Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 4(2011), Pages 213 - 222.

A UNIQUE COMMON TRIPLE FIXED POINT THEOREM IN PARTIALLY ORDERED CONE METRIC SPACES

(COMMUNICATED BY SIMEON RICH)

K.P.R.RAO AND G.N.V.KISHORE

Abstract. The notion of triple fixed point was introduced by V.Berinde and M.Borcut [8] and obtained some fixed point theorems in partially ordeded metric spaces. In this paper, we obtain a unique common triple fixed point theorem in partially ordered cone metric spaces in which cone is not necessarilly normal and also mention one supported example.

1. Introduction

In 2007, Huang and Zhang [10] introduced the concept of cone metric spaces by using ordered Banach space instead of the set of real numbers as a codomain and established Banach contraction principle. Later several authors proved fixed and common fixed point theorems in cone and partially ordered metric spaces. Some interesting references are [1, 3, 4, 5, 6, 11, 12, 15, 16, 19, 21, 22, 23].

The notion of coupled fixed point is introduced by Bhaskar and Lakshmikantham [9] and studied some fixed point theorems in partially ordered metric spaces.
Recently some of authors proved coupled and common coupled fixed point theorems in partially ordered cone metric spaces see [2, 7, 13, 14, 17, 18, 20].

In 2011, V.Berinde and M.Borcut [8] introduced triple fixed point and obtained some fixed point theorems for contractive type maps in partially ordeded metric spaces. The aim of this paper is to study unique common triple fixed point theorem for two maps by using w - compatible maps over partially ordered cone metric spaces,in which the underlying cone is not necessarily normal.

Throughout this paper, let Z^+ denote the set of all positive integers.

Definition 1.1. [10] *Let E be a real Banach space and P be a subset of E. P is called a cone if and only if :*
(i) P is closed, non - empty and $P \neq \{0\}$;
(ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

2000 *Mathematics Subject Classification.* 54H25, 47H10, 54E50.
Key words and phrases. cone metric, w - compatible maps, triple fixed point, complete space.

Submitted September 28, 2011. Published November 26, 2011.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $x - y \in P$. We shal write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{int } P$, int P denotes the interior of P.

The cone P is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow ||x|| \leq K||y||.$$

The least positive number satisfying above is called the normal constant of P. There are no normal cones with normal constant $K < 1$ (see [19]).

Example 1.2. [19] *Let* $E = C_R^2[0,1]$ *with the norm* $||f|| = ||f||_\infty + ||f'||_\infty$ *and consider the cone* $P = \{f \in E : f \geq 0\}$. *For each* $K > 1$, *put* $f(x) = x$ *and* $g(x) = x^{2K}$. *Then* $0 \leq g \leq f$, $||f|| = 2$ *and* $||g|| = 2K + 1$. *Since* $K||f|| < ||g||, K$ *is not the normal constant of* P. *Therefore,* P *is a non - normal cone.*

Definition 1.3. [10] *Let* X *be a nonempty set. Suppose the mapping* $d : X \times X \to E$ *satisfies*
(*i*) $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;
(*ii*) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(*iii*) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.
Then d *is called a cone metric on* X *and* (X, d) *is called a cone metric space.*

Definition 1.4. [10] *Let* (X, d) *be a cone metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *If for every* $c \in E$ *with* $0 << c$, *there is* $n_0 \in Z^+$ *such that* $d(x_n, x) << c$ *for all* $n \geq n_0$, *then* $\{x_n\}$ *is said to be convergent to* x *and* x *is called the limit of* $\{x_n\}$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

If for every $c \in E$ *with* $0 << c$, *there is* $n_0 \in Z^+$ *such that* $d(x_n, x_m) << c$ *for all* $n, m \geq n_0$, *then* $\{x_n\}$ *is called Cauchy sequence in* X. *If every Cauchy sequence is convergent in* X *then* X *is called a complete cone metric space.*

Remark. *Let* E *be an ordered Banach space with cone* P. *Then*
(1) *if* $u \leq v$ *and* $v << w$ *then* $u << w$,
(2) *if* $u << v$ *and* $v << w$ *then* $u << w$,
(3) *if* $0 \leq u << c$ *for each* $c \in intP$, *then* $u = 0$,
(4) $c \in intP$ *if and only if* $[-c, c]$ *is a neighborhood of* 0,
(5) *if* P *is a solid cone and if a sequence* $\{x_n\}$ *is convergent in a cone metric space* (X, d), *then the limit of* $\{x_n\}$ *is unique.*

Definition 1.5. [9] *Let* $(X, \preceq)$ *be a partially ordered set and* $F : X \times X \to X$. *Then the map* F *is said to have mixed monotone property if* $F(x, y)$ *is monotone non - decreasing in* x *and is monotone non - increasing in* y; *that is, for any* $x, y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X$$

and

$$y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.$$

Inspired by Definition 1.5, Lakshmikantham and Ćirić in [17] introduced the concept of a g - mixed monotone mapping.

Definition 1.6. [17] *Let* $(X, \preceq)$ *be a partially ordered set and* $F : X \times X \to X$. *Then the map* F *is said to have mixed* g *- monotone property if* $F(x, y)$ *is monotone*

g - non - decreasing in x and is monotone g - non - increasing in y; that is, for any $x, y \in X$,

$$gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.$$

Recently V.Berinde and M.Borcut [8] introduced the notion of triple fixed point of a mapping as folloows.

Definition 1.7. [8] *An element $(x, y, z) \in X \times X \times X$ is called a triple fixed point of mapping $F : X \times X \times X \to X$ if $x = F(x, y, z),\ y = F(y, x, y)$ and $z = F(z, y, x)$.*

In [8] the authors obtained the following theorem.

Theorem 1.8. *(Theorem.7.[8]): Let $(X, \leq)$ be a partially order set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which*

$$d\left(F(x, y, z), F(u, v, w)\right) \leq jd(x, u) + kd(y, v) + ld(z, w),$$

$\forall\ x \geq u,\ y \leq v,\ z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), y = F(y, x, y) \text{ and } z = F(z, y, x).$$

Now we give the following definitions.

Definition 1.9. *An element $(x, y) \in X \times X \times X$ is called*
(i) a triple coincident point of mapping $F : X \times X \times X \to X$ and $f : X \to X$ if $fx = F(x, y, z),\ fy = F(y, x, y)$ and $fz = F(z, y, x)$;
(ii) a common triple fixed point of mapping $F : X \times X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y, z),\ y = fy = F(y, x, y)$ and $z = fz = F(z, y, x)$.

Definition 1.10. *The mappings $F : X \times X \times X \to X$ and $f : X \to X$ are called w - compatible if $f(F(x, y, z)) = F(fx, fy, fz),\ f(F(y, x, y)) = F(fy, fx, fy)$ and $f(F(z, y, x)) = F(fz, fy, fx)$ whenever $fx = F(x, y, z),\ fy = F(y, x, y)$ and $fz = F(z, y, x)$.*

Now we prove our main result.

2. Main Result

Theorem 2.1. *Let $(X, \preceq, d)$ be a partially ordered cone metric space and let $T : X \times X \times X \to X$ and $f : X \to X$ be mappings satisfying*
(i) $d\left(T(x, y, z), T(u, v, w)\right) \leq jd(fx, fu) + kd(fy, fv) + ld(fz, fw)$
$\forall x, y, z, u, v, w \in X$ with $fx \succeq fu, fy \preceq fv, fz \succeq fw$ and $j, k, l \in [0, 1)$ with $j + k + l < 1$,
(ii) $T(X \times X \times X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X,
(iii) T has the mixed f - monotone property,
(iv) (a) If a non - decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
(b) If a non - increasing sequence $\{x_n\} \to x$, then $x \preceq x_n$ for all n.

If there exist $x_0, y_0, z_0 \in X$ *such that* $fx_0 \succeq T(x_0, y_0, z_0), fy_0 \preceq T(y_0, x_0, y_0)$ *and* $fz_0 \succeq T(z_0, y_0, x_0)$*, then* T *and* f *have triple coincidence point in* $X \times X \times X$*.*

Proof. Let $x_0, y_0, z_0 \in X$ such that $fx_0 \succeq T(x_0, y_0, z_0), fy_0 \preceq T(y_0, x_0, y_0)$ and $fz_0 \succeq T(z_0, y_0, x_0)$.
Since $T(X \times X \times X) \subseteq f(X)$, we choose $x_1, y_1, z_1 \in X$ such that

$$\begin{aligned} fx_1 &= T(x_0, y_0, z_0) \preceq fx_0, \\ fy_1 &= T(y_0, x_0, y_0) \succeq fy_0 \text{ and} \\ fz_1 &= T(z_0, y_0, x_0) \preceq fz_0. \end{aligned}$$

Now choose $x_2, y_2, z_2 \in X$ such that

$$\begin{aligned} fx_2 &= T(x_1, y_1, z_1), \\ fy_2 &= T(y_1, x_1, y_1) \text{ and} \\ fz_2 &= T(z_1, y_1, x_1). \end{aligned}$$

Since T has the mixed f - monotone property we have

$$\begin{aligned} fx_0 &\succeq fx_1 \succeq fx_2, \\ fy_0 &\preceq fy_1 \preceq fy_2 \text{ and} \\ fz_0 &\succeq fz_1 \succeq fz_2. \end{aligned}$$

Continuing this process, we can construct three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$\begin{aligned} fx_{n+1} &= T(x_n, y_n, z_n), \\ fy_{n+1} &= T(y_n, x_n, y_n) \text{ and} \\ fz_{n+1} &= T(z_n, y_n, x_n), n = 0, 1, 2, \cdots \end{aligned}$$

with

$$\begin{aligned} fx_0 &\succeq fx_1 \succeq fx_2 \succeq \cdots, \\ fy_0 &\preceq fy_1 \preceq fy_2 \preceq \cdots \text{ and} \\ fz_0 &\succeq fz_1 \succeq fz_2 \succeq \cdots \end{aligned}$$

To simplify we denote
$d_n^x = d(fx_{n-1}, fx_n),\ d_n^y = d(fy_{n-1}, fy_n)$ and $d_n^z = d(fz_{n-1}, fz_n)$.
Then by (i) we obtain

$$\begin{aligned} d_2^x &= d(fx_1, fx_2) \\ &= d(fx_2, fx_1) \\ &= d(T(x_2, y_2, z_2), T(x_1, y_1, z_1)) \\ &\leq jd(fx_2, fx_1) + kd(fy_2, fy_1) + ld(fz_2, fz_1) \\ &= jd_1^x + kd_1^y + ld_1^z. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} d_2^y &\leq kd_1^x + (j+l)d_1^y \\ d_2^z &\leq ld_1^x + kd_1^y + jd_1^z. \end{aligned}$$

Also

$$\begin{aligned} d_3^x &\leq (j^2 + k^2 + l^2)d_1^x + (2jk + 2lk)d_1^y + (2jl)d_1^z \\ d_3^y &\leq (2jk + lk)d_1^x + [(j+l)^2 + k^2]d_1^y + kld_1^z \\ d_3^z &\leq (2jl + k^2)d_1^x + 2[jk + lk]d_1^y + (j^2 + l^2)d_1^z. \end{aligned}$$

In order to simplify we consider the matrix

$$A = \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \text{ denoted by } \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & b_1 & h_1 \end{pmatrix}$$

and further denote

$$A^2 = \begin{pmatrix} j^2+k^2+l^2 & 2jk+2lk & 2jl \\ 2jk+lk & (j+l)^2+k^2 & kl \\ 2jl+k^2 & 2jk+2lk & j^2+l^2 \end{pmatrix} \text{ by } \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & b_2 & h_2 \end{pmatrix}$$

where

$$a_2+b_2+c_2 = d_2+e_2+f_2 = g_2+b_2+h_2 = (j+k+l)^2 < (j+k+l) < 1.$$

Now we prove by the induction that

$$A^n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \tag{2.1}$$

where

$$a_n+b_n+c_n = d_n+e_n+f_n = g_n+b_n+h_n = (j+k+l)^n < (j+k+l) < 1.$$

Clearly (2.1) is true for $n=1$ and $n=2$.
Assume that (2.1) is true for some n.
Consider

$$\begin{aligned} A^{n+1} &= A^n.A \\ &= \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \\ &= \begin{pmatrix} ja_n+kb_n+lc_n & ka_n+(j+l)b_n+kc_n & la_n+jc_n \\ jd_n+ke_n+lf_n & kd_n+(j+l)e_n+kf_n & ld_n+jf_n \\ jg_n+kb_n+lh_n & kg_n+(j+l)b_n+kh_n & lg_n+jh_n \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} a_{n+1}+b_{n+1}+c_{n+1} &= ja_n+kb_n+lc_n+ka_n+(j+l)b_n+kc_n+la_n+jc_n \\ &= (j+k+l)a_n+(j+k+l)b_n+(j+k+l)c_n \\ &= (j+k+l)(a_n+b_n+c_n) \\ &= (j+k+l)(j+k+l)^n \\ &= (j+k+l)^{n+1} \\ &< (j+k+l) \\ &< 1. \end{aligned}$$

Similarly we have

$$d_{n+1}+e_{n+1}+f_{n+1} = g_{n+1}+b_{n+1}+h_{n+1} = (j+k+l)^{n+1} < (j+k+l) < 1.$$

Thus (2.1) is true for $n+1$.
Hence by induction, (2.1) is true for all n.
Therefore,

$$\begin{pmatrix} d^x_{n+1} \\ d^y_{n+1} \\ d^z_{n+1} \end{pmatrix} \leq \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \begin{pmatrix} d^x_1 \\ d^y_1 \\ d^z_1 \end{pmatrix} \; for\ all\ n = 1,2,3,\cdots$$

That is

$$\left.\begin{aligned} d^x_{n+1} &\leq a_n d^x_1 + b_n d^y_1 + c_n d^z_1 \\ d^y_{n+1} &\leq d_n d^x_1 + e_n d^y_1 + f_n d^z_1 \\ d^z_{n+1} &\leq g_n d^x_1 + b_n d^y_1 + h_n d^z_1 \end{aligned}\right\} \tag{2.2}$$

for all $n = 1, 2, 3, \cdots$
Let $m, n \in N$ with $m > n$.

$$\begin{aligned} d(fx_m, fx_n) &\leq d(fx_m, fx_{m-1}) + d(fx_{m-1}, fx_{m-2}) + \cdots + d(fx_{n+2}, fx_{n+1}) + d(fx_{n+1}, fx_n) \\ &\leq a_{m-1}d_1^x + b_{m-1}d_1^y + c_{m-1}d_1^z + a_{m-2}d_1^x + b_{m-2}d_1^y + c_{m-2}d_1^z \\ &\quad + \cdots + a_{n+1}d_1^x + b_{n+1}d_1^y + c_{n+1}d_1^z + a_n d_1^x + b_n d_1^y + c_n d_1^z \\ &= [a_{m-1} + a_{m-2} + a_{m-3} + \cdots + a_n]d_1^x \\ &\quad + [b_{m-1} + b_{m-2} + b_{m-3} + \cdots + b_n]d_1^y \\ &\quad + [c_{m-1} + c_{m-2} + c_{m-3} + \cdots + c_n]d_1^z \\ &\leq (\mu^{m-1} + \mu^{m-2} + \cdots + \mu^{n+1} + \mu^n)d_1^x \\ &\quad + (\mu^{m-1} + \mu^{m-2} + \cdots + \mu^{n+1} + \mu^n)d_1^y \\ &\quad + (\mu^{m-1} + \mu^{m-2} + \cdots + \mu^{n+1} + \mu^n)d_1^z \\ &= (\mu^{m-1} + \mu^{m-2} + \cdots + \mu^{n+1} + \mu^n)(d_1^x + d_1^y + d_1^z) \\ &\leq \frac{\mu^n}{1-\mu}(d_1^x + d_1^y + d_1^z) \end{aligned} \tag{2.3}$$

where $\mu = j + k + l < 1$.
It follows from (2.3) that for $c \in E$, $0 << c$ and large n,
we have $\frac{\mu^n}{1-\mu}(d_1^x + d_1^y + d_1^z) << c$.
Thus

$$d(fx_m, fx_n) << c.$$

Hence $\{fx_n\}$ is a Cauchy sequence in the metric space (X, d).
Similarly $\{fy_n\}$ and $\{fz_n\}$ are also Cauchy sequences in the cone metric space (X, d).
Suppose $f(X)$ is complete.
Since $\{fx_n\} \subseteq f(X), \{fy_n\} \subseteq f(X)$ and $\{fz_n\} \subseteq f(X)$ are Cauchy sequences in the complete cone metric space $(f(X), d)$, it follows that the sequences $\{fx_n\}, \{fy_n\}$ and $\{fz_n\}$ are converge to some α, β and γ in $(f(X), d)$ respectively.
There exist $x, y, z \in X$ such that $\alpha = fx, \beta = fy$ and $\gamma = fz$.
Since $\{fx_n\}, \{fy_n\}$ and $\{fz_n\}$ are Cauchy sequences in X and $\{fx_n\} \to \alpha, \{fy_n\} \to \beta$ and $\{fz_n\} \to \gamma$, it follows that $\{fx_{n+1}\} \to \alpha, \{fy_{n+1}\} \to \beta$ and $\{fz_{n+1}\} \to \gamma$.
Since $\{fx_n\}$ is a non - increasing sequence and $\{fx_n\} \to fx$ we have $fx \preceq fx_n$, $\{fy_n\}$ is a non - decreasing sequence and $\{fy_n\} \to fy$ we have $fy_n \preceq fy$ and $\{fz_n\}$ is a non - increasing sequence and $\{fz_n\} \to fz$ we have $fz \preceq fz_n$ for all n.
Now,

$$\begin{aligned} d(T(x,y,z), \alpha) &\leq d(T(x,y,z), fx_{n+1}) + d(fx_{n+1}, \alpha) \\ &= d(T(x,y,z), T(x_n, y_n, z_n)) + d(fx_{n+1}, \alpha) \\ &\leq jd(fx, fx_n) + kd(fy, fy_n) + ld(fz, fz_n) + d(fx_{n+1}, \alpha) \\ &= jd(\alpha, fx_n) + kd(\beta, fy_n) + ld(\gamma, fz_n) + d(fx_{n+1}, \alpha) \\ &<< j\tfrac{c}{4j} + k\tfrac{c}{4k} + l\tfrac{c}{4l} + \tfrac{c}{4} = c. \end{aligned}$$

It follows that $\alpha = T(x, y, z)$.
Similarly $\beta = T(y, x, y)$ and $\gamma = T(z, y, x)$.
Thus

$$\alpha = fx = T(x,y,z), \beta = fy = T(y,x,y) \text{ and } \gamma = fz = T(z,y,x).$$

Hence (x, y, z) is a triple coincidence point of T and f. □

Theorem 2.2. *In addition to the hypothesis of Theorem 2.1. Suppose that for every* $(x, y, z), (x^*, y^*, z^*) \in X \times X \times X$ *there exists* $(u, v, w) \in X \times X \times X$ *such that*

$(T(u,v,w),T(v,u,v),T(w,v,u))$ is comparable to $(T(x,y,z),T(y,x,y),T(z,y,x))$ and $(T(x^*,y^*,z^*),T(y^*,x^*,y^*),T(z^*,y^*,x^*))$. If (x,y,z) and (x^*,y^*,z^*) are triple coincidence points of T and f, then

$$\begin{aligned} T(x,y,z) &= fx = fx^* = T(x^*,y^*,z^*), \\ T(y,x,y) &= fy = fy^* = T(y^*,x^*,y^*) \text{ and} \\ T(z,y,x) &= fz = fz^* = T(z^*,y^*,x^*). \end{aligned}$$

Moreover if (T,f) is w - compatible, then T and f have a unique common triple fixed point in $X \times X \times X$.

Proof. From Theorem 2.1, there exists $(x,y,z) \in X \times X \times$ such that

$$T(x,y,z) = fx = \alpha, T(y,x,y) = fy = \beta \text{ and } T(z,y,x) = fz = \gamma.$$

Thus the existence of triple coincidence point of T and f is conformed.
Now let (x^*,y^*,z^*) be another triple coincidence point of T and f.
That is

$$T(x^*,y^*,z^*) = fx^*, T(y^*,x^*,y^*) = fy^* \text{ and } T(z^*,y^*,x^*) = fz^*.$$

By additional assumption, there is $(u,v,w) \in X \times X \times X$ such that $(T(u,v,w),T(v,u,v),T(w,v,u))$ is comparable to $(T(x,y,z),T(y,x,y),T(z,y,x))$ and $(T(x^*,y^*,z^*),T(y^*,x^*,y^*),T(z^*,y^*,x^*))$.
Let $u_0 = u, v_0 = v, w_0 = w, x_0 = x, y_0 = y, z_0 = z, x_0^* = x^*, y_0^* = y^*$ and $z_0^* = z^*$.
Since $T(X \times X \times X) \subseteq f(X)$, we can construct the sequences $\{fu_n\},\{fv_n\},\{fw_n\},\{fx_n\},\{fy_n\},\{fz_n\},\{fx_n^*\},\{fy_n^*\}$ and $\{fz_n^*\}$.

$$\begin{aligned} fu_{n+1} &= T(u_n,v_n,w_n), fv_{n+1} = T(v_n,u_n,v_n), fw_{n+1} = T(w_n,v_n,u_n), \\ fx_{n+1} &= T(x_n,y_n,z_n), fy_{n+1} = T(y_n,x_n,y_n), fz_{n+1} = T(z_n,y_n,x_n), \\ fx_{n+1}^* &= T(x_n^*,y_n^*,z_n^*), fy_{n+1}^* = T(y_n^*,x_n^*,y_n^*) \text{ and } fz_{n+1}^* = T(z_n^*,y_n^*,x_n^*), n = 0,1,2,\cdots \end{aligned}$$

Since $(fx,fy,fz) = (T(x,y,z),T(y,x,y),T(z,y,x)) = (fx_1,fy_1,fz_1)$ and $(T(u,v,w),T(v,u,v),T(w,v,u)) = (fu_1,fv_1,fw_1)$ are comparable, then $fx \succeq fu_1, fy \preceq fv_1$ and $fz \succeq fw_1$.
One can show that $fx \succeq fu_n, fy \preceq fv_n$ and $fz \succeq fw_n$ for all n.
As in Theorem 2.1 we conclude that

$$\begin{pmatrix} d(fx,fu_{n+1}) \\ d(fy,fv_{n+1}) \\ d(fz,fw_{n+1}) \end{pmatrix} \leq \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \begin{pmatrix} d(fx,fu_1) \\ d(fy,fv_1) \\ d(fz,fw_1) \end{pmatrix}$$

where $a_n + b_n + c_n = d_n + e_n + f_n = g_n + b_n + h_n = (j+k+l)^n < (j+k+l) < 1$.
Thus

$$\begin{aligned} d(fx,fu_{n+1}) &\leq a_n d(fx,fu_1) + b_n d(fy,fv_1) + c_n d(fz,fw_1) \\ &\leq [a_n + b_n + c_n][d(fx,fu_1) + d(fy,fv_1) + d(fz,fw_1)] \\ &\leq \mu^n[d(fx,fu_1) + d(fy,fv_1) + d(fz,fw_1)] \qquad (2.4) \end{aligned}$$

where $\mu = j + k + l < 1$.
Let $0 << c$ be given choose a natural number n_0 such that $\mu^n[d(fx,fu_1) + d(fy,fv_1) + d(fz,fw_1)] << c$ for all $n > n_0$.
Thus

$$d(fx,fu_{n+1}) << c.$$

Therefore $\{fu_{n+1}\}$ converges to fx in $(f(X),d)$.
Similarly we may show that $\{fv_{n+1}\}$ converges to fy and $\{fw_{n+1}\}$ converges to fz.

Analogously we can show that $\{fu_{n+1}\}$ converges to fx^*, $\{fv_{n+1}\}$ converges to fy^* and $\{fw_{n+1}\}$ converges to fz^* in $(f(X),d)$.
Since the cone P is closed and $\{fu_{n+1}\}$ converges to fx and fx^*, we get $fx = fx^*$. Similarly $fy = fy^*$ and $fz = fz^*$.
Thus we have that if (x,y,z) and (x^*,y^*,z^*) are triple coincidence points of T and f, then

$$\begin{aligned}\alpha &= T(x,y,z) = fx = fx^* = T(x^*,y^*,z^*),\\ \beta &= T(y,x,y) = fy = fy^* = T(y^*,x^*,y^*) \text{ and}\\ \gamma &= T(z,y,x) = fz = fz^* = T(z^*,y^*,x^*).\end{aligned}$$

Since (T,f) is w - compatible, then

$$\begin{aligned}f\alpha &= f(fx) = f(T(x,y,z)) = T(fx,fy,fz) = T(\alpha,\beta,\gamma),\\ f\beta &= f(fy) = f(T(y,x,y)) = T(fy,fx,fy) = T(\beta,\alpha,\beta) \text{ and}\\ f\gamma &= f(fz) = f(T(z,y,x)) = T(fz,fy,fx) = T(\gamma,\beta,\alpha).\end{aligned}$$

Hence the triple (α,β,γ) is also triple coincidence point of T and f. Thus we have

$$f\alpha = fx, f\beta = fy \text{ and } f\gamma = fz.$$

Therfore

$$\alpha = f\alpha = T(\alpha,\beta,\gamma), \beta = f\beta = T(\beta,\alpha,\beta) \text{ and } \gamma = f\gamma = T(\gamma,\beta,\alpha).$$

Thus (α,β,γ) is common triple fixed point of T and f.
To prove uniqueness, let (p,s,t) be any common triple fixed point of T and f.
Then $p = fp = T(p,s,t), s = fs = T(s,p,s)$ and $t = ft = T(t,s,p)$.
Since the triple (p,s,t) is a triple coincidence point of T and f.
We have

$$fp = fx, fs = fy \text{ and } ft = fz.$$

Thus

$$p = fp = fx = \alpha, s = fs = fy = \beta \text{ and } t = ft = fz = \gamma.$$

Hence the common triple fixed point is unique.

□

Example 2.3. *Let $X = R^+ = [0,+\infty)$ and the order relation $\preceq$, be defined by $x \preceq y \Leftrightarrow \{(x = y)$ or $(x,y \in [0,1]$ with $x \le y)\}$. Let $E = C^1_R[0,1]$ with the norm $||x|| = ||x||_\infty + ||x'||_\infty$ and consider the cone $P = \{x \in E : x(t) \ge 0$ on $[0,1]\}$ (this cone is not normal). Define $d : X \times X \to E$ by $d(x,y) = |x-y|\varphi$, where $\varphi : [0,1] \to R$ such that $\varphi(t) = e^t$. It is easy to see that d is a cone metric on X.*

Consider the mappings $T : X \times X \times X \to X$ and $f : X \to X$ are defined as $F(x,y,z) = \frac{x+y+z}{12}$ and $f(x) = \frac{x}{2}$ respectively.
Clearly

$$\begin{aligned}d(T(x,y,z),T(u,v,w)) &= |\tfrac{x+y+z}{12} - \tfrac{u+v+w}{12}|\varphi\\ &= |\tfrac{x}{12} + \tfrac{y}{12} + \tfrac{z}{12} - \tfrac{u}{12} - \tfrac{v}{12} - \tfrac{w}{12}|\varphi\\ &\le \left(|\tfrac{x}{12} - \tfrac{u}{12}| + |\tfrac{y}{12} - \tfrac{v}{12}| + |\tfrac{z}{12} - \tfrac{w}{12}|\right)\varphi\\ &= \tfrac{1}{6}|\tfrac{x}{2} - \tfrac{u}{2}|\varphi + \tfrac{1}{6}|\tfrac{y}{2} - \tfrac{v}{2}|\varphi + \tfrac{1}{6}|\tfrac{z}{2} - \tfrac{w}{2}|\varphi\\ &= \tfrac{1}{6}d(fx,fu) + \tfrac{1}{6}d(fy,fv) + \tfrac{1}{6}d(fz,fw).\end{aligned}$$

Also all conditions of Theorem 2.1 and Theorem 2.2 are hold.
Clearly $(0,0,0)$ is the unique common triple fixed point of T and f.

References

[1] A. Azam, M. Arshad, I. Beg, *Common fixed points of two maps in cone metric spaces*, Rend. Cir. Mat. Palermo,**57** , 2008, 433-441.

[2] M.Abbas, M.Alikhan and S.Radenović, *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Applied Mathematics and Computation, **Volume 217**, Issue 1, 1 September 2010, Pages 195-202.

[3] I. Altun, B. Damjanovic and D. Djoric, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Applied Mathematics Letters, **Volume 23**, Issue 3, March 2010, Pages 310-316.

[4] A. Amini-Harindi, M. Fakhar, *Fixed point theory in cone metric spaces via scalarization method*, Comput. Math. Appl., **59**, 2010, 3529- 3534.

[5] T. Abdeljawad, E. Karapínar, *Quasicone metric spaces and generalizations of Caristi Kirk's theorem*, Fixed Point Theory Appl. **Volume 2009**, 2009, Article ID 574387, 9 pages

[6] I. Arandjelovic, Z. Kadelburg, S. Radenovic, *Boyd - Wong type common fixed point results in cone metric spaces*, Applied Mathematics and Computation, **217**, 2011, 7167-7171.

[7] H. Aydi, B. Samet and C. Vetro, *Coupled fixed point results in cone metric spaces for W - compatible mappings* , Fixed Point Theory and Applications **2011**,2011:27 doi: 10.1186/1687 - 1812 - 2011 -27.

[8] V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Analysis, **Volume 74**, Issue 15, October 2011, Pages 4889-4897.

[9] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis. **65**, 2006, 1379-1393.

[10] LG. Huang and X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl.,**332** , 2007, 1468-1476.

[11] S. Jankovic, Z. Kadelburg and Radenovic, *On cone metric spaces: A survey*, Nonlinear Analysis, **74**, 2011, 2591-2601.

[12] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, *Common fixed point theorems for weakly compatible pairs in cone metric spaces*, Fixed point theory and Applications,**2009**, 2009, ID:643840, 13 pages.

[13] E. Karapínar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Computers and Mathematics with Applications. **59**, 2010, 3656 - 3668.

[14] E. Karapínar, *Couple fixed point on cone metric spaces*, Gazi University J. Sci. **24(1)**, 2011, 51 -58.

[15] Z. Kadelburg, M. Pavlović and S. Radenović, *Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces*, Applied Mathematics with Applications, **59**, 2010, 3148 - 3159.

[16] Z. Kadelburg, S. Radenovic, V. Rakocevic, *Remarks on quasi-contraction on a cone metric space*, Applied Mathematics Letters, **22**, 2009, 1674-1679.

[17] V. Lakshmikantham and Lj. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis. **70**, 2009, 4341-4349.

[18] M. O. Olatinwo, *Coupled fixed point theorems in cone metric spaces*, Ann Univ Ferrara, DOI 10.1007/s11565 -010-0111-3

[19] Sh. Rezapour, R. Hamlbarani, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. **347**, 2008, 719 - 724.

[20] W. Shatanawi, *Partially ordered cone metric spaces and coupled fixed point results*, **60** , 2010, 2508 - 2515.

[21] B. S. Choudhury, N. Metiya, *Fixed points of weak contractions in cone metric spaces*, Nonlinear analysis, **72** , 2010, 1589-1593.

[22] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Ciric. Mat. Palermo, **56** , 2007, 464-468.

[23] Wei-Shih Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Analysis. TMA, **72**, 2010, 2259-2261.

K. P. R. Rao,
Department of Applied Mathematics, A.N.U.-Dr.M.R.Appa Row Campus, Nuzvid-521 201, Andhra Pradesh, India
E-mail address: kprrao2004@yahoo.com

G.N.V.Kishore,
Department of Mathematics, Swarnandhra Institute of Engineering and Technology, Seetharampuram, Narspur- 534 280 , West Godavari District, Andhra Pradesh, India
E-mail address: kishore.apr2@gmail.com