Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 4(2011), Pages 201-212.

DOUBLE TRIGONOMETRIC SERIES AND ZYGMUND CLASSES OF FUNCTIONS WITH TWO VARIABLES

(COMMUNICATED BY HÜSEIN BOR)

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Abstract. In the present paper, we generalize Zygmund classes of functions with two variables defined by Móricz by means of modulus of continuity, and give the necessary and sufficient conditions for double sine, sine-cosine, cosine-sine and double cosine series so that their sums belong to the generalized Zygmund classes. Some new results of Fülöp [1] and [2] on double trigonometric series are extended.

1. Introduction

Given a double sequence $\{a_{ij}, i, j, = 1, 2, ...\}$ of nonnegative numbers satisfying

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} < \infty, \tag{1.1}$$

then the following double trigonometric series (called double sine, sine-cosine, cosine-sine,and double cosine series, respectively)

$$f_{11}(x, y) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \sin ix \sin jy,$$

$$f_{12}(x, y) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \sin ix \cos jy,$$

$$f_{21}(x, y) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \cos ix \sin jy,$$

$$f_{22}(x, y) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \cos ix \cos jy$$

are continuous, due to uniform convergence.

2000 *Mathematics Subject Classification.* 42A16; 42B05.
Key words and phrases. Double trigonometric series; Zygmund classes.

Submitted Oct 17, 2011. Published Nov 20, 2011.

For any continuous function $f(x,y)$, $2\pi-$periodic in each variable, in symbols $f \in C\left(\mathbb{T}^2\right)$, Móricz defined the double Zygmund classes $\Lambda_*(2)$ and little Zygmund classes $\lambda_*(2)$ as follows (see [1] and [2] for details):

$$\Lambda_*(2) := \left\{f(x,y) : |\Delta^{2,2}(f;x,y;h,k)| \le Khk, \quad h>0, k>0\right\},$$
$$\lambda_*(2) := \left\{f(x,y) : \lim_{h\to 0, k\to 0} |\Delta^{2,2}(f;x,y;h,k)| = 0, \quad h>0, k>0\right\},$$

where

$$\begin{aligned}\Delta^{2,2}(f;x,y;h,k) :=& f(x+h,y+k)+f(x-h,y+k)+f(x+h,y-k)+f(x-h,y-k)\\ &-2f(x,y+k)-2f(x,y-k)-2f(x+h,y)-2f(x-h,y)+4f(x,y).\end{aligned}$$

Recently, Fülöp ([1], [2]) obtained the necessary and sufficient conditions for the double trigonometric series belonging to the the double Zygmund classes $\Lambda_*(2)$ and the little Zygmund classes $\lambda_*(2)$. Combining the results of Fülöp in [1] and [2], we actually have

Theorem F. Let $\{a_{ij}, i, j, = 1, 2, ...\}$ be a double sequence of nonnegative numbers such that satisfying (1.1), then

(1) $f_{pq} \in \Lambda_*(2), p, q = 1, 2,$ if and only if

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} a_{ij} = O\left(\frac{1}{mn}\right), \qquad m,n = 1,2,....$$

(2) $f_{pq} \in \lambda_*(2), p, q, = 1, 2,$ if and only if

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} a_{ij} = o\left(\frac{1}{mn}\right), \qquad m,n \to \infty.$$

Let $\omega(h,k)$ be a modulus of continuity, that is, $\omega(h,k)$ is a continuous function on the square $[0,2\pi]\times[0,2\pi]$,nondecreasing in each variable,and possesses the following properties:

$$\begin{aligned}&\omega(0,0) = 0,\\ &\omega(t_1+t_2,t_3) \le \omega(t_1,t_3)+\omega(t_2,t_3),\\ &\omega(t_1,t_2+t_3) \le \omega(t_1,t_2)+\omega(t_1,t_3).\end{aligned}$$

By adopting the modulus of smoothness, we can further generalize the double Zygmund classes $\Lambda_*(2)$ and the little Zygmund classes $\lambda_*(2)$ as follows

$$\Lambda_*^{\omega}(2) := \left\{f : |\Delta^{2,2}(f;x,y;h,k)| = O\left(\omega(h,k)\right) \quad h>0, k>0\right\},$$
$$\lambda_*^{\omega}(2) := \left\{f : |\Delta^{2,2}(f;x,y;h,k)| = o\left(\omega(h,k)\right) \quad h>0, k>0\right\}.$$

When $\omega(h,k) = hk$, $\Lambda_*^{\omega}(2)$ and $\lambda_*^{\omega}(2)$ reduce to $\Lambda_*(2)$ and $\lambda_*(2)$ respectively.

The main purpose of the present paper is to generalize Theorem F to the generalized classes $\Lambda_*^{\omega}(2)$ and $\lambda_*^{\omega}(2)$. Our main results are presented in Section 2, and the proofs are given in Section 3.

2. Main results

In what follows, we always assume that $\{a_{ij}\}$ is a double sequence of nonnegative numbers satisfying (1.1). We first give a sufficient condition for $f_{pq} \in \Lambda_*^\omega(2), p, q = 1, 2$.

Theorem 2.1. *If*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^2 j^2 a_{ij} = O\left(m^2 n^2 \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.1}$$

$$\sum_{i=1}^{m}\sum_{j=n+1}^{\infty} i^2 a_{ij} = O\left(m^2 \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.2}$$

$$\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} j^2 a_{ij} = O\left(n^2 \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.3}$$

$$\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij} = O\left(\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.4}$$

then $f_{pq} \in \Lambda_*^\omega(2), p, q = 1, 2$.

We have the following necessary conditions for $f_{pq} \in \Lambda_*^\omega(2), p, q = 1, 2:$

Theorem 2.2. *If* $f_{pq} \in \Lambda_*^\omega(2), p, q = 1, 2,$ *then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^{\beta(p)} j^{\gamma(q)} a_{ij} = O\left(m^{\beta(p)} n^{\gamma(q)} \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.5}$$

where $\beta(1) = 3, \beta(2) = 2$.

If $\omega(u, v)$ satisfies some further conditions, we can obtain the necessary and sufficient conditions for $f_{pq} \in \Lambda_*^\omega(2), p, q = 1, 2$. In fact, we have the following Theorem 2.3-Theorem 2.5.

Theorem 2.3. *If* $\omega\left(\frac{1}{m}, \frac{1}{n}\right)$ *satisfies the following conditions*

$$\sum_{i=m}^{\infty} \frac{1}{i}\omega\left(\frac{1}{i}, \frac{1}{n}\right) = O\left(\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.6}$$

$$\sum_{j=n}^{\infty} \frac{1}{j}\omega\left(\frac{1}{m}, \frac{1}{j}\right) = O\left(\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.7}$$

for all $m, n = 1, 2, ...,$ *then* $f_{22} \in \Lambda_*^\omega(2)$ *if and only if (2.1) holds.*

Theorem 2.4. *If* $\omega\left(\frac{1}{m}, \frac{1}{n}\right)$ *satisfies (2.6), (2.7) and the following two further conditions*

$$\sum_{i=1}^{m} i\omega\left(\frac{1}{i}, \frac{1}{n}\right) = O\left(m^2\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.8}$$

$$\sum_{j=1}^{n} j\omega\left(\frac{1}{m}, \frac{1}{j}\right) = O\left(n^2\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.9}$$

for all $m, n = 1, 2, ...,$ *then* $f_{11} \in \Lambda_*^\omega(2)$ *if and only if (2.4) holds.*

Theorem 2.5. *(i) If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (2.6), (2.7), (2.9), then $f_{21}\in\Lambda_*^{\omega}(2)$ if and only if (2.2) holds.*

(ii) If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (2.6), (2.7) and (2.8), then $f_{12}\in\Lambda_*^{\omega}(2)$ if and only if (2.3) holds.

Now, we give some useful corollaries of Theorem 2.3-Theorem 2.5.
A nonnegative sequence $\{a_n\}$ is said to be almost decreasing (increasing), if there is a positive constant C such that

$$a_m \le Ca_n \ (a_m \ge Ca_n) \quad \text{for all } m \ge n.$$

Corollary 2.6. *If there are μ_1,ν_1 $(0<\mu_1,\nu_1\le 2)$ such that $\left\{m^{\mu_1}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ and $\left\{n^{\nu_1}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ are almost decreasing on m and n respectively, then*

(i). $f_{22}\in\Lambda_*^{\omega}(2)$ if and only if (2.1) holds.

(ii). If there are μ_2,ν_2 $(0<\mu_2,\nu_2<2)$ such that $\left\{m^{\mu_2}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ and $\left\{n^{\nu_2}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ are almost increasing on m and n respectively, then $f_{11}\in HH^{\omega}$ if and only if (2.4) holds.

(iii). If there is a ν_3 $(0<\nu_3<2)$ such that $\left\{n^{\nu_3}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ is almost increasing on n, then $f_{21}\in\Lambda_*^{\omega}(2)$ if and only if (2.2) holds.

(iv). If there are μ_3 $(0<\mu_3<2)$ such that $\left\{m^{\mu_3}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ is almost increasing on m, then $f_{12}\in\Lambda_*^{\omega}(2)$ if and only if (2.3) holds.

By taking $\omega(h,k)=h^{\alpha}k^{\beta}$, from Corollary 2.6, we immediately have the following

Corollary 2.7. *(i)If $\omega(h,k)=h^{\alpha}k^{\beta}$ $\quad(0<\alpha,\beta\le 2)$, then $f_{22}\in\Lambda_*^{\omega}(2)$ if and only if*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^2j^2a_{ij}=O\left(m^{2-\alpha}n^{2-\beta}\right).$$

(ii) If $\omega(h,k)=h^{\alpha}k^{\beta}$ $\quad(0<\alpha,\beta<2)$, then $f_{11}\in\Lambda_*^{\omega}(2)$ if and only if

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} a_{ij}=O\left(m^{-\alpha}n^{-\beta}\right).$$

(iii)If $\omega(h,k)=h^{\alpha}k^{\beta}$ $\quad(0<\alpha\le 2, 0<\beta<2)$, then $f_{21}\in\Lambda_*^{\omega}(2)$ if and only if

$$\sum_{i=1}^{m}\sum_{j=n}^{\infty} i^2a_{ij}=O\left(m^{2-\alpha}n^{-\beta}\right).$$

(iv)If $\omega(h,k)=h^{\alpha}k^{\beta}$ $\quad(0<\alpha<2, 0<\beta\le 2)$, then $f_{12}\in\Lambda_*^{\omega}(2)$ if and only if

$$\sum_{i=m}^{\infty}\sum_{j=1}^{n} j^2a_{ij}=O\left(m^{-\alpha}n^{2-\beta}\right).$$

For the little double Zygmund classes $\lambda_*^{\omega}(2)$, we have the following Theorem 2.8 and Theorem 2.9.

Theorem 2.8. *If*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^2 j^2 a_{ij} = o\left(m^2 n^2 \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.10}$$

$$\sum_{i=1}^{m}\sum_{j=n+1}^{\infty} i^2 a_{ij} = o\left(m^2 \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.11}$$

$$\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} j^2 a_{ij} = o\left(n^2 \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.12}$$

$$\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij} = o\left(\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{2.13}$$

then $f_{pq} \in \lambda_*^{\omega}(2), p, q = 1, 2.$

Theorem 2.9. *If* $f_{pq} \in \lambda_*^{\omega}(2), p, q = 1, 2,$ *then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^{\beta(p)} j^{\beta(q)} a_{ij} = o\left(m^{\beta(p)} n^{\beta(q)} \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

where $\beta(1) = 3, \beta(2) = 2.$

Remark 1. When 'O' is replaced by 'o', the corresponding results of Theorem 2.3-Theorem 2.5, Corollary 2.6 and Corollary 2.7 still hold.

3. Auxiliary results

Lemma 3.1. *If* $\omega\left(\frac{1}{m}, \frac{1}{n}\right)$ *satisfies (2.6), (2.7), then, for any* $\delta, \eta \geq 2$,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^{\delta} j^{\eta} a_{ij} = O\left(m^{\delta} n^{\eta} \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{3.1}$$

implies

$$\sum_{i=1}^{m}\sum_{j=n+1}^{\infty} i^{\delta} a_{ij} = O\left(m^{\delta} \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{3.2}$$

$$\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} j^{\eta} a_{ij} = O\left(n^{\eta} \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \tag{3.3}$$

$$\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij} = O\left(\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right). \tag{3.4}$$

Proof. (A) Let N be integer for which $2 \leq n < N$, by Abel's transformation and (3.1), we have

$$\begin{aligned}\sum_{i=1}^{m} i^{\delta} \sum_{j=n}^{N} a_{ij} &= \sum_{i=1}^{m} i^{\delta} \sum_{j=n}^{N} j^{-\eta}(j^{\eta} a_{ij}) \\ &= \sum_{i=1}^{m} i^{\delta}\left(-\frac{1}{n^{\eta}}\sum_{j=1}^{n-1} j^{\eta} a_{ij} + \sum_{j_1=n}^{N-1}\left(\frac{1}{j_1^{\eta}} - \frac{1}{(j_1+1)^{\eta}}\right)\sum_{j=1}^{j_1} j^{\eta} a_{ij} + \frac{1}{N^{\eta}}\sum_{j=1}^{N} j^{\eta} a_{ij}\right)\end{aligned}$$

$$
\begin{aligned}
&= O\left(\sum_{j_1=n}^{N-1}\frac{1}{j_1^{\eta+1}}\sum_{i=1}^{m}\sum_{j=1}^{j_1} i^\delta j^\eta a_{ij} + \frac{1}{N^\eta}\sum_{i=1}^{m}\sum_{j=1}^{N} i^\delta j^\eta a_{ij}\right)\\
&= O\left(m^\delta \sum_{j_1=n}^{N-1}\frac{1}{j_1}\omega\left(\frac{1}{m},\frac{1}{j_1}\right) + m^\delta\omega\left(\frac{1}{m},\frac{1}{n}\right)\right).
\end{aligned}
$$

Letting N tend to ∞, by (2.7), we get (3.2).
By using (2.6) instead of (2.7), we have (3.3).
Let M and N be integers for which $2 \le n < N, 2 \le m < M$,by using Abel's transformation twice,we obtain that

$$
\begin{aligned}
\sum_{i=m}^{M}\sum_{j=n}^{N} a_{ij} \le{}& \sum_{j_1=n}^{N-1}\frac{1}{j_1^{\eta+1}}\sum_{i_1=n}^{M-1}\frac{1}{i_1^{\delta+1}}\sum_{j=1}^{j_1}\sum_{i=1}^{i_1} j^\eta i^\delta a_{ij} + \frac{1}{M^\delta}\sum_{j_1=n}^{N-1}\frac{1}{j_1^{\eta+1}}\sum_{j=1}^{j_1}\sum_{i=1}^{M} j^\eta i^\delta a_{ij}\\
&+ \frac{1}{N^\eta}\sum_{i_1=n}^{M-1}\frac{1}{i_1^{\delta+1}}\sum_{j=1}^{N}\sum_{i=1}^{i_1} j^\eta i^\delta a_{ij} + \frac{1}{M^\delta N^\eta}\sum_{j=1}^{N}\sum_{i=1}^{M} j^\eta i^\delta a_{ij}.
\end{aligned}
$$

Letting M and N tend to ∞, by (3.1), (2.6) and (2.7), we obtain

$$
\begin{aligned}
\sum_{i=m}^{\infty}\sum_{j=n}^{\infty} a_{ij} &= O\left(\sum_{i_1=m}^{\infty}\sum_{j_1=n}^{\infty}\frac{1}{i_1 j_1}\omega\left(\frac{1}{i_1},\frac{1}{j_1}\right) + \sum_{j_1=n}^{\infty}\frac{1}{j_1}\omega\left(\frac{1}{m},\frac{1}{j_1}\right) + \sum_{i_1=m}^{\infty}\frac{1}{i_1}\omega\left(\frac{1}{i_1},\frac{1}{n}\right) + \omega\left(\frac{1}{m},\frac{1}{n}\right)\right)\\
&= O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),
\end{aligned}
$$

which proves (3.4). □

Lemma 3.2. *If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (2.8) and (2.9), then, for any $\delta,\eta \ge 2$, (3.4) implies (3.1)-(3.3).*

Proof. Let M and N be integers for which $1 \le m < M, 1 \le n < N$, by Abel's transformation, we conclude that

$$
\begin{aligned}
\sum_{i=1}^{m} i^\delta \sum_{j=1}^{n} j^\eta a_{ij} &= \sum_{i=1}^{m} i^\delta\left(\sum_{j=j_1}^{n}\left(j_1^\eta - (j_1-1)^\eta\right)\sum_{j=j_1}^{N} a_{ij} - n^\eta \sum_{j=n+1}^{N}\right)\\
&\le \sum_{j_1=1}^{n}\eta j_1^{\eta-1}\sum_{j=j_1}^{N}\sum_{i=1}^{m} i^\delta a_{ij}\\
&= \sum_{j_1=1}^{n}\eta j_1^{\eta-1}\sum_{j=j_1}^{N}\left(\sum_{i_1=1}^{m}\left(i_1^\delta - (i_1-1)^\delta\right)\sum_{i=i_1}^{M} a_{ij} - m^\delta\sum_{i=m+1}^{M} a_{ij}\right)\\
&\le \sum_{j_1=1}^{n}\eta j_1^{\eta-1}\sum_{i_1=1}^{m}\delta i_1^{\delta-1}\sum_{j=j_1}^{N}\sum_{i=i_1}^{M} a_{ij}.
\end{aligned}
$$

Letting M and N tend to ∞, by (3.4), (2.6) and (2.7), we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n} i^{\delta} j^{\eta} a_{ij} &= O\left(\sum_{j_1=1}^{n}\sum_{i_1=1}^{m} i_1^{\delta-1} j_1^{\eta-1}\omega\left(\frac{1}{i_1},\frac{1}{j_1}\right)\right)\\ &= O\left(m^{\delta-2}n^{\eta-2}\sum_{j_1=1}^{n} j_1 \sum_{i_1=1}^{m} i_1\omega\left(\frac{1}{i_1},\frac{1}{j_1}\right)\right)\\ &= O\left(m^{\delta}n^{\eta-2}\sum_{j_1=1}^{n} j_1\omega\left(\frac{1}{m},\frac{1}{j_1}\right)\right)\\ &= O\left(m^{\delta}n^{\eta}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),\end{aligned}$$

which proves (3.1)
By Abel's transformation again, we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=n}^{\infty} i^{\delta} a_{ij} &\leq \sum_{j=n}^{\infty}\left(\sum_{i_1=1}^{m}\left(i_1^{\delta}-(i_1-1)^{\delta}\right)\sum_{i=i_1}^{M} a_{ij} - m^{\delta}\sum_{i=m+1}^{M} a_{ij}\right)\\ &\leq \sum_{i_1=1}^{m}\delta i_1^{\delta-1}\sum_{i=i_1}^{M}\sum_{j=n}^{\infty} a_{ij}.\end{aligned}$$

Letting M tend to ∞, by (2.8), we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=n}^{\infty} i^{\delta} a_{ij} &= O\left(m^{\delta-2}\sum_{i_1=1}^{m} i_1\omega\left(\frac{1}{i_1},\frac{1}{n}\right)\right)\\ &= O\left(m^{\delta}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right).\end{aligned}$$

Thus, (3.2) is proved.
In a similar way to the proof of (3.2), we have (3.3). $\square$

In similar ways to the proofs of Lemma 3.1 and Lemma 3.2, we can prove the following two lemmas.

Lemma 3.3. *If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (2.6) and (2.9), then, for any $\delta,\eta \geq 2$, (3.2) implies (3.1), (3.3) and (3.4).*

Lemma 3.4. *If $\omega\left(\frac{1}{m},\frac{1}{n}\right)$ satisfies (2.7), (2.8), then, for any $\delta,\eta \geq 2$, (3.3) implies (3.1), (3.2) and (3.4).*

Remark 2. When 'O' is replaced by 'o', the corresponding results of Lemma 3.1-Lemma 3.4 still hold.

4. Proof of the Theorem

Proof of Theorem 2.1 Write $m := \left[\frac{1}{h}\right], n := \left[\frac{1}{k}\right]$ for given $h > 0, k > 0$. Direct calculation yields that

$$\begin{aligned}\left|\Delta^{2,2}(f_{11};x,y;h,k)\right| &= 4\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin ix\sin jy(1-\cos ih)(1-\cos jk)\right| \\ &\le 16\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin ix\sin jy\sin^2\frac{ih}{2}\sin^2\frac{jk}{2}\right|\end{aligned}$$

$$\begin{aligned}&\le 16\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin^2\frac{ih}{2}\sin^2\frac{jk}{2} \\ &= 16\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}+\sum_{i=1}^{m}\sum_{j=n+1}^{\infty}+\sum_{i=m+1}^{\infty}\sum_{j=1}^{n}+\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\right\} a_{ij}\sin^2\frac{ih}{2}\sin^2\frac{jk}{2} \\ &=: S_1+S_2+S_3+S_4.\end{aligned}$$

By (2.1), we have

$$S_1 \le h^2k^2\sum_{i=1}^{m}\sum_{j=1}^{n} i^2j^2a_{ij} = O\left(\omega(h,k)\right),$$

By (2.2) and (2.3), we have

$$S_2 \le 4h^2\sum_{i=1}^{m}\sum_{j=n+1}^{\infty} i^2a_{ij} = O\left(\omega(h,k)\right),$$

and

$$S_3 \le 4k^2\sum_{i=m+1}^{\infty}\sum_{j=1}^{n} j^2a_{ij} = O\left(\omega(h,k)\right),$$

respectively. Finally,by (2.4), we have

$$S_4 \le 16\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij} = O\left(\omega(h,k)\right).$$

Combining all the estimates above, the proof of Theorem 2.1 for $f_{11} \in \Lambda_*(\omega)$ is complete.
All the other cases can be proved exactly in the same way.

Proof of Theorem 2.2 We prove the result by considering the following many cases.
Case 1. $p = q = 1$. Since $f_{11} \in \Lambda_*^{\omega}(2)$, there exists a constant C such that

$$\begin{aligned}\left|\Delta^{2,2}(f_{11};x,y;h,k)\right| &= 4\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin ix\sin jy(1-\cos ih)(1-\cos jk)\right| \\ &\le C\omega(h,k),\ \ h>0,k>0.\end{aligned}$$

Noting that f_{11} is uniform convergent (due to (1.1)), we can integrate with respect to x on $(0,h)$ and y on $(0,k)$ to obtain that

$$4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\frac{(1-\cos ih)^2}{i}\frac{(1-\cos jk)^2}{j}\leq Chk\omega(h,k).$$

In other words,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{ij}}{ij}(1-\cos ih)^2(1-\cos jk)^2=16\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{ij}}{ij}\sin^4\frac{ih}{2}\sin^4\frac{jk}{2}\leq Chk\omega(h,k). \tag{4.1}$$

Since

$$\sin t\geq\frac{2t}{\pi},\qquad 0\leq t\leq\frac{\pi}{2}, \tag{4.2}$$

then

$$\sin^4\frac{ih}{2}\geq\left(\frac{2}{\pi}\frac{ih}{2}\right)^4=\frac{i^4h^4}{\pi^4},\quad i=1,2,...,m:=\left[\frac{1}{h}\right], \tag{4.3}$$

$$\sin^4\frac{jk}{2}\geq\left(\frac{2}{\pi}\frac{jk}{2}\right)^4=\frac{j^4k^4}{\pi^4},\quad j=1,2,...,n:=\left[\frac{1}{k}\right]. \tag{4.4}$$

By (4.2)-(4.4), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}i^3j^3a_{ij}=O\left(\frac{\omega(h,k)}{h^3k^3}\right)=O\left(m^3n^3\omega\left(\frac{1}{m},\frac{1}{n}\right)\right), \tag{4.5}$$

which proves Theorem 2.2 in the case when $p=q=1$.

Case 2. $p=1,q=2$. Since $f_{12}\in\Lambda_*^{\omega}(2)$, there exists a constant C such that

$$\begin{aligned}\left|\Delta^{2,2}(f_{12};x,0;h,k)\right|&=4\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\sin ix(1-\cos ih)(1-\cos jk)\right|\\&\leq C\omega(h,k).\end{aligned}$$

By the uniform convergence of f_{12} (due to (1.1)), we can integrate with respect to x on $(0,h)$ and obtain that

$$4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}(1-\cos ih)\frac{(1-\cos jk)^2}{j}\leq Ch\omega(h,k),$$

that is

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{ij}}{i}\sin^4\frac{ih}{2}\sin^2\frac{jk}{2}\leq Ch\omega(h,k). \tag{4.6}$$

By (4.2) and (4.6), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{a_{ij}}{i}\frac{i^4h^4}{\pi^4}\frac{j^2k^2}{\pi^2}\leq Ch\omega(h,k), \tag{4.7}$$

where $m:=\left[\frac{1}{h}\right],n:=\left[\frac{1}{k}\right]$. Therefore,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}i^3j^2a_{ij}=0\left(m^3n^2\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),$$

which proves Theorem 2.2 in this case.

Case 3. $p=2, q=1$. In a way similar to Case 2, we see that Theorem 2 holds in this case.
Case 4. $p=q=2$. Since $f_{22}\in\Lambda_*^{\omega}(2)$, there exists a constant C such that

$$\begin{aligned}\left|\Delta^{2,2}(f_{22};0,0;h,k)\right| &= 4\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}(1-\cos ih)(1-\cos jk)\right| \\ &= 16\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\sin^2\frac{ih}{2}\sin^2\frac{jk}{2} \\ &\le C\omega(h,k).\end{aligned}$$

Making use of inequality (4.2), we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\frac{i^2h^2}{\pi^2}\frac{j^2k^2}{\pi^2}\le C\omega(h,k), \tag{4.8}$$

where $m:=\left[\frac{1}{h}\right], n:=\left[\frac{1}{k}\right]$. Therefore,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}i^2j^2a_{ij}=O\left(m^2n^2\omega\left(\frac{1}{m},\frac{1}{n}\right)\right).$$

Thus, Theorem 2.2 holds for $p=q=2$.

Proof of Theorem 2.3 The necessity follows from Theorem 2.2, while the sufficiency follows from Lemma 3.1 with $\delta=\eta=\beta(2)=2$ and Theorem 2.1.

Proof of Theorem 2.4 The necessity follows from Theorem 2.2 and Lemma 3.1 ((3.1) $\Rightarrow$ (3.4)), while the sufficiency follows from Lemma 3.2 and Theorem 2.3.

Proof of Theorem 2.5 (i) Proof of the first part. The necessity follows from Theorem 2.2 and Lemma 3.1 ((3.1) $\Rightarrow$ (3.2)), while the sufficiency follows from Lemma 3.3 and Theorem 2.1.
(ii) The proof of the second part can be done in a way similar to that of the first part.

Proof of Corollary 2.6. (1) If there are μ_1,ν_1 $(0<\mu_1,\nu_1\le 2)$ such that $\left\{m^{\mu_1}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ and $\left\{n^{\nu_1}\omega\left(\frac{1}{m},\frac{1}{n}\right)\right\}$ are almost decreasing on m and n respectively, then

$$\begin{aligned}\sum_{i=m}^{\infty}i^{-1}\omega\left(\frac{1}{i},\frac{1}{n}\right) &= \sum_{i=m}^{\infty}i^{-1-\mu_1}\left(i^{\mu_1}\omega\left(\frac{1}{i},\frac{1}{n}\right)\right) \\ &= O\left(m^{\mu_1}\omega\left(\frac{1}{m},\frac{1}{n}\right)\sum_{i=m}^{\infty}i^{-1-\mu_1}\right) \\ &= O\left(\omega\left(\frac{1}{m},\frac{1}{n}\right)\right),\end{aligned}$$

which implies (2.6). Similarly, we have (2.7). Therefore, (i) follows from Theorem 2.3.

(ii) If there are also μ_2, ν_2 $(0 < \mu_2, \nu_2 < 2)$ such that $\left\{m^{\mu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ and $\left\{n^{\nu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right\}$ are almost increasing on m and n respectively, then

$$\begin{aligned}\sum_{i=1}^{m} i\omega\left(\frac{1}{i}, \frac{1}{n}\right) &= \sum_{i=1}^{m} i^{1-\mu_2}\left(i^{\mu_2}\omega\left(\frac{1}{i}, \frac{1}{n}\right)\right)\\ &= O\left(m^{\mu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\sum_{i=1}^{m} i^{1-\mu_2}\right)\\ &= O\left(m^2\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right),\end{aligned}$$

and

$$\sum_{j=1}^{n} j\omega\left(\frac{1}{m}, \frac{1}{j}\right) = O\left(n^{\nu_2}\omega\left(\frac{1}{m}, \frac{1}{n}\right)\sum_{j=1}^{n} j^{1-\nu_2}\right) = O\left(n^2\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right).$$

Thus, (ii) follows from Theorem 2.4.
Similarly, (iii) and (iv) follows from Theorem 2.5.
Proof of Theorem 2.8 The proof of this part is similar to that of Theorem 2.1.

Proof of Theorem 2.9 (i) If $p = q = 1$, that is, $f_{11} \in \lambda_*^{\omega}(2)$, then, for every $\varepsilon > 0$, there exist $h_0 > 0, k_0 > 0$ such that

$$\begin{aligned}\left|\Delta^{2,2}(f_{11}; x, y; h, k)\right| &= 4\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin ix \sin jy(1-\cos ih)(1-\cos jk)\right|\\ &\le \varepsilon\omega(h,k), \qquad 0 < h < h_0, 0 < k < k_0.\end{aligned}$$

Similar to (4.5), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^3 j^3 a_{ij} \le \varepsilon\frac{1}{h^3k^3}\omega(h,k) \le \varepsilon\left(m^3n^3\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right),$$

provided that $0 < h < h_0, 0 < k < k_0$, and $m = \left[\frac{1}{h}\right] \ge \left[\frac{1}{h_0}\right], n = \left[\frac{1}{k}\right] \ge \left[\frac{1}{k_0}\right]$.Since $\varepsilon > 0$ is arbitrary, then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^3 j^3 a_{ij} = o\left(m^3n^3\omega\left(\frac{1}{m}, \frac{1}{n}\right)\right), \qquad m, n \to \infty.$$

Here $\beta = \gamma = 3$.
(ii) If $p = 1, q = 2$, that is, $f_{12} \in \lambda_*^{\omega}(2)$, then, for every $\varepsilon > 0$ there exist $h_0 > 0, k_0 > 0$ such that

$$\begin{aligned}\left|\Delta^{2,2}(f_{11}; x, 0; h, k)\right| &= 4\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\sin ix(1-\cos ih)(1-\cos jk)\right|\\ &\le \varepsilon\omega(h,k), \qquad 0 < h < h_0, 0 < k < k_0.\end{aligned}$$

Similar to (4.7), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{a_{ij}}{i}\frac{i^4h^4}{\pi^4}\frac{j^2k^2}{\pi^2} \le \varepsilon h\omega(h,k),$$

provided that $0 < h < h_0, 0 < k < k_0$, and $m := \left[\frac{1}{h}\right] \geq \left[\frac{1}{h_0}\right], n := \left[\frac{1}{k}\right] \geq \left[\frac{1}{k_0}\right]$. Since $\varepsilon > 0$ is arbitrary, then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} i^3 j^2 a_{ij} = o\left(m^3 n^2 \omega\left(\frac{1}{m}, \frac{1}{n}\right)\right) \qquad m, n \to \infty.$$

The proofs of the other two cases $p = 2, q = 1$ and $p = q = 2$,can be done similarly. We omit the details here.

Acknowledgments. Research of the second author is supported by NSF of China (10901044), Qianjiang Rencai Program of Zhejiang Province (2010R10101), SRF for ROCS, SEM, and Program for excellent Young Teachers in HZNU).

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