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APPLICATION OF HÖLDER'S INEQUALITY AND CONVOLUTIONS

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Abstract. In this paper we introduce a new subclass $M_p(n,\ \alpha, c)$ of analytic and multivalent functions in the unit disk which includes the class $S_p(n, \alpha)$ of multivalent starlike functions of order α and the class $T_p(n, \alpha)$ of multivalent convex functions of order α . Using generalized Bernardi Libera integral operator and Hölder's inequality, some interesting properties of convolution for the class $M_p(n, \alpha, c)$ are considered

1. Introduction

Let $A_p(n)$ be class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathcal{N})$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathcal{C}; |z| < 1\}$. Let $S_p(n, \alpha)$ be the subclass of $A_p(n)$ consisting of functions $f(z)$ which satisfy

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{U})$$

for some $\alpha(0 \leq \alpha < p)$. A function $f(z) \in S_p(n, \alpha)$ is known as starlike of order α in U.

Further, let $T_p(n, \alpha)$ be the subclass of $A_p(n)$ consisting of functions $f(z)$ satisfying $\frac{zf'(z)}{p} \in S_p(n, \alpha)$, that is,

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{U})$$

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for some α $(o \leq \alpha < p)$. A function $f(z)$ in $T_p(n,\alpha)$ is known as convex of order α in U.

These classes $A_p(n)$, $S_p(n,\alpha)$ and $T_p(n,\alpha)$ were studied earlier by Owa [9] respectively. Also Nishiwaki and Owa [6] have given the following lemmas which provide the sufficient conditions for functions $f(z) \in A_p(n)$ to be in the classes $S_p(n,\alpha)$ and $T_p(n,\alpha)$ respectively.

Lemma 1.1. *If* $f(z) \in A_p(n)$ *satisfies*

$$\sum_{k=p+n}^{\infty} (k-\alpha)\,|a_k| \leq p-\alpha \tag{1.1}$$

for some α $(o \leq \alpha < p)$, *then* $f(z) \in S_p(n,\alpha)$

Lemma 1.2. *If* $f(z) \in A_p(n)$ *satisfies*

$$\sum_{k=p+n}^{\infty} k(k-\alpha)\,|a_k| \leq p(p-\alpha) \tag{1.2}$$

for some α $(o \leq \alpha < p)$, *then* $f(z) \in T_p(n,\alpha)$.

Remark 1. We note that Silverman [10] has given Lemma (1.1) and Lemma (1.2) in the case of $p = 1$ and $n = 1$. Also Srivastava, Owa and Chatterjea [11] have given the coefficient inequalities in the case of $p = 1$.

In view of lemmas (1.1) and (1.2) Nishiwaki and Owa [6] introuced the subclasses $S_p^*(n,\alpha)$, $T_p^*(n,\alpha)$ consisting of functions $f(z)$ which satisfy the coefficient inequalities (1.1) and (1.2) respectively.

Now, we introduce subclass $M_p^*(n,\alpha,c)$ consisting of functions $f(z) \in A_p(n)$ which satisfy the following coefficient inequality.

$$\sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^c (k-\alpha)\,|a_k| \leq p-\alpha \tag{1.3}$$

for some $c \geq 0$ and α $(o \leq \alpha < p)$.

Obviously $M_p^*(n,\alpha,0) \equiv S_p^*(n,\alpha)$ and $M_p^*(n,\alpha,1) \equiv T_p^*(n,\alpha)$.

For functions $f_j(z) \in A_p(n)$ given by

$$f_j(z) = z^p + \sum_{k=p+n}^{\infty} a_{k,j} z^k (j = 1, 2, ..., m), \tag{1.4}$$

we define

$$G_m(z) = z^p + \sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}\right) z^k, \tag{1.5}$$

and

$$H_m(z) = z^p + \sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} (a_{k,j})^{p_j}\right) z^k \quad (p_j > 0) \tag{1.6}$$

where $G_m(z)$ denotes the convolution of $f_j(z)$ $(j = 1, 2, ..., m)$. Therefore $H_m(z)$ is the generalization of convolutions. In the case $p_j = 1$, we have $G_m(z) := H_m(z)$. The generalization of the convolutions was considered by Choi, Kim and Owa [2], and Nishiwaki and Owa [6].

Further for functions $f_j(z) \in A_p(n)$ given by (1.4), generalized Libera integral operator is defined as follows :

$$B_{j,p}(z) = \frac{p+c_j}{z^{c_j}} \int_0^z t^{c_j-1} f(t)\ dt \quad (c_j > -p) \tag{1.7}$$

(For $c_j = 1$, we obtain multivalent Libera operator. For $p = 1$, we get generalized Bernardi-Libera- Livingston integral operator defined recently by Gordji et al. [4] and for $p = 1$, $c_j = 1$, Libera [5] studied the above operator. The operator (1.7) also includes the Alexander operator [1] for $p = 1$ and $c_j = 0$)

Using the operator (1.7), we find that the convolution integral of $B_{1,p}$ and $B_{2,p}$ as

$$(B_{1,p} * B_{2,p})(z) = \frac{p+c_1}{z^{c_1}} \int_0^z t^{c_1-1} f(t)dt * \ \frac{p+c_2}{z^{c_2}} \int_0^z t^{c_2-1} f(t)dt$$
$$= z^p + \sum_{k=p+n}^{\infty} \frac{(p+c_1)(p+c_2)}{(k+c_1)(k+c_2)} a_{k,1} a_{k,2} z^k$$

The convolution integral was studied by Duren [3].
Hence the convolution integral of $B_{1,p}, B_{2,p}, ..., B_{m,p}$ is given by

$$(B_{1,p} * B_{2,p} * ... * B_{m,p})(z) = \frac{p+c_1}{z^{c_1}} \int_0^z t^{c_1-1} f(t)dt * \ ... * \frac{p+c_m}{z^{c_m}} \int_0^z t^{c_m-1} f(t)dt$$

$$= z^p + \sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} \frac{p+c_j}{k+c_j} a_{k,j} \right) z^k \tag{1.8}$$

For functions $f_j(z) \in A_p(n)$ $(j = 1, 2, ..., m)$ given by (1.4), the familiar Hölder's inequality assumes the form

$$\sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} |a_{k,j}| \right) \leq \prod_{j=1}^{m} \left(\sum_{k=p+n}^{\infty} |a_{k,j}|^{p_j} \right)^{\frac{1}{p_j}} \tag{1.9}$$

where $p_j > 1$ and $\sum_{j=1}^{m} \frac{1}{p_j} \geq 1 \quad (j = 1, 2, ..., m)$

Nishiwaki, Owa and Srivastava [8] have given some results of Hölder's-type inequalities for subclass of uniformly starlike functions. Also applying these inequalities, Nishiwaki and Owa [6] have obtained some interesting properties of generalizations of convolutions for functions $f(z)$ in the classes $S_p^*(n, \alpha)$ and $T_p^*(n, \alpha)$. Again Nishiwaki and Owa [7] have given application of convolution integral for certain subclasses by using Hölder's inequalities . Motivated essentially by these papers, we discuss some applications of Hölder's inequalities for $H_m(z)$ defined by (1.6) and convolution integral defined by (1.8) for the subclass $M_p^*(n, \alpha, c)$.

2. Main Results

Theorem 1. *If* $f_j(z) \in M_p^*(n, \alpha_j, c)$ *for each* $j = 1, 2, ..., m$, *then* $H_m(z) \in M_p^*(n, \beta, c)$ *with*

$$\beta = \inf_{k \geq p+n} \left\{ p - \frac{k^c(k-p) \prod_{j=1}^{m} [p^c(p-\alpha_j)]^{p_j}}{p^c \prod_{j=1}^{m} [k^c(k-\alpha_j)]^{p_j} - k^c \prod_{j=1}^{m} [p^c(p-\alpha_j)]^{p_j}} \right\} \tag{2.1}$$

where $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m} \frac{1}{q_j} \geq 1$.

Proof . Since $f_j(z) \in M_p^*(n, \alpha_j, c)$, therefore by equation (1.3), we get

$$\sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^c \left(\frac{k-\alpha_j}{p-\alpha_j}\right) |a_{k,j}| \leq 1 \quad (j = 1, 2, ..., m)$$

which implies

$$\left\{ \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^c \left(\frac{k-\alpha_j}{p-\alpha_j}\right) |a_{k,j}| \right\}^{\frac{1}{q_j}} \leq 1) \tag{2.2}$$

with $q_j > 1$ and $\sum_{j=1}^{m} \frac{1}{q_j} \geq 1$.

From (2.2), we have

$$\prod_{j=1}^{m} \left\{ \sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^c \left(\frac{k-\alpha_j}{p-\alpha_j}\right) |a_{k,j}| \right\}^{\frac{1}{q_j}} \leq 1$$

Applying Hölder's inequality (1.9), we find that

$$\sum_{k=p+n}^{\infty} \left\{ \prod_{j=1}^{m} \left(\frac{k}{p}\right)^{\frac{c}{q_j}} \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\} \leq 1 \tag{2.3}$$

Note that we have to find the largest β such that

$$\sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^c \left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^{m} |a_{k,j}|^{p_j} \right) \leq 1$$

that is,

$$\sum_{k=p+n}^{\infty} \left(\frac{k}{p}\right)^c \left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^{m} |a_{k,j}|^{p_j} \right) \leq \sum_{k=p+n}^{\infty} \left\{ \prod_{j=1}^{m} \left(\frac{k}{p}\right)^{\frac{c}{q_j}} \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\}$$

Therefore we need to find largest β such that

$$\left(\frac{k}{p}\right)^c \left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^{m} |a_{k,j}|^{p_j} \right) \leq \prod_{j=1}^{m} \left(\frac{k}{p}\right)^{\frac{c}{q_j}} \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}}$$

which is equivalent to

$$\left(\frac{k}{p}\right)^c \left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^{m} |a_{k,j}|^{p_j-\frac{1}{q_j}}\right) \leq \prod_{j=1}^{m} \left(\frac{k}{p}\right)^{\frac{c}{q_j}} \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}}$$

for all $k \geq p+n$. Since

$$\prod_{j=1}^{m} \left\{ \left(\frac{k}{p}\right)^c \left(\frac{k-\alpha_j}{p-\alpha_j}\right) |a_{k,j}| \right\}^{p_j-\frac{1}{q_j}} \leq 1 \quad \left(p_j - \frac{1}{q_j} \geq 0\right)$$

,
therefore

$$\prod_{j=1}^{m} |a_{k,j}|^{p_j-\frac{1}{q_j}} \leq \frac{1}{\prod_{j=1}^{m} \left\{ \left(\frac{k}{p}\right)^c \left(\frac{k-\alpha_j}{p-\alpha_j}\right) \right\}^{p_j-\frac{1}{q_j}}}$$

This implies that

$$\left(\frac{k}{p}\right)^c \left(\frac{k-\beta}{p-\beta}\right) \leq \prod_{j=1}^{m} \left\{ \left(\frac{k}{p}\right)^c \left(\frac{k-\alpha_j}{p-\alpha_j}\right) \right\}^{p_j}$$

for all $k \geq p+n$. Therefore β should be

$$\beta \leq p - \frac{k^c(k-p) \prod_{j=1}^{m} (p^c(p-\alpha_j))^{p_j}}{p^c \prod_{j=1}^{m} (k^c(k-\alpha_j))^{p_j} - k^c \prod_{j=1}^{m} (p^c(p-\alpha_j))^{p_j}}$$

for all $k \geq p+n$ This completes the proof of the theorem.

Taking $p_j = 1$ in Theorem 1, we obtain

Corollary 2.1. *If* $f_j(z) \in M_p^*(n, \alpha_j, c)$ *for each* $j = 1, 2, ..., m$, *then* $G_m(z) \in M_p^*(n, \beta, c)$ *with*

$$\beta = p - \frac{np^{(m-1)c} \prod_{j=1}^{m} (p-\alpha_j)}{(p+n)^{(m-1)c} \prod_{j=1}^{m} (p+n-\alpha_j) - p^{(m-1)c} \prod_{j=1}^{m} (p-\alpha_j)} \tag{2.4}$$

Proof . In view of Theorem 1, we obtain

$$\beta \leq \inf_{k \geq p+n} \left\{ p - \frac{k^c(k-p) \prod_{j=1}^{m} p^c(p-\alpha_j)}{p^c \prod_{j=1}^{m} k^c(k-\alpha_j) - k^c \prod_{j=1}^{m} p^c(p-\alpha_j)} \right\}$$

Let $F(c, k, m)$ be the right-hand side of the above inequality. Further let us define $G(c, k, m)$ be the numerator of $F'(c, k, m)$. Then

$$G(c, k, m) = -p^{(m-1)c} \prod_{j=1}^{m} (p-\alpha_j) \left\{ k^{(m-1)c} \prod_{j=1}^{m} (k-\alpha_j) - p^{(m-1)c} \prod_{j=1}^{m} (p-\alpha_j) \right\}$$

$$+(k-p)(pk)^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)[\tfrac{(m-1)c}{k}\prod_{j=1}^{m}(k-\alpha_j)$$

$$+\{(k-\alpha_2)(k-\alpha_3)\cdots(k-\alpha_m)+\cdots+(k-\alpha_1)(k-\alpha_2)\cdots(k-\alpha_{m-1})\}]$$

$$=-p^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)\left\{k^{(m-1)c}\prod_{j=1}^{m}(k-\alpha_j)-p^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)\right\}$$

$$+(k-p)(pk)^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)\left[\tfrac{(m-1)c}{k}\prod_{j=1}^{m}(k-\alpha_j)+\left\{\sum_{j=1}^{m}\frac{\prod_{j=1}^{m}(k-\alpha_j)}{(k-\alpha_j)}\right\}\right].$$

$$=(pk)^{(m-1)c}\prod_{j=1}^{m}\{(p-\alpha_j)(k-\alpha_j)\}\left\{(k-p)\left(\sum_{j=1}^{m}\frac{1}{k-\alpha_j}+\frac{(m-1)c}{k}\right)-1\right\}$$

$$+p^{2(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)^2\geq 0$$

Thus $F(c,k,m)$ is increasing function for all $k\geq p+n$.
This means that

$$\beta=\ F(p+n)=p-\frac{np^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)}{(p+n)^{(m-1)c}\prod_{j=1}^{m}(p+n-\alpha_j)-p^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)}$$

On taking $c=0$ and $c=1$ in Theorem 1, we get the Theorem 2.1, 2.6 and 2.8 obtained recently by Nishiwaki and Owa [6]. Corollaries 2.2, 2.3, 2.5, 2.7 and 2.9 due to them are also special cases of our Theorem 1 and corollary 2.1

Theorem 2. *If* $f_j(z)\in M_p^*(n,\alpha_j,c)$ *for each* $j=1,2,.....,m$ *then* $(B_1*B_2.....*B_m)(z)\in M_p^*(n,\beta,c)$ *with*

$$\beta=p-\frac{np^{(m-1)c}\prod_{j=1}^{m}(p+c_j)(p-\alpha_j)}{(p+n)^{(m-1)c}\prod_{j=1}^{m}(p+n+c_j)(p+n-\alpha_j)-p^{(m-1)c}\prod_{j=1}^{m}(p+c_j)(p-\alpha_j)} \tag{2.5}$$

Proof . Since $f_j(z)\in M_p^*(n,\alpha_j,c)$, then from (2.3) , we get

$$\sum_{k=p+n}^{\infty}\left\{\prod_{j=1}^{m}\left(\frac{k}{p}\right)^{\frac{c}{q_j}}\left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}}|a_{k,j}|^{\frac{1}{q_j}}\right\}\leq 1$$

Note that we have to find the largest β such that

$$\sum_{k=p+n}^{\infty}\left(\frac{k}{p}\right)^{c}\left(\frac{k-\beta}{p-\beta}\right)\prod_{j=1}^{m}\left(\frac{p+c_j}{k+c_j}\right)|a_{k,j}|\leq 1$$

that is

$$\sum_{k=p+n}^{\infty}\left(\frac{k}{p}\right)^{c}\left(\frac{k-\beta}{p-\beta}\right)\prod_{j=1}^{m}\left(\frac{p+c_j}{k+c_j}\right)|a_{k,j}|\leq\sum_{j=1}^{m}\left\{\prod_{j=1}^{m}\left(\frac{k}{p}\right)^{\frac{c}{q_j}}\left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}}|a_{k,j}|^{\frac{1}{q_j}}\right\}$$

Using the same procedure of Theorem 1, we can easily prove that for all $k \geq p+n$, β should be

$$\beta \leq p - \frac{k^c(k-p)\prod_{j=1}^{m}(p+c_j)p^c(p-\alpha_j)}{p^c\prod_{j=1}^{m}(k+c_j)k^c(k-\alpha_j) - k^c\prod_{j=1}^{m}(p+c_j)p^c(p-\alpha_j)} \tag{2.6}$$

Let $F(c,k,m)$ be the right hand side of the above inequality. Further let us define $G(c,k,m)$ be the numerator of $F'(c,k,m)$. Then

$$G(c,k,m) = -p^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)(p+c_j)\left\{k^{(m-1)c}\prod_{j=1}^{m}(k-\alpha_j)(k+c_j) - p^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)(p+c_j)\right\}$$

$$+(k-p)(pk)^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)(p-c_j)[\frac{(\mathrm{m}-1)\,\mathrm{c}}{k}\prod_{j=1}^{m}(k-\alpha_j)(k+c_j)$$

$$+\{(k-\alpha_2)(k-\alpha_3)\cdots(k-\alpha_m)(k+c_1)\cdots(k+c_m)+\cdots+(k-\alpha_1)$$

$$(k-\alpha_2)\cdots(k-\alpha_{m-1})(k+c_1)\cdots(k+c_m)+(k-\alpha_1)\cdots(k-\alpha_m)(k+c_2)\cdots(k+c_m)+$$

$$\cdots+(k-\alpha_1)\cdots(k-\alpha_m)(k+c_1)\cdots(k+c_{m-1})\}]$$

$$= -p^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)(p+c_j)\left\{k^{(m-1)c}\prod_{j=1}^{m}(k-\alpha_j)(k+c_j) - p^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)(p+c_j)\right\}$$

$$+(k-p)(pk)^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)(p+c_j)\left[\frac{(\mathrm{m}-1)\mathrm{c}}{\mathrm{k}}\prod_{\mathrm{j}=1}^{\mathrm{m}}(\mathrm{k}-\alpha_{\mathrm{j}})(\mathrm{k}+\mathrm{c_j})\right.$$

$$\left.+\left\{\sum_{j=1}^{m}\frac{\prod_{j=1}^{m}(k-\alpha_j)(k+c_j)}{k-\alpha_j}+\sum_{j=1}^{m}\frac{\prod_{j=1}^{m}(k-\alpha_j)(k+c_j)}{k+c_j}\right\}\right]$$

$$= (pk)^{(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)(p+c_j)(k-\alpha_j)(k+c_j)$$

$$\times\left\{(k-p)\left(\sum_{j=1}^{m}\frac{1}{k-\alpha_j}\sum_{j=1}^{m}\frac{1}{k+c_j}+\frac{(m-1)c}{k}\right)-1\right\}+p^{2(m-1)c}\prod_{j=1}^{m}(p-\alpha_j)^2(p+c_j)^2 \geq 0.$$

Thus $F(c,k,m)$ is increasing function for all $k \geq p+n$.
This means that

$$\beta = F(p+n) = p - \frac{np^{(m-1)c}\prod_{j=1}^{m}(p+c_j)(p-\alpha_j)}{(p+n)^{(m-1)c}\prod_{j=1}^{m}(p+n+c_j)(p+n-\alpha_j) - p^{(m-1)c}\prod_{j=1}^{m}(p+c_j)(p-\alpha_j)}$$

If we take $c_j = 1$ in Theorem 2, we get

Corollary 2.2. *If* $f_j(z) \in M_p^*(n, \alpha_j, c)$ *for each* $j = 1, 2, \ldots.., m$ *then* $(B_1 * B_2 \ldots.. * B_m)(z) \in M_p^*(n, \beta, c)$ *with*

$$\beta = p - \frac{np^{(m-1)c}(p+1)^m \prod_{j=1}^{m}(p-\alpha_j)}{(p+n)^{(m-1)c}(p+n+1)^m \prod_{j=1}^{m}(p+n-\alpha_j) - p^{(m-1)c}(p+1)^m \prod_{j=1}^{m}(p-\alpha_j)}$$

If we take $p = 1$ in Theorem 2 and Corollary 2.2, we deduce that

Corollary 2.3. *If* $f_j(z) \in M^*(n, \alpha_j, c)$ *for each* $j = 1, 2, \ldots.., m$ *then* $(B_1 * B_2 \ldots.. * B_m)(z) \in M^*(n, \beta, c)$ *with*

$$\beta = 1 - \frac{n \prod_{j=1}^{m}(1+c_j)(1-\alpha_j)}{(n+1)^{(m-1)c} \prod_{j=1}^{m}(n+1+c_j)(n+1-\alpha_j) - \prod_{j=1}^{m}(1+c_j)(1-\alpha_j)}$$

Corollary 2.4. *If* $f_j(z) \in M^*(n, \alpha_j, c)$ *for each* $j = 1, 2, \ldots.., m$ *then* $(B_1 * B_2 \ldots.. * B_m)(z) \in M^*(n, \beta, c)$ *with*

$$\beta = 1 - \frac{n2^m \prod_{j=1}^{m}(1-\alpha_j)}{(n+1)^{(m-1)c}(n+2)^m \prod_{j=1}^{m}(n+1-\alpha_j) - 2^m \prod_{j=1}^{m}(1-\alpha_j)}$$

Further taking $c_j = 0$ in Corollary 2.3, we get

Corollary 2.5. *If* $f_j(z) \in M_p^*(n, \alpha_j, c)$ *for each* $j = 1, 2, \ldots.., m$ *then* $(B_1 * B_2 \ldots.. * B_m)(z) \in M_p^*(n, \beta, c)$ *with*

$$\beta = 1 - \frac{n \prod_{j=1}^{m}(1-\alpha_j)}{(n+1)^{(m-1)c+m} \prod_{j=1}^{m}(n+1-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}$$

Remark. All results due to Nishiwaki and Owa [7] can be deduced as special cases if we put $c = 0$ and $c = 1$ in Theorem 2 and Corollaries 2.2-2.5.

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References

[1] J.W.Alexander, *Functions which map the interior of the unit circle upon simple regions* , Ann. Math. **17** (1915), 12-22.

[2] J.H.Choi, Y.C.Kim and S.Owa, *Generalizations of Hadmard products of functions with negative coefficients* , J. Math. Anal. Appl. **199** (1996),495-501.

[3] P.L.Duren, *"Comprehensive Studies in Mathematics" 259, Springer-verlag, New York-Berlin-Heiderberg-Tokyo* ,Univalent Functions (1983).

[4] M.E.Gordji, D.Alimohammadi, A.Ebadian, *Some applications of the generalized Bernardi-Libera- Livingston integral operator on univalent functions*, Jour. Inequal. Pure and Appl. Math.(JIPAM) **10** (2009) Issue 4, Article 100, 12 pp.

[5] R.J.Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755-758.
[6] J.Nishiwaki and S.Owa, *An application of Hölder's inequality for convolutions*, Jour. Inequal. Pure Appl. Math.(JIPAM) **10** (2009), Issue 4, Article 98, 7 pp.
[7] J.Nishiwaki and S.Owa, *An application of convolutions integral*, Fractional Calculus and Appl. Anal. **13** (2010), 395-402.
[8] J.Nishiwaki, S.Owa and H.M.Srivastava, *Convolution and Hölder-type inequalities for a certain class of analytic functions* , Math. Inequal. Appl. **11** (2008),717-727.
[9] S.Owa, *On certain classes of p-valent functions with negative coefficients* , Simon. Stevin **59** (1985), 385-402.
[10] H.Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975),109-116.
[11] H.M.Srivastava, S.Owa and S.K.Chatterjea, *A note on certain classes of starlike functions*, Rend. Sem. Mat. Univ. Padova **77** (1987), 115-124.

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