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ON BANACH ALGEBRA VALUED FUNCTIONS OF BOUNDED GENERALIZED VARIATION OF ONE AND SEVERAL VARIABLES

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Abstract. Here, it is observe that $\Lambda BV^{(p)}(\sigma,\mathbb{B})$, the class of functions of bounded $p-\Lambda-$ variation from a non-empty compact subset σ of $\mathbb{R}$ into a commutative unital Banach algebra $\mathbb{B}$, is a commutative unital Banach algebra. Moreover, $(\Lambda^1,...,\Lambda^N)^*BV^{(p)}(\Pi_{i=1}^N\sigma_i,\mathbb{B})$, the class of $N-$variables functions of bounded $p-(\Lambda^1,...,\Lambda^N)^*-$variation from $\Pi_{i=1}^N\sigma_i$ into $\mathbb{B}$, is a Banach space, where σ_i are non-empty compact subsets of $\mathbb{R}$, for all $i=1$ *to* N.

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1. Introduction.

Berkson and Gillespie [3] proved that $BV_H([a,b]\times[c,d],\mathbb{C})$, the class of functions of bounded variation (in the sense of Hardy) over $[a,b]\times[c,d]$, is a Banach algebra with respect to the pointwise operations and the variation norm (also see [1], [6] and [7]). Here, first we prove that $\Lambda BV^{(p)}(\sigma,\mathbb{B})$, the class of functions of $p-\Lambda-$ bounded variation from a non-empty compact subset σ of $\mathbb{R}$ into a commutative unital Banach algebra $\mathbb{B}$, is a commutative unital Banach algebra with respect to the pointwise operations and the Λ_p-variation norm. Finally, we show that $(\Lambda^1,\Lambda^2,...,\Lambda^N)^*BV^{(p)}(\prod_{i=1}^N\sigma_i,\mathbb{B})$, the class of N-variables functions of bounded $p-(\Lambda^1,\Lambda^2,...,\Lambda^N)^*-$variation over $\prod_{i=1}^N\sigma_i$ is a Banach space with respect to the pointwise linear operations and $(\Lambda^1,\Lambda^2,...,\Lambda^N)_p$-variation norm, where $\sigma_1,\ \sigma_2,...,\sigma_N$ are non-empty compact subsets of $\mathbb{R}$.

2. The class $\Lambda BV^{(p)}(\sigma,\mathbb{B})$.

Let σ be any non-empty compact subset of $\mathbb{R}$ and $I=[a,b]$ be the smallest closed interval containing σ. Let $\amalg(\sigma)$ be a class of all partitions of σ. That is $\amalg(\sigma)=\{t\ :\ t=\{t_i\}_{i=0}^n$ is an increasing finite sequence in $\sigma\}$.

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Definition 2.1. *For a given non-empty compact subset σ of $\mathbb{R}$, a Banach algebra $\mathbb{B}$, a non-decreasing sequence of positive numbers $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \frac{1}{\lambda_i}$ diverges and $p \geq 1$. A function $f : \sigma \to \mathbb{B}$ is said to be of bounded $p - \Lambda$ variation over σ (that is, $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$) if*

$$V_{\Lambda_p}(f, \sigma, \mathbb{B}) = \sup_{t \in \Pi(\sigma)} V_{\Lambda_p}(f, \sigma, \mathbb{B}, t) < \infty,$$

where

$$V_{\Lambda_p}(f, \sigma, \mathbb{B}, t) = (\sum_{i=1}^{n} \frac{||\triangle f(t_i)||_{\mathbb{B}}^p}{\lambda_i})^{1/p}, \text{ in which } \triangle f(t_i) = f(t_i) - f(t_{i-1}).$$

In the above definition, for $\sigma = [a, b]$ one gets the class $\Lambda BV^{(p)}([a, b], \mathbb{B})$; for $p = 1$ and $\lambda_n = 1$, for all n, one gets the class BV$(\sigma, \mathbb{B})$ [1] and for $\lambda_n = 1$, for all n, one gets the class $BV^{(p)}(\sigma, \mathbb{B})$. For $\mathbb{B} = \mathbb{C}$, we omit writing $\mathbb{C}$, the class $\Lambda BV^{(p)}(\sigma, \mathbb{B})$ reduces to the class $\Lambda BV^{(p)}(\sigma)$.

Definition 2.2. *For a given function $f : \sigma \to \mathbb{B}$, where σ is a non-empty compact subset of $\mathbb{R}$. Define the function $E_f : I \to \mathbb{B}$ by $E_f(x) = f(\alpha(x))$, where*

$$\alpha(x) = \begin{cases} x, & \text{if } x \in \sigma, \\ \sup \{t \ : \ [x, t] \subset I \setminus \sigma\}, & \text{otherwise.} \end{cases}$$

Obviously, E_f is an extension of f and is constant on the gaps in σ.

We prove the following theorems.

Theorem 2.1. *If $f, g \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$, where $\mathbb{B}$ is a commutative unital Banach algebra and σ is a non empty compact subset of $\mathbb{R}$. Then the following hold:*
(i) f and g are bounded.
(ii) $V_{\Lambda_p}(f + g, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B}) + V_{\Lambda_p}(g, \sigma, \mathbb{B})$.
(iii) $V_{\Lambda_p}(\alpha f, \sigma, \mathbb{B}) = |\alpha| V_{\Lambda_p}(f, \sigma, \mathbb{B})$, for any $\alpha \in \mathbb{C}$.
(iv) $V_{\Lambda_p}(f, \sigma', \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B})$, if a compact set $\sigma' \subset \sigma$.
(v) $V_{\Lambda_p}(fg, \sigma, \mathbb{B}) \leq ||f||_\infty V_{\Lambda_p}(g, \sigma, \mathbb{B}) + ||g||_\infty V_{\Lambda_p}(f, \sigma, \mathbb{B})$, where

$$||f||_\infty = \sup_{x \in \sigma} ||f(x)||_{\mathbb{B}} \ < \ \infty.$$

(vi) If $\sigma = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are non empty compact subsets of $\mathbb{R}$ such that $\sigma_1 \subset [a, c], \sigma_2 \subset [c, b]$ and $\sigma_1 \cap \sigma_2 = \{c\}$, then

$$V_{\Lambda_p}(f, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma_1, \mathbb{B}) + V_{\Lambda_p}(f, \sigma_2, \mathbb{B}).$$

Corollary 2.2. *Let σ_1 and σ_2 be non empty compact subsets of $\mathbb{R}$ such that $\sigma_1 \subset \sigma_2$. If $f \in \Lambda BV^{(p)}(\sigma_2, \mathbb{B})$ then $f|\sigma_1 \in \Lambda BV^{(p)}(\sigma_1, \mathbb{B})$ and $||f|\sigma_1||_{\Lambda_p(\sigma_1, \mathbb{B})} \leq ||f||_{\Lambda_p(\sigma_2, \mathbb{B})}$, where*

$$||f||_{\Lambda_p(\sigma, \mathbb{B})} = ||f||_\infty + V_{\Lambda_p}(f, \sigma, \mathbb{B}).$$

Theorem 2.3. *Let σ be a non empty compact subset of $\mathbb{R}$ and $\mathbb{B}$ be a commutative unital Banach algebra. If $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ then $V_{\Lambda_p}(f, \sigma, \mathbb{B}) = V_{\Lambda_p}(E_f, I, \mathbb{B})$, where $I = [a, b]$ is the smallest closed interval containing σ.*

Corollary 2.4. *For a given function $f : \sigma \to \mathbb{B}$. $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ if and only if $E_f \in \Lambda BV^{(p)}(I, \mathbb{B})$.*

Corollary 2.5. *The map* $F : \Lambda BV^{(p)}(\sigma,\mathbb{B}) \to \Lambda BV^{(p)}(I,\mathbb{B})$, *defined as* $F(f) = E_f$ *for all* $f \in \Lambda BV^{(p)}(\sigma,\mathbb{B})$, *is a linear isometry.*

Theorem 2.6. $(\Lambda BV^{(p)}(\sigma,\mathbb{B}), \|.\|_{\Lambda_p(\sigma,\mathbb{B})})$ *is a commutative unital Banach algebra with respect to the pointwise operations, where* $\mathbb{B}$ *is a commutative unital Banach algebra and* σ *is a non empty compact subset of* $\mathbb{R}$.

Corollary 2.7. $(\Lambda BV^{(p)}([a,b]), \|.\|_{\Lambda_p([a,b])})$ *is a commutative unital Banach algebra with respect to the pointwise operations.*

Proof of Theorem 2.1. For any $x \in \sigma$,

$$||f(x)||_{\mathbb{B}} \;\leq\; ||f(a)||_{\mathbb{B}} \;+\; (\lambda_1)^{(1/p)}\; V_{\Lambda_p}(f,\sigma,\mathbb{B}). \tag{1}$$

Thus, Theorem 2.1(i) follows.

Proof of the remaining part of the Theorem is obvious.

Proof of Theorem 2.3. $I = [a,b]$ is the smallest closed interval containing σ and $E_f|\sigma = f$ implies $V_{\Lambda_p}(f,\sigma,\mathbb{B}) \leq V_{\Lambda_p}(E_f, I,\mathbb{B})$.

For any $t \in \amalg(I)$ define $s = t \cap \sigma \in \amalg(\sigma)$. For the simplicity of the proof, suppose that there is only one $t_k \in t \setminus s$ and $(t_{k-1}, t_k) \cap \sigma = (t_k, t_{k+1}) \cap \sigma = \phi$. Then, $E_f(t_k) = f(\alpha(t_k))$ and

$$\sum_t \frac{||\triangle E_f(t_i)||^p_{\mathbb{B}}}{\lambda_i} = \sum_{i=1}^{k-2} \frac{||\triangle E_f(t_i)||^p_{\mathbb{B}}}{\lambda_i} + \frac{||\triangle E_f(t_{k-1})||^p_{\mathbb{B}}}{\lambda_{k-1}} + \frac{||\triangle E_f(t_k)||^p_{\mathbb{B}}}{\lambda_k} + \sum_{i\geq k+1} \frac{||\triangle E_f(t_i)||^p_{\mathbb{B}}}{\lambda_i}.$$

For $s^* = s \cup \{\alpha(t_k)\}$ we get

$$\sum_t \frac{||\triangle E_f(t_i)||^p_{\mathbb{B}}}{\lambda_i} \leq \sum_{s^*} \frac{||\triangle f(s_i)||^p_{\mathbb{B}}}{\lambda_i}.$$

Hence, the result follows from $V_{\Lambda_p}(E_f, I,\mathbb{B}) \leq V_{\Lambda_p}(f,\sigma,\mathbb{B})$.

Proof of Theorem 2.6. Let $\{f_k\}$ be any Cauchy sequence in the given normed linear space. Therefore it converges uniformly to some function say f. For any $t \in \amalg(\sigma)$, we get

$$\begin{aligned} V_{\Lambda_p}(f_k,\sigma,\mathbb{B},t) &\leq V_{\Lambda_p}(f_k - f_l,\sigma,\mathbb{B},t) + V_{\Lambda_p}(f_l,\sigma,\mathbb{B},t) \\ &\leq V_{\Lambda_p}(f_k - f_l,\sigma,\mathbb{B}) + V_{\Lambda_p}(f_l,\sigma,\mathbb{B}). \end{aligned}$$

This implies,

$$V_{\Lambda_p}(f_k,\sigma,\mathbb{B}) \leq V_{\Lambda_p}(f_k - f_l,\sigma,\mathbb{B}) + V_{\Lambda_p}(f_l,\sigma,\mathbb{B})$$

and

$$|V_{\Lambda_p}(f_k,\sigma,\mathbb{B}) - V_{\Lambda_p}(f_l,\sigma,\mathbb{B})| \leq V_{\Lambda_p}(f_k - f_l,\sigma,\mathbb{B}) \to 0 \; as \; k,l \to \infty.$$

Hence, $\{V_{\Lambda_p}(f_k,\sigma,\mathbb{B})\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore

$$\begin{aligned} V_{\Lambda_p}(f,\sigma,\mathbb{B},t) &= \lim_{k\to\infty} V_{\Lambda_p}(f_k,\sigma,\mathbb{B},t) \\ &\leq \lim_{k\to\infty} V_{\Lambda_p}(f_k,\sigma,\mathbb{B}) \leq M < \infty. \end{aligned}$$

Thus, $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$.

Since $\{f_k\}$ is a Cauchy sequence, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}, t) < \epsilon, \ for \ all \ k, l \geq n_0.$$

Letting $l \to \infty$ and taking supremum on the both sides of the above inequality we get $V_{\Lambda_p}(f_k - f, \sigma, \mathbb{B}) \leq \epsilon, \ for \ all \ k \geq n_0$.

Thus, $||f_k - f||_{\Lambda_p(\sigma,\mathbb{B})} \to 0 \ as \ k \to \infty$.

Hence, $(\Lambda BV^{(p)}(\sigma, \mathbb{B}), \|.\|_{\Lambda_p(\sigma,\mathbb{B})})$ is a Banach space.

For any $g, h \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$,

$$\begin{aligned} ||g.h||_{\Lambda_p(\sigma,\mathbb{B})} &= ||g.h||_\infty + V_{\Lambda_p}(g.h, \sigma, \mathbb{B}) \\ &\leq ||g||_\infty \ ||h||_\infty + ||g||_\infty \ V_{\Lambda_p}(h, \sigma, \mathbb{B}) + ||h||_\infty \ V_{\Lambda_p}(g, \sigma, \mathbb{B}) \\ &\leq (||g||_\infty + V_{\Lambda_p}(g, \sigma, \mathbb{B}))(||h||_\infty + V_{\Lambda_p}(h, \sigma, \mathbb{B})) \\ &= ||g||_{\Lambda_p(\sigma,\mathbb{B})} \ ||h||_{\Lambda_p(\sigma,\mathbb{B})}. \end{aligned}$$

This completes the proof.

3. Generalizations to several variables.

For the sake of simplicity, first we prove these results for functions of two variables and then we extend the results for functions of several variables.

Let σ_1 and σ_2 be two non empty compact subsets of $\mathbb{R}$ and let $\mathbf{R} = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ be the smallest closed rectangle containing $\sigma = \sigma_1 \times \sigma_2$.

Definition 3.1. *Let $\mathbb{L}$ be the class of all non-decreasing sequences $\Lambda' = \{\lambda_n'\}$ $(n = 1, 2, ...)$ of positive numbers such that $\sum_n \frac{1}{\lambda_n'}$ diverges. For given a Banach algebra $\mathbb{B}$, $p \geq 1$, $\Lambda = (\Lambda^1, \Lambda^2)$ and $\sigma = \sigma_1 \times \sigma_2$; where $\Lambda^1 = \{\lambda_n^1\}, \ \Lambda^2 = \{\lambda_n^2\} \in \mathbb{L}$ and σ_1, σ_2 are non empty compact subsets of $\mathbb{R}$. A function $f : \sigma \to \mathbb{B}$ is said to be of bounded $p - \Lambda-$variation (that is, $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$), if*

$$V_{\Lambda_p}(f, \sigma, \mathbb{B}) = \sup_D V_{\Lambda_p}(f, \sigma, \mathbb{B}, D) < \infty,$$

where

$$V_{\Lambda_p}(f, \sigma, \mathbb{B}, D) = \Big(\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{||\triangle f(s_i, t_j)||_{\mathbb{B}}^p}{\lambda_i^1 \ \lambda_j^2}\Big)^{1/p}$$

in which

$$\triangle f(s_i, t_j) = f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1}), \ and$$

$D = s \times t$ is a rectangular grid on σ obtained from any two partitions $s = \{s_i\}_{i=0}^n \in \Pi(\sigma_1)$ and $t = \{t_j\}_{j=0}^m \in \Pi(\sigma_2)$.

If $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ is such that the marginal functions $f(a_1, \ .) \in \Lambda^2 BV^{(p)}(\sigma_2, \mathbb{B})$ and $f(., a_2) \in \Lambda^1 BV^{(p)}(\sigma_1, \mathbb{B})$ then f is said to be of bounded $p - \Lambda^*-$variation over σ (that is, $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$). If $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ then each of the marginal functions $f(., t) \in \Lambda^1 BV^{(p)}(\sigma_1, \mathbb{B})$ and $f(s, .) \in \Lambda^2 BV^{(p)}(\sigma_2, \mathbb{B})$, where $t \in \sigma_2$ and $s \in \sigma_1$ are fixed.

For $\mathbb{B} = \mathbb{C}$, we omit writing $\mathbb{C}$, the class $\Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ reduces to the class $\Lambda^* BV^{(p)}(\sigma)$. For $\sigma = \mathbf{R}$ the class $\Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ reduces to the class $\Lambda^* BV^{(p)}(\mathbf{R}, \mathbb{B})$. For $p = 1$, we omit writing p, the classes $\Lambda BV^{(p)}(\mathbf{R})$ and $\Lambda^* BV^{(p)}(\mathbf{R})$ reduce to classes $\Lambda BV(\mathbf{R})$ and $\Lambda^* BV(\mathbf{R})$ respectively.

Definition 3.2. *For a given function $f : \sigma \to \mathbb{B}$, where $\sigma = \sigma_1 \times \sigma_2$. Define the function $E_f : \mathbf{R} \to \mathbb{B}$ by $E_f(x_1, x_2) = f(\alpha(x_1), \alpha(x_2))$ where*

$$\alpha(x_i) = \begin{cases} x_i, & \text{if } x_i \in \sigma_i, \\ sup\ \{t \ : \ [x_i, t] \subset [a_i, b_i] \setminus \sigma_i\}, & \text{otherwise}, \end{cases}$$

for $i = 1, 2$.

Theorem 3.1. *If $f, g \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$, where $\mathbb{B}$ is a commutative unital Banach algebra and $\sigma = \sigma_1 \times \sigma_2$, in which σ_1 and σ_2 are non empty compact subsets of $\mathbb{R}$. Then the following hold:*
(i) f and g are bounded.
(ii) $V_{\Lambda_p}(f + g, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B}) + V_{\Lambda_p}(g, \sigma, \mathbb{B})$.
(iii) $V_{\Lambda_p}(\alpha f, \sigma, \mathbb{B}) = |\alpha| V_{\Lambda_p}(f, \sigma, \mathbb{B})$, for any $\alpha \in \mathbb{C}$.
(iv) $V_{\Lambda_p}(f, \sigma', \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B})$, if $\sigma' = \sigma_1' \times \sigma_2' \subset \sigma$, in which σ_1' and σ_2' are non empty compact subsets of $\mathbb{R}$.

Corollary 3.2. *Let σ_1, σ_2, τ_1 and τ_2 be non empty compact subsets of $\mathbb{R}$ such that $\sigma = \sigma_1 \times \sigma_2 \subset \tau = \tau_1 \times \tau_2$. If $f \in \Lambda^* BV^{(p)}(\tau, \mathbb{B})$ then $f|\sigma \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ and $||f|\sigma||_{\Lambda_p(\sigma,\mathbb{B})} \leq ||f||_{\Lambda_p(\tau,\mathbb{B})}$, where*

$$\|f\|_{\Lambda_p(\sigma,\mathbb{B})} = \|f\|_\infty + V_{\Lambda_p}(f, \sigma, \mathbb{B}) + V_{(\Lambda^1)_p}(f(., a_2), \sigma_1, \mathbb{B}) + V_{(\Lambda^2)_p}(f(a_1, .), \sigma_2, \mathbb{B}).$$

Theorem 3.3. *Let σ_1 and σ_2 be non empty compact subsets of $\mathbb{R}$ such that $\sigma = \sigma_1 \times \sigma_2$ and $\mathbb{B}$ be a commutative unital Banach algebra. If $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ then $V_{\Lambda_p}(f, \sigma, \mathbb{B}) = V_{\Lambda_p}(E_f, \mathbf{R}, \mathbb{B})$, where $\mathbf{R}$ is the smallest closed rectangle containing σ.*

Corollary 3.4. *For a given function $f : \sigma \to \mathbb{B}$. $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ if and only if $E_f \in \Lambda^* BV^{(p)}(\mathbf{R}, \mathbb{B})$.*

Corollary 3.5. *The map $F : \Lambda^* BV^{(p)}(\sigma, \mathbb{B}) \to \Lambda^* BV^{(p)}(\mathbf{R}, \mathbb{B})$, defined as $F(f) = E_f$ for all $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$, is a linear isometry.*

Theorem 3.6. *$(\Lambda^* BV^{(p)}(\sigma, \mathbb{B}), \|.\|_{\Lambda_p(\sigma,\mathbb{B})})$ is a Banach space with respect to the pointwise operations, where $\mathbb{B}$ is a commutative unital Banach algebra and $\sigma = \sigma_1 \times \sigma_2$ in which σ_1 and σ_2 are non empty compact subsets of $\mathbb{R}$.*

Corollary 3.7. *$(\Lambda^* BV^{(p)}([a_1, b_1] \times [a_2, b_2]), \|.\|_{\Lambda_p([a_1,b_1]\times[a_2,b_2])})$ is a Banach space with respect to the pointwise operations.*

Proof of Theorem 3.1. For any $(x, y) \in \sigma$,

$$\begin{aligned} &||f(x, y)||_{\mathbb{B}} \\ &\leq ||f(a_1, a_2)||_{\mathbb{B}} + ||f(x, y) - f(x, a_2) - f(a_1, y) + f(a_1, a_2)||_{\mathbb{B}} + ||f(x, a_2)||_{\mathbb{B}} + ||f(a_1, y)||_{\mathbb{B}} \\ &\leq 3||f(a_1, a_2)||_{\mathbb{B}} + (\lambda_1^1\ \lambda_1^2)^{1/p}\ V_{\Lambda_p}(f, \sigma, \mathbb{B}) + (\lambda_1^1)^{1/p}\ V_{(\Lambda^1)_p}(f(., a_2), \sigma_1, \mathbb{B}) \\ &\qquad + (\lambda_1^2)^{1/p}\ V_{(\Lambda^2)_p}(f(a_1, .), \sigma_2, \mathbb{B}). \end{aligned}$$

Thus, Theorem 3.1(i) follows.

Proof of remaining part of the Theorem is obvious.

Proof of Theorem 3.3. **R** is the smallest closed rectangle containing $\sigma = \sigma_1 \times \sigma_2$ and $E_f|\sigma = f$ implies $V_{\Lambda_p}(f, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(E_f, \mathbf{R}, \mathbb{B})$.

Let $R = s \times t$ be any rectangle grid of $[a_1, b_1] \times [a_2, b_2]$, where $s = \{s_i\}_{i=1}^m \in \Pi([a_1, b_1])$ and $t = \{t_j\}_{j=1}^n \in \Pi([a_2, b_2])$. Consider $u = s \cap \sigma_1 \in \Pi(\sigma_1)$ and $v = t \cap \sigma_2 \in \Pi(\sigma_2)$. Then $D = u \times v$ is a rectangle grid of $\sigma = \sigma_1 \times \sigma_2$. For the simplicity of the proof, suppose there is only one $(s_k, t_l) \in R$ such that $(s_k, t_l) \in R \setminus D$, where $(s_{k-1}, s_k) \cap \sigma_1 = (s_k, s_{k+1}) \cap \sigma_1 = \phi$ and $(t_{l-1}, t_l) \cap \sigma_2 = (t_l, t_{l+1}) \cap \sigma_2 = \phi$. Then

$$\sum_{s\times t} \frac{||\triangle E_f(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2} = \sum_{i=1}^{k-2}\sum_{j=1}^{l-2} \frac{||\triangle E_f(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2} + \frac{||\triangle E_f(s_{k-1},t_{l-1})||_{\mathbb{B}}^p}{\lambda_{k-1}^1\lambda_{l-1}^2}$$

$$+\frac{||\triangle E_f(s_{k-1},t_l)||_{\mathbb{B}}^p}{\lambda_{k-1}^1\lambda_l^2} + \frac{||\triangle E_f(s_k,t_{l-1})||_{\mathbb{B}}^p}{\lambda_k^1\lambda_{l-1}^2} + \frac{||\triangle E_f(s_k,t_l)||_{\mathbb{B}}^p}{\lambda_k^1\lambda_l^2}$$

$$+\sum_{i\geq k+1}^{m}\sum_{j\geq l+1}^{n} \frac{||\triangle E_f(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2}.$$

Since $E_f(s_k, t_l) = f(\alpha(s_k, t_l))$, for $(u \times v)^* = (u \times v) \cup \{\alpha(s_k, t_l)\}$, we get

$$\sum_{s\times t} \frac{||\triangle E_f(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2} \leq \sum_{(u\times v)^*} \frac{||\triangle f(u_i,v_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2}.$$

Hence, the result follows from $V_{\Lambda_p}(E_f, \mathbf{R}, \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B})$.

Proof of Theorem 3.6. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\Lambda^*BV^{(p)}(\sigma, \mathbb{B})$. Therefore it converges uniformly to some function say f on σ. From Theorem 2.6, we get

$$\lim_{k\to\infty} V_{(\Lambda^2)_p}((f_k(a_1,.) - f(a_1,.)), \sigma_2, \mathbb{B}) = 0 \tag{2}$$

and

$$\lim_{k\to\infty} V_{(\Lambda^1)_p}((f_k(.,a_2) - f(.,a_2)), \sigma_1, \mathbb{B}) = 0. \tag{3}$$

Now, for any rectangular grid D of σ

$$V_{\Lambda_p}(f_k, \sigma, \mathbb{B}, D) \leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}, D) + V_{\Lambda_p}(f_l, \sigma, \mathbb{B}, D)$$

$$\leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}) + V_{\Lambda_p}(f_l, \sigma, \mathbb{B}).$$

This implies,

$$V_{\Lambda_p}(f_k, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}) + V_{\Lambda_p}(f_l, \sigma, \mathbb{B})$$

and

$|V_{\Lambda_p}(f_k, \sigma, \mathbb{B}) - V_{\Lambda_p}(f_l, \sigma, \mathbb{B})|$

$$\leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}) \to 0 \; as \; k, l \to \infty.$$

Hence, $\{V_{\Lambda_p}(f_k, \sigma, \mathbb{B})\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore

$V_{\Lambda_p}(f,\sigma,\mathbb{B},D) = (\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{||\triangle f(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2})^{1/p}$

$$= \lim_{k\to\infty}(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{||\triangle f_k(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2})^{1/p} \leq \lim_{k\to\infty} V_{\Lambda_p}(f_k,\sigma,\mathbb{B}) \leq M < \infty.$$

This together with (2) and (3) implies $f \in \Lambda^* BV^{(p)}(\sigma,\mathbb{B})$. Moreover,

$V_{\Lambda_p}(f_k - f,\sigma,\mathbb{B},D) = (\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{||\triangle (f_k-f)(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2})^{1/p}$

$$= \lim_{l\to\infty}(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{||\triangle (f_k-f_l)(s_i,t_j)||_{\mathbb{B}}^p}{\lambda_i^1\lambda_j^2})^{1/p}$$
$$\leq \lim_{l\to\infty} V_{\Lambda_p}(f_k-f_l,\sigma,\mathbb{B}) \to 0 \ as\ k\to\infty.$$

Thus the result follows form (2) and (3).

Finally, we extends these results for functions of several variables as follow.

Let $f(\mathbf{x}) = f(x_1,...,x_N)$ be a function from $\mathbb{R}^N$ into a Banach algebra $\mathbb{B}$. Given $\mathbf{x} = (x_1,...,x_N) \in \mathbb{R}^N$ and $\mathbf{h} = (h_1,...,h_N) \in \mathbb{R}^N$, define (see [5])

$$\triangle f(\mathbf{x};\mathbf{h}) = T_{\mathbf{h}}f(\mathbf{x}) - f(\mathbf{x}) = \triangle f(x_1,...,x_N;h_1,...,h_N)$$
$$= \sum_{\eta_1=0}^{1}...\sum_{\eta_N=0}^{1}(-1)^{\eta_1+...+\eta_N} f(x_1+\eta_1h_1,...,x_N+\eta_Nh_N).$$

Let $\sigma_1,\ \sigma_2,...,\sigma_N$ be non-empty compact subsets of $\mathbb{R}$ and $\mathbf{R} = \prod_{i=1}^{N}[a_i,b_i] \subset \mathbb{R}^N$ be the smallest closed parallelepiped containing $\sigma = \prod_{i=1}^{N}\sigma_i$.

Definition 3.3. *Let $\mathbb{L}$, p and $\mathbb{B}$ be as in the definition 3.1. For given $\Lambda = (\Lambda^1,\Lambda^2,...,\Lambda^N)$; where $\Lambda^i = \{\lambda_n^i\} \in \mathbb{L},\ \forall\ i = 1$ to N; and $\sigma = \prod_{i=1}^{N}\sigma_i$. A function $f : \sigma \to \mathbb{B}$ is said to be of bounded $p-\Lambda-$variation (that is, $f \in \Lambda BV^{(p)}(\sigma,\mathbb{B})$) if*

$V_{\Lambda_p}(f,\sigma,\mathbb{B})$

$$= \sup_{P}(\sum_{r_1=1}^{s_1}...\sum_{r_N=1}^{s_N}\frac{||\triangle f(x_1^{r_1-1},...,x_N^{r_N-1};h_1^{r_1},...,h_N^{r_N})||_{\mathbb{B}}^p}{\lambda_{r_1}^1...\lambda_{r_N}^N})^{1/p} < \infty,$$

where the supremum is extended over all partitions $P = P_1 \times P_2 \times ... \times P_N$ of the σ, $P_j = \{a_j = x_j^0 < x_j^1 < ... < x_j^{s_j} = b_j\}$ and $s_j \geq 1$; $r_j = 1,2,...,s_j$; $h_j^{r_j} = x_j^{r_j} - x_j^{r_j-1}$; $j = 1,2,...,N$.

Moreover, a function $f \in \Lambda BV^{(p)}(\sigma,\mathbb{B})$ is said to be of bounded $p-\Lambda^*-$variation (that is, $f \in \Lambda^* BV^{(p)}(\sigma,\mathbb{B})$), if for each of its marginal functions

$f(x_1,...,x_{i-1},a_i,x_{i+1},...,x_N) \in (\Lambda^1,...,\Lambda^{i-1},\Lambda^{i+1},...,\Lambda^N)^* BV^{(p)}(\sigma(a_i),\mathbb{B})$,
$\forall i = 1,2,...,N$, *where* $\sigma(a_i) = \{(x_1,...,x_{i-1},x_{i+1},...,x_N) \in \prod_{\substack{j=1\\ j\neq i}}^{N}\sigma_j\}$.

Definition 3.4. *For a given function $f : \sigma \to \mathbb{B}$, where $\sigma = \prod_{i=1}^{N} \sigma_i$. Define the function $E_f : \mathbf{R} \to \mathbb{B}$ by $E_f(\boldsymbol{x}) = f(\alpha(x_1), \alpha(x_2), ..., \alpha(x_N))$, $\boldsymbol{x} = (x_1, x_2, ...x_N)$, where*

$$\alpha(x_i) = \begin{cases} x_i, & \text{if } \; x_i \in \sigma_i, \\ \sup \{t \;:\; [x_i, t] \subset [a_i, b_i] \setminus \sigma_i\}, & \text{otherwise,} \end{cases}$$

for all $i = 1, 2, ..., N$.

Theorem 3.8. *If $f, g \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$, where $\mathbb{B}$ is a commutative unital Banach algebra and $\sigma = \prod_{i=1}^{N} \sigma_i$ in which σ_i are non empty compact subsets of $\mathbb{R}$, for all $i = 1$ to N . Then the following hold:*
(i) f and g are bounded.
(ii) $V_{\Lambda_p}(f + g, \sigma, \mathbb{B}) \le V_{\Lambda_p}(f, \sigma, \mathbb{B}) + V_{\Lambda_p}(g, \sigma, \mathbb{B})$.
(iii) $V_{\Lambda_p}(\alpha f, \sigma, \mathbb{B}) = |\alpha| V_{\Lambda_p}(f, \sigma, \mathbb{B})$, for any $\alpha \in \mathbb{C}$.
(iv) $V_{\Lambda_p}(f, \sigma', \mathbb{B}) \le V_{\Lambda_p}(f, \sigma, \mathbb{B})$, if $\sigma' = \prod_{i=1}^{N} \sigma_i' \subset \sigma$, in which σ_i' are non empty compact subsets of $\mathbb{R}$, for all $i = 1$ to N.

Corollary 3.9. *Let σ_i and τ_i be non empty compact subsets of $\mathbb{R}$, for all $i = 1$ to N, such that $\sigma = \prod_{i=1}^{N} \sigma_i \subset \tau = \prod_{i=1}^{N} \tau_i$. If $f \in \Lambda^* BV^{(p)}(\tau, \mathbb{B})$ then $f|\sigma \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ and $||f|\sigma||_{\Lambda_p(\sigma,\mathbb{B})} \le ||f||_{\Lambda_p(\tau,\mathbb{B})}$, where*

$$\begin{aligned} \|f\|_{\Lambda_p(\sigma,\mathbb{B})} &= \|f\|_\infty + V_{\Lambda_p}(f, \sigma, \mathbb{B}) \\ &\quad + \sum_{i=1}^{N} V_{(\Lambda^1, ...\Lambda^{i-1}, \Lambda^{i+1}, ..., \Lambda^N)_p}(f(..., a_i, ...), \prod_{\substack{j=1 \\ j \ne i}}^{N} \sigma_j, \mathbb{B}). \end{aligned}$$

Theorem 3.10. *Let σ_i be non empty compact subsets of $\mathbb{R}$, for all $i = 1$ to N, such that $\sigma = \prod_{i=1}^{N} \sigma_i$ and $\mathbb{B}$ be a commutative unital Banach algebra. If $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ then $V_{\Lambda_p}(f, \sigma, \mathbb{B}) = V_{\Lambda_p}(E_f, \mathbf{R}, \mathbb{B})$, where $\mathbf{R}$ is the smallest closed parallelepiped containing $\sigma = \prod_{i=1}^{N} \sigma_i$.*

Corollary 3.11. *For a given function $f : \sigma \to \mathbb{B}$. $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$ if and only if $E_f \in \Lambda^* BV^{(p)}(\mathbf{R}, \mathbb{B})$.*

Corollary 3.12. *The map $F : \Lambda^* BV^{(p)}(\sigma, \mathbb{B}) \to \Lambda^* BV^{(p)}(\mathbf{R}, \mathbb{B})$, defined as $F(f) = E_f$ for all $f \in \Lambda^* BV^{(p)}(\sigma, \mathbb{B})$, is a linear isometry.*

Theorem 3.13. *$(\Lambda^* BV^{(p)}(\sigma, \mathbb{B}), \|.\|_{\Lambda_p(\sigma,\mathbb{B})})$ is a Banach space with respect to the pointwise operations, where $\mathbb{B}$ is a commutative unital Banach algebra and $\sigma = \prod_{i=1}^{N} \sigma_i$ in which σ_i are non empty compact subsets of $\mathbb{R}$ for, all $i = 1$ to N.*

Corollary 3.14. *$(\Lambda^* BV^{(p)}(\prod_{i=1}^{N} [a_i, b_i]), \|.\|_{\Lambda_p(\prod_{i=1}^{N} [a_i, b_i])})$ is a Banach space with respect to the pointwise operations.*

Proof of all these extended theorems are similarly follows by induction arguments on N.

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