Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 1 (2012), Pages 168-173.

AN ADDENDUM TO: "A COMMON FIXED POINT THEOREM IN INTUITIONISTIC FUZZY METRIC SPACE USING SUBCOMPATIBLE MAPS"

(COMMUNICATED BY NASEER SHAHZAD)

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Abstract. The aim of this note is to point out a fallacy in the proof of Theorem 3.1 contained in the recent paper (Int. J. Contemp. Math. Sci. 5 (2010), 2699-2707) proved in intuitionistic fuzzy metric spaces employing the newly introduced notion of sub-compatible pair of mappings wherein our claim is also substantiated with the aid of an appropriate example. We also rectify the erratic theorem in two ways.

In order to avoid repetition and also due to paucity of the space, we assume the terminology and the notations utilized in [6] rather than presenting the same again. For more recent developments, we refer the readers to [1, 3, 9] and references cited therein.

The following definitions are essentially contained in [6].

Definition 0.1. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A pair of self maps (A, S) defined on X is said to be compatible iff*

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$$

and

$$\lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0$$

wherein $\{x_n\}$ are sequences in X with

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,\ z \in X.$$

Definition 0.2. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A pair of self maps (A, S) defined on X is said to be reciprocally continuous if $\lim_{n\to\infty} ASx_n = Az$, $\lim_{n\to\infty} SAx_n = Sz$, wherein $\{x_n\}$ are sequences in X with $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.*

2000 *Mathematics Subject Classification.* 54H25, 47H10.

Key words and phrases. Intuitionistic fuzzy metric space, Compatible maps, subcompatible maps, subsequential continuity, reciprocal continuity.

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Submitted November 10, 2011. Published January 20, 2012.

Definition 0.3. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A pair of self maps (A, S) defined on X is said to be subcompatible iff there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,\ z \in X,$$

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$$

and

$$\lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0.$$

Definition 0.4. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A pair of self maps (A, S) defined on X is said to be subsequentially continuous iff there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,\ \ t \in X$$

and

$$\lim_{n\to\infty} ASx_n = At,\ \lim_{n\to\infty} SAx_n = St.$$

Motivated by [2], utilizing the preceeding two definitions, Manro et al. [6] proved the following common fixed point theorem for two pairs of subcompatible as well as subsequentially continuous maps in intuitionistic fuzzy metric space.

Theorem A.(cf.[6]) Let A, B, S and T be four self maps of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ with continuous t-norm $*$ and continuous t-conorm $\diamond$ defined by $t * t \geq t$ and $(1-t) \diamond (1-t) \leq (1-t)$ for all $t \in [0, 1]$. If the pairs (A, S) and (B, T) are subcompatible as well as subsequentially continuous, then

a) A and S have a coincidence point,
b) B and T have a coincidence point.

Further, for all x, y in X, $k \in (0, 1), t > 0$, let

$$M(Ax, By, kt) \geq M(Sx, Ty, t)*M(Ax, Sx, t)*M(By, Ty, t)*M(By, Sx, 2t)*M(Ax, Ty, t)$$

and

$$N(Ax, By, kt) \leq N(Sx, Ty, t)\diamond N(Ax, Sx, t)\diamond N(By, Ty, t)\diamond N(By, Sx, 2t)\diamond N(Ax, Ty, t).$$

Then A, B, S and T have a unique common fixed point.

Unfortunately, Theorem A is not true in its present form but can be recovered either by replacing subcompatible pairs with compatible pairs or by replacing subsequential continuity of the pairs with reciprocal continuity of the pairs (e.g. [5]). The error crept in due to the fact that the sequences satisfying the requirements of Definitions 0.2 and 0.3 need not be the same as utilized in the proofs of [6]. To substantiate this viewpoint, we furnish the following example which disproves Theorem A.

Example 0.1. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space (as defined in Example 2.1 in* [6]*) where $X = [0, \infty)$. Set $A = B$, $S = T$ and define A,*

$S : X \to X$ by

$$A(x) = \begin{cases} 0 \text{ if } x = 0, \\ 1 + x \text{ if } x \in (0, 1], \\ 2x - 1 \text{ if } x \in (1, \infty), \end{cases} \quad \text{and } S(x) = \begin{cases} 1 - x \text{ if } x \in [0, 1), \\ 3x - 2 \text{ if } x \in [1, \infty). \end{cases}$$

Notice that A and S are discontinuous at $x = 1$. Let us consider the sequence $x_n = 1 + \frac{1}{n}$ for $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(2 + \frac{2}{n} - 1\right) = 1 = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} \left(3 + \frac{3}{n} - 2\right),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1 + \frac{2}{n}\right) = \lim_{n\to\infty} \left(3 + \frac{6}{n} - 2\right) = 1$$

and

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1 + \frac{3}{n}\right) = \lim_{n\to\infty} \left(2 + \frac{6}{n} - 1\right) = 1.$$

Thus, for all $t > 0$, we have

$$\lim_{n\to\infty} M(SA(x_n), AS(x_n), t) = 1 \text{ and } \lim_{n\to\infty} N(SA(x_n), AS(x_n), t) = 0,$$

so that the pair (A, S) is subcompatible.

Next, choose $x_n = \frac{1}{n}$ for $n = 1, 2.....$ Then we have

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1,$$

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1,$$

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} \left(2 - \frac{1}{n}\right) = 2 = A(1)$$

and

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \left(1 + \frac{3}{n}\right) = 1 = S(1).$$

Thus the pair (A, S) is subsequentially continuous as well as subcompatible so that all the conditions of Theorem A (upto coincidence point) are satisfied. But the maps in the pair do not have a coincidence or common fixed point which shows that Theorems A is not true in its present form.

However, Theorem A can be corrected in two ways as follows:

Theorem 0.1. *Let A, B, S and T be self maps of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. If the pairs (A, S) and (B, T) are compatible as well as subsequentially continuous, then*

a) the pair (A, S) has a coincidence point,
b) the pair (B, T) has a coincidence point.

Further, for all x, y in X, $k \in (0, 1)$andt > 0, let

$$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, 2t) * M(Ax, Ty, t)$$

and

$$N(Ax, By, kt) \leq N(Sx, Ty, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t) \diamond N(By, Sx, 2t) \diamond N(Ax, Ty, t).$$

Then A, B, S and T have a unique common fixed point.

Theorem 0.2. *The conclusions of Theorem 0.1 remain valid if we replace compatibility with subcompatibility and subsequential continuity with reciprocal continuity besides retaining rest of the hypotheses.*

There is no need to give the proofs of both corrected theorems as the proof furnished in [6] survives in respect of both theorems (except the noted fallacy).

Now, we furnish two illustrative examples to highlight the utility of Theorem 0.1 and Theorem 0.2 which exhibit that even corrected results brought about noted improvements.

Example 0.2. *Consider $(X, M, N, *, \diamond)$ as defined in Example 0.1 with $X = [0, \infty)$. Set $A = B$ and $S = T$. Define $A, S : X \to X$ as follows:*

$$Ax = \begin{cases} x/3 \text{ if } x \in [0,1], \\ \\ 2x - 1 \text{ if } x \in (1, \infty), \end{cases} \qquad Sx = \begin{cases} x/2 \text{ if } x \in [0,1], \\ \\ 3x - 2 \text{ if } x \in (1, \infty). \end{cases}$$

In respect of the sequence $x_n = \dfrac{1}{n}$ in X,

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \frac{1}{3n} = 0 = \lim_{n\to\infty} \frac{1}{2n} = \lim_{n\to\infty} S(x_n),$$

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(\frac{1}{2n}\right) = \lim_{n\to\infty} \frac{1}{6n} = 0 = A(0),$$

and

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(\frac{1}{3n}\right) = \lim_{n\to\infty} \frac{1}{6n} = 0 = S(0),$$

so that for all $s > 0$, we have

$$\lim_{n\to\infty} M(ASx_n, SAx_n, s) = 1 \text{ and } \lim_{n\to\infty} N(ASx_n, SAx_n, s) = 0.$$

In respect of another sequence $x_n = 1 + \dfrac{1}{n}$,

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(2 + \frac{2}{n} - 1\right) = 1,$$

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} \left(3 + \frac{3}{n} - 2\right) = 1,$$

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1 + \frac{3}{n}\right) = \lim_{n\to\infty} \left(2 + \frac{6}{n} - 1\right) = 1 \neq A(1),$$

and

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1 + \frac{2}{n}\right) = \lim_{n\to\infty} \left(3 + \frac{6}{n} - 2\right) = 1 \neq S(1),$$

so that for all $s > 0$, we have

$$\lim_{n\to\infty} M(ASx_n, SAx_n, s) = 1 \text{ and } \lim_{n\to\infty} N(ASx_n, SAx_n, s) = 0.$$

Therefore, the pair (A, S) is compatible as well as subsequentially continuous but not reciprocally continuous. Thus, all the conditions of Theorem 0.1 (upto coincidence point) are satisfied and $x = 0$ is a coincidence point of the pair (A, S). Notice *that this example cannot be covered by those fixed point theorems which involve both compatibility and reciprocal continuity. (e.g. relevant results contained in references [8],[12] and [13] of* [6]*).*

Example 0.3. *Consider $(X, M, N, *, \diamond)$ as defined in Example 0.1 and let $X = \mathbb{R}$. Set $A = B$ and $S = T$. Define $A, S : X \to X$ as follows:*

$$Ax = \begin{cases} x + 1 \ \textit{if}\ x \in (-\infty, 1), \\ \\ 2x - 1 \ \textit{if}\ x \in [1, \infty), \end{cases} \qquad Sx = \begin{cases} x/2 \ \textit{if}\ x \in (-\infty, 1), \\ \\ 3x - 2 \ \textit{if}\ x \in [1, \infty). \end{cases}$$

In respect of the sequence $x_n = 1 + \dfrac{1}{n}$,

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(2 + \frac{2}{n} - 1\right) = 1,$$

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} \left(3 + \frac{3}{n} - 2\right) = 1,$$

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1 + \frac{3}{n}\right) = \lim_{n\to\infty} \left(2 + \frac{6}{n} - 1\right) = 1 = A(1),$$

and

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1 + \frac{2}{n}\right) = \lim_{n\to\infty} \left(3 + \frac{6}{n} - 2\right) = 1 = S(1)$$

so that for all $s > 0$,

$$\lim_{n\to\infty} M(ASx_n, SAx_n, s) = 1 \ \textit{and} \ \lim_{n\to\infty} N(ASx_n, SAx_n, s) = 0.$$

Next, in respect of the sequence $x_n = \dfrac{1}{n} - 2$,

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(\frac{1}{n} - 2 + 1\right) = -1,$$

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} \left(\frac{1}{2n} - 1\right) = -1,$$

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(\frac{1}{2n} - 1\right) = \lim_{n\to\infty} \left(\frac{1}{2n} - 1 + 1\right) = 0 = A(-1),$$

and

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(\frac{1}{n} - 1\right) = \lim_{n\to\infty} \left(\frac{1}{2n} - \frac{1}{2} - \frac{1}{2}\right) = -\frac{1}{2} = S(-1),$$

so that all $s > 0$, we have

$$\lim_{n\to\infty} M(ASx_n, SAx_n, s) \neq 1 \ \textit{and} \ \lim_{n\to\infty} N(ASx_n, SAx_n, s) \neq 0.$$

Thus, the pair (A, S) is reciprocally continuous as well as subcompatible but not compatible so that all the conditions of Theorem 0.2 (upto coincidence point) are satisfied and $x = 1$ is a coincidence point of the pair (A, S). Notice that this example cannot be covered by those fixed point theorems which involve both compatibility and reciprocal continuity (e.g. relevant results contained in references [8],[12] and [13] of [6]*).*

Acknowledgments The authors are thankful to referees for their fruitful comments. The third author is supported by Università degli Studi di Palermo, Local University Project R. S. ex 60%.

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