1]. Let (X, p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s) .

(b) (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

Note 2.4. If $\{x_n\}$ is converges to x in (X, p) , then $\lim_{n \rightarrow \infty} p(x_n, y) \leq p(x, y) \forall y \in X$.

Proof. Since $\{x_n\}$ converges to x we have $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x)$.

Now $p(x_n, y) \leq p(x_n, x) + p(x, y) - p(x, x)$

Letting $n \rightarrow \infty$,

$\lim_{n \rightarrow \infty} p(x_n, y) \leq \lim_{n \rightarrow \infty} p(x_n, x) + p(x, y) - p(x, x)$.

Thus $\lim_{n \rightarrow \infty} p(x_n, y) \leq p(x, y)$. □

3. MAIN RESULT

Theorem 3.1. Let (X, p) be a partial metric space and let $S, T, f, g : X \rightarrow X$ be such that

$$\psi(p(Sx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)), \forall x, y \in X, \tag{3.1}$$

where $\psi : [0, \infty) \rightarrow [0, \infty)$ is continuous , non-decreasing and $\phi : [0, \infty) \rightarrow [0, \infty)$ is lower semi continuous with $\phi(t) > 0$ for $t > 0$ and

$$M(x, y) = \max \{p(fx, gy), p(fx, Sx), p(gy, Ty), \frac{1}{2} [p(fx, Ty) + p(gy, Sx)]\}$$

$$S(X) \subseteq g(X), T(X) \subseteq f(X) \tag{3.2}$$

$$\text{either } f(X) \text{ or } g(X) \text{ is a complete subspace of } X \tag{3.3}$$

and

$$\text{the pairs } (f, S) \text{ and } (g, T) \text{ are weakly compatible.} \tag{3.4}$$

Then S, T, f and g have a unique common fixed point in X .

Proof. Let $x_0 \in X$. From (3.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Sx_{2n} = gx_{2n+1}, y_{2n+1} = Tx_{2n+1} = fx_{2n+2}, n = 0, 1, 2, \dots$

Case(i): Suppose $y_{2m} = y_{2m+1}$ for some m .

Assume that $y_{2m+1} \neq y_{2m+2}$.

$$M(x_{2m+2}, x_{2m+1}) = \max \left\{ \begin{array}{l} p(y_{2m+1}, y_{2m}), p(y_{2m+1}, y_{2m+2}), p(y_{2m}, y_{2m+1}), \\ \frac{1}{2} [p(y_{2m+1}, y_{2m+1}) + p(y_{2m}, y_{2m+2})] \end{array} \right\}$$

But $p(y_{2m+1}, y_{2m}) = p(y_{2m+1}, y_{2m+1}) \leq p(y_{2m+1}, y_{2m+2})$. *from*(p_2)
and

$$\begin{aligned} & \frac{1}{2} [p(y_{2m+1}, y_{2m+1}) + p(y_{2m}, y_{2m+2})] \\ & \leq \frac{1}{2} [p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2})] \text{ from } (p_4) \\ & \leq \frac{1}{2} [p(y_{2m+1}, y_{2m+2}) + p(y_{2m+1}, y_{2m+2})] \\ & = p(y_{2m+1}, y_{2m+2}). \end{aligned}$$

Hence $M(x_{2m+2}, x_{2m+1}) = p(y_{2m+1}, y_{2m+2})$.

From (3.1),

$$\begin{aligned} \psi(p(y_{2m+2}, y_{2m+1})) &= \psi(p(Sx_{2m+2}, Tx_{2m+1})) \\ &\leq \psi(M(x_{2m+2}, x_{2m+1})) - \phi(M(x_{2m+2}, x_{2m+1})) \\ &= \psi(p(y_{2m+2}, y_{2m+1})) - \phi(p(y_{2m+2}, y_{2m+1})) \\ &< \psi(p(y_{2m+2}, y_{2m+1})) \text{ since } \phi(t) > 0 \text{ if } t > 0. \end{aligned}$$

It is a contradiction. Hence $y_{2m+2} = y_{2m+1}$.

Continuing in this way, we can conclude that $y_n = y_{n+k}$ for all $k > 0$. Thus $\{y_n\}$ is a Cauchy sequence.

Case(ii) Assume that $y_n \neq y_{n+1}$ for all n .

Denote $p_n = p(y_n, y_{n+1})$.

$$\begin{aligned} \psi(p_{2n}) &= \psi(p(y_{2n}, y_{2n+1})) \\ &= \psi(p(Sx_{2n}, Tx_{2n+1})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) - \phi(M(x_{2n}, x_{2n+1})). \end{aligned}$$

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max \left\{ \begin{array}{l} p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n+1}), \\ \frac{1}{2} [p(y_{2n-1}, y_{2n+1}) + p(y_{2n}, y_{2n})] \end{array} \right\} \\ &= \max \{ p_{2n-1}, p_{2n} \} \text{ from } (p_4) \end{aligned}$$

Hence $\psi(p_{2n}) \leq \psi(\max \{p_{2n-1}, p_{2n}\}) - \phi(\max \{p_{2n-1}, p_{2n}\})$.

If p_{2n} is maximum then $\psi(p_{2n}) \leq \psi(p_{2n}) - \phi(p_{2n}) < \psi(p_{2n})$.

Hence

$$\begin{aligned} \psi(p_{2n}) &\leq \psi(p_{2n-1}) - \phi(p_{2n-1}) \\ &< \psi(p_{2n-1}). \end{aligned} \tag{3.5}$$

Since ψ is increasing we have $p_{2n} < p_{2n-1}$.

Similarly, we can show that $p_{2n-1} < p_{2n-2}$.

Thus $p_n < p_{n-1}, n = 1, 2, 3, \dots$

Thus $\{p_n\}$ is a non increasing sequence of non negative real numbers and must converge to a real number, say, $l \geq 0$.

Letting $n \rightarrow \infty$ in (3.5), we get

$\psi(l) \leq \psi(l) - \phi(l)$ so that $\phi(l) \leq 0$. Hence $l = 0$.

Thus

$$\lim_{n \rightarrow \infty} p(y_n, y_{n+1}) = 0 \tag{3.6}$$

Hence from (p_2),

$$\lim_{n \rightarrow \infty} p(y_n, y_n) = 0 \tag{3.7}$$

From (3.6) and (3.7), we have

$$\lim_{n \rightarrow \infty} p^s(y_n, y_{n+1}) = 0 \quad (3.8)$$

Now we prove that $\{y_{2n}\}$ is a Cauchy sequence in (X, p^s) . On contrary suppose that $\{y_{2n}\}$ is not Cauchy.

There exists an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$,

$$p^s(y_{2m_k}, y_{2n_k}) \geq \epsilon \quad (3.9)$$

and

$$p^s(y_{2m_k}, y_{2n_k-2}) < \epsilon \quad (3.10)$$

From (3.9),

$$\begin{aligned} \epsilon &\leq p^s(y_{2m_k}, y_{2n_k}) \\ &\leq p^s(y_{2m_k}, y_{2n_k-2}) + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}) \\ &< \epsilon + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}) \text{ from (3.10)} \end{aligned}$$

Letting $k \rightarrow \infty$ and using (3.8), we have

$$\lim_{k \rightarrow \infty} p^s(y_{2m_k}, y_{2n_k}) = \epsilon. \quad (3.11)$$

Hence from definition of p^s and from (3.7), we have

$$\lim_{k \rightarrow \infty} p(y_{2m_k}, y_{2n_k}) = \frac{\epsilon}{2}. \quad (3.12)$$

Letting $k \rightarrow \infty$ and using (3.11) and (3.8) in

$$|p^s(y_{2n_k+1}, y_{2m_k}) - p^s(y_{2m_k}, y_{2n_k})| \leq p^s(y_{2n_k+1}, y_{2n_k})$$

we get

$$\lim_{k \rightarrow \infty} p^s(y_{2n_k+1}, y_{2m_k}) = \epsilon. \quad (3.13)$$

Hence we have

$$\lim_{k \rightarrow \infty} p(y_{2n_k+1}, y_{2m_k}) = \frac{\epsilon}{2} \quad (3.14)$$

Letting $k \rightarrow \infty$ and using (3.11) and (3.8) in

$$|p^s(y_{2n_k}, y_{2m_k-1}) - p^s(y_{2n_k}, y_{2m_k})| \leq p^s(y_{2m_k-1}, y_{2m_k})$$

we get

$$\lim_{k \rightarrow \infty} p^s(y_{2n_k}, y_{2m_k-1}) = \epsilon. \quad (3.15)$$

Hence we have

$$\lim_{k \rightarrow \infty} p(y_{2n_k}, y_{2m_k-1}) = \frac{\epsilon}{2} \quad (3.16)$$

Letting $k \rightarrow \infty$ and using (3.15) and (3.8) in

$$|p^s(y_{2m_k-1}, y_{2n_k+1}) - p^s(y_{2m_k-1}, y_{2n_k})| \leq p^s(y_{2n_k+1}, y_{2n_k})$$

we get

$$\lim_{k \rightarrow \infty} p^s(y_{2m_k-1}, y_{2n_k+1}) = \epsilon. \quad (3.17)$$

Hence we have

$$\lim_{k \rightarrow \infty} p(y_{2m_k-1}, y_{2n_k+1}) = \frac{\epsilon}{2} \quad (3.18)$$

$$\begin{aligned}
& \psi(p(y_{2m_k}, y_{2n_k+1})) \\
&= \psi(p(Sx_{2m_k}, Tx_{2n_k+1})) \\
&\leq \psi\left(\max\left\{p(y_{2m_k-1}, y_{2n_k}), p(y_{2m_k-1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k+1}), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[p(y_{2m_k-1}, y_{2n_k+1}) + p(y_{2n_k}, y_{2m_k})]\right\}\right) \\
&\quad - \phi\left(\max\left\{p(y_{2m_k-1}, y_{2n_k}), p(y_{2m_k-1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k+1}), \right.\right. \\
&\quad \left.\left.\frac{1}{2}[p(y_{2m_k-1}, y_{2n_k+1}) + p(y_{2n_k}, y_{2m_k})]\right\}\right).
\end{aligned}$$

Letting $k \rightarrow \infty$ and using (3.14), (3.16), (3.6), (3.18) and (3.12), we get

$$\begin{aligned}
\psi\left(\frac{\epsilon}{2}\right) &\leq \psi\left(\max\left\{\frac{\epsilon}{2}, 0, 0, \frac{1}{2}\left[\frac{\epsilon}{2} + \frac{\epsilon}{2}\right]\right\}\right) - \phi\left(\max\left\{\frac{\epsilon}{2}, 0, 0, \frac{1}{2}\left[\frac{\epsilon}{2} + \frac{\epsilon}{2}\right]\right\}\right) \\
\psi\left(\frac{\epsilon}{2}\right) &\leq \psi\left(\frac{\epsilon}{2}\right) - \phi\left(\frac{\epsilon}{2}\right) < \psi\left(\frac{\epsilon}{2}\right).
\end{aligned}$$

It is a contradiction. Hence $\{y_{2n}\}$ is Cauchy.

Letting $n, m \rightarrow \infty$ in

$$\begin{aligned}
|p^s(y_{2n+1}, y_{2m+1}) - p^s(y_{2n}, y_{2m})| &\leq p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m}, y_{2m+1}) \\
\text{we get } \lim_{n, m \rightarrow \infty} p^s(y_{2n+1}, y_{2m+1}) &= 0.
\end{aligned}$$

Hence $\{y_{2n+1}\}$ is Cauchy.

Thus $\{y_n\}$ is a Cauchy sequence in (X, p^s) . Hence, we have $\lim_{n, m \rightarrow \infty} p^s(y_n, y_m) = 0$.

Now, from the definition of p^s and from (3.7), we have

$$\lim_{n, m \rightarrow \infty} p(y_n, y_m) = 0. \quad (3.19)$$

Suppose $f(X)$ is complete.

Since $\{y_{2n+1}\} \subseteq f(X)$ is a Cauchy sequence in the complete metric space $(f(X), p^s)$, it follows that $\{y_{2n+1}\}$ converges in $(f(X), p^s)$.

Thus $\lim_{n \rightarrow \infty} p^s(y_{2n+1}, v) = 0$ for some $v \in f(X)$.

There exists $t \in X$ such that $v = f(t)$.

Since $\{y_n\}$ is Cauchy in X and $\{y_{2n+1}\} \rightarrow v$, it follows that $\{y_{2n}\} \rightarrow v$.

From Lemma 2.3(b), we have

$$p(v, v) = \lim_{n \rightarrow \infty} p(y_{2n+1}, v) = \lim_{n \rightarrow \infty} p(y_{2n}, v) = \lim_{n, m \rightarrow \infty} p(y_n, y_m). \quad (3.20)$$

From (3.19) and (3.20), we have

$$p(v, v) = \lim_{n \rightarrow \infty} p(y_{2n+1}, v) = \lim_{n \rightarrow \infty} p(y_{2n}, v) = 0. \quad (3.21)$$

We now prove that $\lim_{n \rightarrow \infty} p(St, y_{2n}) = p(St, v)$.

$$p^s(St, y_{2n}) = 2p(St, y_{2n}) - p(St, St) - p(y_{2n}, y_{2n}).$$

Letting $n \rightarrow \infty$, we get

$$p^s(St, v) = 2 \lim_{n \rightarrow \infty} p(St, y_{2n}) - p(St, St) - 0 \text{ from (3.7)}.$$

$$2p(St, v) - p(St, St) - p(v, v) = 2 \lim_{n \rightarrow \infty} p(St, y_{2n}) - p(St, St).$$

$$p(St, v) = \lim_{n \rightarrow \infty} p(St, y_{2n}) \text{ from (3.21)}.$$

Let $St \neq v$.

$$\begin{aligned}
p(St, v) &\leq p(St, Tx_{2n+1}) + p(Tx_{2n+1}, v) - p(Tx_{2n+1}, Tx_{2n+1}) \\
&\leq p(St, Tx_{2n+1}) + p(y_{2n+1}, v)
\end{aligned}$$

$$\psi(p(St, v)) \leq \psi(p(St, Tx_{2n+1}) + p(y_{2n+1}, v)).$$

Letting $n \rightarrow \infty$, we have

$$\begin{aligned}
\psi(p(St, v)) &\leq \psi\left(\lim_{n \rightarrow \infty} p(St, Tx_{2n+1}) + 0\right) \\
&= \lim_{n \rightarrow \infty} \psi(p(St, Tx_{2n+1})) \\
&\leq \lim_{n \rightarrow \infty} [\psi(M(t, x_{2n+1})) - \phi(M(t, x_{2n+1}))].
\end{aligned}$$

$$M(t, x_{2n+1}) = \max\{p(v, y_{2n}), p(v, St), p(y_{2n}, y_{2n+1}), \frac{1}{2}[p(v, y_{2n+1}) + p(y_{2n}, St)]\} \\ \rightarrow p(v, St) \text{ as } n \rightarrow \infty, \text{ from (3.21), (3.6).}$$

Thus $\psi(p(St, v)) \leq \psi(p(St, v)) - \phi(p(St, v)) < \psi(p(St, v))$.

Hence $St = v$. Thus $St = v = ft$.

Since the pair (f, S) is weakly compatible, we have $fv = Sv$.

Suppose $Sv \neq v$

As in above, using the metric p^s and (3.7), (3.21), we can show that

$$p(Sv, v) = \lim_{n \rightarrow \infty} p(Sv, y_{2n}).$$

$$p(Sv, v) \leq p(Sv, Tx_{2n+1}) + p(Tx_{2n+1}, v) - p(Tx_{2n+1}, Tx_{2n+1}) \\ \leq p(Sv, Tx_{2n+1}) + p(y_{2n+1}, v)$$

$$\psi(p(Sv, v)) \leq \psi(p(Sv, Tx_{2n+1}) + p(y_{2n+1}, v)).$$

Letting $n \rightarrow \infty$, we have that

$$\psi(p(Sv, v)) \leq \psi\left(\lim_{n \rightarrow \infty} p(St, Tx_{2n+1}) + 0\right) \\ = \lim_{n \rightarrow \infty} \psi(p(St, Tx_{2n+1})) \\ \leq \lim_{n \rightarrow \infty} [\psi(M(v, x_{2n+1})) - \phi(M(v, x_{2n+1}))].$$

$$M(v, x_{2n+1}) = \max\left\{p(Sv, y_{2n}), p(Sv, Sv), p(y_{2n}, y_{2n+1}), \frac{1}{2}[p(Sv, y_{2n+1}) + p(y_{2n}, Sv)]\right\} \\ \rightarrow p(Sv, v) \text{ as } n \rightarrow \infty, \text{ from (3.6) and } (p_2).$$

Thus $\psi(p(Sv, v)) \leq \psi(p(Sv, v)) - \phi(p(Sv, v)) < \psi(p(Sv, v))$.

Hence $Sv = v$. Thus

$$fv = Sv = v. \quad (3.22)$$

Since $S(X) \subseteq g(X)$, there exists $w \in X$ such that $v = Sv = gw$.

Suppose $v \neq Tw$.

$$\psi(p(v, Tw)) = \psi(Sv, Tw) \\ \leq \psi(\max\{p(v, v), p(v, v), p(v, Tw), \frac{1}{2}[p(v, Tw) + p(v, v)]\}) \\ - \phi(\max\{p(v, v), p(v, v), p(v, Tw), \frac{1}{2}[p(v, Tw) + p(v, v)]\}) \\ = \psi(p(v, Tw)) - \phi(p(v, Tw)) \text{ from (3.21)} \\ < \psi(p(v, Tw)).$$

Hence $Tw = v$. Thus $gw = Tw = v$.

Since (g, T) is weakly compatible, we have $gv = Tv$.

Suppose $Tv \neq v$.

$$\psi(p(v, Tv)) = \psi(p(Sv, Tv)) \\ \leq \psi\left(\max\left\{p(v, Tv), p(v, v), p(Tv, Tv), \frac{1}{2}[p(v, Tv) + p(Tv, v)]\right\}\right) \\ - \phi\left(\max\left\{p(v, Tv), p(v, v), p(Tv, Tv), \frac{1}{2}[p(v, Tv) + p(Tv, v)]\right\}\right) \\ \leq \psi(p(v, Tv)) - \phi(p(v, Tv)) \text{ from (3.21) and } (p_2) \\ < \psi(p(v, Tv)).$$

Hence $Tv = v$. Thus

$$gv = Tv = v. \quad (3.23)$$

From (3.22) and (3.23), v is a common fixed point of f, g, S and T .

Let z be another common fixed point of f, g, S and T .

Suppose $v \neq z$.

$$\begin{aligned}
\psi(p(v, z)) &= \psi(p(Sv, Tz)) \\
&\leq \psi \left(\max \left\{ p(v, z), p(v, v), p(z, z), \frac{1}{2} [p(v, z) + p(z, v)] \right\} \right) \\
&\quad - \phi \left(\max \left\{ p(v, z), p(v, v), p(z, z), \frac{1}{2} [p(v, z) + p(z, v)] \right\} \right) \\
&= \psi(p(v, z)) - \phi(p(v, z)) \text{ from } (p_2) \\
&< \psi(p(v, z))
\end{aligned}$$

Hence $v = z$. Thus v is the unique common fixed point of f, g, S and T . \square

The following two simple examples illustrate our Theorem 3.1.

Example 3.2. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $f, g, S, T : X \rightarrow X$, $f(x) = \frac{x}{2}$, $g(x) = \frac{x}{3}$, $S(x) = \frac{x}{4}$ and $T(x) = \frac{x}{6}$, $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = t$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = \frac{t}{2}$. Then all conditions (3.1), (3.2), (3.3) and (3.4) are satisfied and 0 is unique common fixed point of f, g, S and T .

Example 3.3. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $f, g, S, T : X \rightarrow X$, $f(x) = \frac{x}{x+1}$, $g(x) = \frac{x}{x+2}$, $S(x) = \frac{x^2}{2x+2}$ and $T(x) = \frac{x^2}{2x+4}$, $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = t$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = \frac{t}{2}$. Then all conditions (3.2), (3.3) and (3.4) are satisfied and

$$\begin{aligned}
p(Sx, Ty) &= \max\left\{\frac{x^2}{2x+2}, \frac{y^2}{2y+4}\right\} \\
&\leq \frac{1}{2} \max\left\{\frac{x}{x+1}, \frac{y}{y+2}\right\} \\
&= \frac{1}{2} p(fx, gy) \\
&\leq \frac{1}{2} \max\{p(fx, gy), p(fx, Sx), p(gy, Ty), \frac{1}{2}[p(fx, Ty) + p(gy, Sx)]\}.
\end{aligned}$$

Clearly 0 is unique common fixed point of f, g, S and T .

Corollary 3.4. Theorem 3.1 holds with the condition (3.1) is replaced by

$$p(Sx, Ty) \leq \varphi \left(\max \left\{ p(fx, gy), p(fx, Sx), p(gy, Ty), \frac{1}{2} [p(fx, Ty) + p(gy, Sx)] \right\} \right) \quad (3.24)$$

$\forall x, y \in X$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is continuous and $\varphi(t) < t$ for $t > 0$.

Proof. Define $\psi(t) = t$ and $\phi(t) = t - \varphi(t) \forall t \geq 0$.

Then the condition (3.24) implies the condition (3.1). \square

Corollary 3.5. Let (X, p) be a complete partial metric space and $F : X \rightarrow X$ be a map such that

$$p(Fx, Fy) \leq \varphi \left(\max \left\{ p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2} [p(x, Fy) + p(y, Fx)] \right\} \right),$$

$\forall x, y \in X$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is continuous and $\varphi(t) < t$ for $t > 0$.

Then F has a unique fixed point in X .

Remark. Altun, Sola, Simsek [12] proved the corollary 3.5. with an additional condition on φ , namely, φ is non-decreasing.

Acknowledgments. The authors are thankful to the referees for their valuable suggestions.

REFERENCES

- [1] S.G. Matthews, *Partial metric topology*, in: *Proc. 8th Summer Conference on General Topology and Applications*, in: *Ann. New York Acad. Sci.* **Vol. 728**, 1994, pp. 183 - 197.
- [2] R. Heckmann, *Approximation of metric spaces by partial metric spaces*, *Appl. Categ. Structures.* **No.1-2, 7**, 1999, 71 - 83.

- [3] S.J. O'Neill, *Partial metrics, valuations and domain theory*, in: Proc. 11th Summer Conference on General Topology and Applications. **Vol 806** of Annals of the New York Academy of Sciences, 1996, pp. 304 - 315, The New York Academy of Sciences, New York, NY, USA.
- [4] S. Romaguera, M. Schellekens, *Partial metric monoids and semi valuation spaces*, Topology and Applications. **153**, no.5-6, 2005, 948 - 962.
- [5] S. Romaguera, O. Valero, *A quantitative computational model for complete partial metric space via formal balls*, Mathematical Structures in Computer Sciences. **Vol.19**, no.3, 2009, 541 - 563.
- [6] M. Schellekens, *The Smooth completion: a common foundation for denotational semantics and complexity analysis*, Electronic Notes in Theoretical Computer Science. **Vol.1**, 1995, 535 - 556.
- [7] M. Schellekens, *A characterization of partial metrizable domains are quantifiable*, Theoretical Computer Sciences. **Vol 305**, no. 1-3, 2003, 409 - 432.
- [8] P. Waszkiewicz, *Quantitative continuous domains*, Applied Categorical Structures. **Vol 11**, no. 1, 2003, 41 - 67.
- [9] P. Waszkiewicz, *Partial metrizable continuous posets*, Mathematical Structures in Computer Sciences. **Vol 16**, no. 2, 2006, 359 - 372.
- [10] Sandra Oltra, Oscar Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Istit. Mat. Univ. Trieste. **Vol XXXVI**, 2004, 17 - 26.
- [11] Salvador Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory. **Vol.2010**, Article ID 493298, 6 pages, doi:10.1155/2010/493298.
- [12] Ishak Altun, Ferhan Sola, Hakam Simsek, *Generalized contractions on partial metric spaces*, Topology and its Applications. **157**, (2010), 6, 2778 - 2785.

K.P.R.RAO

DEPARTMENT OF APPLIED MATHEMATICS,
ACHARYA NAGARJUNA UNIVERSITY-DR.M.R.APPA ROW CAMPUS,
NUZVID- 521 201, KRISHNA DISTRICT, ANDHRA PRADESH, INDIA
E-mail address: kprrao2004@yahoo.com

G.N.V.KISHORE

DEPARTMENT OF MATHEMATICS,
SWARNANDHRA INSTITUTE OF ENGINEERING AND TECHNOLOGY, SEETHARAMPURAM,
NARSIPUR- 534 280, WEST GODAVARI DISTRICT, ANDHRA PRADESH, INDIA
E-mail address: kishore.apr2@gmail.com