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RATE OF A-STATISTICAL APPROXIMATION OF A MODIFIED Q-BERNSTEIN OPERATORS

(COMMUNICATED BY PROFESSOR SONGXIAO LI)

SHENGGUI LIU

Abstract. In this paper, we discuss properties of convergence for a modification of the q-Bernstein operators. Using the notion of A-statistical approximation, where A is a nonnegative regular summability matrix, we investigate the Korovkin type statistical approximation properties of this modification via A-statistical approximation. For $0 < q \leq 1$, we obtain that the q-Bernstein operators is A-statistical convergence to $f(x)$, and show that the rate of convergence for the modified q-Bernstein operators is better than the q-Bernstein operators on interval $[0, \gamma_n] \subset [0, 1]$ by means of the modulus of continuity.

1. Introduction

Since Bernstein polynomials play an important role in approximation theory and its applications, their various generalizations have been studied [1, 2, 3, 4]. In recent years, due to the intensive development of q-Calculus, the generalizations of Bernstein polynomials connected with q-Calculus have emerged.

We first give some notations on q-analysis that need in this paper. Let $q > 0$, for each nonnegative integer k, the q-factorical $[k]_q!$ are defined by

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1 \\ 1, & k = 0, \end{cases}$$

where

$$[k]_q = \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k, & q = 1. \end{cases}$$

For the integers $n, k, n \geq k \geq 0$, the q-binomial, or the Gaussian coefficient is defined by

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$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We also use the following standard notation:

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{s=0}^{k-1}(1-aq^s), \qquad (a;q)_\infty := \prod_{s=0}^{\infty}(1-aq^s).$$

In [5], Philips introduced the q-Bernstein as follows. For $n \in N$ and $f \in C[0,1]$,

$$B_n(f;q;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\left[\begin{array}{c} n \\ k \end{array}\right]_q x^k (x;q)_{n-k}. \tag{1.1}$$

It is obviously, for $q = 1$, the $B_n(f,q;x)$ is the classical Bernstein polynomial.

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}.$$

For its important role in approximation theory, a lot of interesting results related to the q-Bernstein polynomials have been obtained [5, 6, 7, 8, 9, 10]. In [7], it has shown that q-Bernstein operator is convexity-preserving. From those known results we also know that, for $q \neq 1$, the q-Bernstein polynomials possess many interesting properties, some of which are distinctly different from the classical Bernstein polynomials [11, 12, 13, 14, 15]. For example, for $0 < q < 1$, $f \in C[0,1]$, $B_n(f,q;x)$ does not converge to $f(x)$ as the classical Bernstein operator does when $n \to \infty$, but converges to a limit operator which is defined by

$$B_\infty(f;q;x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)p_{\infty,k}(q,x), & 0 \leq x < 1 \\ f(1), & x = 1, \end{cases}$$

here $p_{\infty,k}(q,x) = \frac{x^k}{(1-q)^k[[k]_q!}(x;q)_\infty$ [10]. In [11], Videnskii replaced q in (1.1) by a sequence q_n in the interval $(0,1]$, and he obtained the following result.

Theory A. For any $f \in C[0,1]$, the following inequality holds

$$\mid B_n(f;q_n;x) - f(x) \mid \leq 2\omega\left(f, \sqrt{\frac{x(1-x)}{[n]_{q_n}}}\right). \tag{1.2}$$

Theorem A shows that for any $f \in C[0,1]$, the sequence $B_n(f;q_n;x)$ convergences uniformly to f if and only if $\lim_{n\to\infty} q_n = 1$.

Let $\{x_n\}_{n\in N}$ be a sequence of numbers. Then, $\{x_n\}_{n\in N}$ is called statistical convergence to a number M if, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{\sharp\{n \leq k : \mid x_n - M \mid \geq \varepsilon\}}{k} = 0,$$

where $\sharp B$ denote the cardinality of the subset B [10] [18]. We denote this statistical limit by

$$st-\lim_{n\to\infty} x_n = M.$$

Let $A=(a_{jn})$ be an infinite summability matrix. Then, the A-transform of x, denoted $Ax:=(Ax)_j$, is given by $(Ax)_j=\sum_{n=1}^{\infty} a_{jn}x_n$, provided the series converges for each j. We say that A is regular if $\lim_{j\to\infty}(Ax)_j=M$ whenever $\lim_{j\to\infty} x_j=M$ [11]. For example, the Cesàro matrix $C_1=(c_{jn})$ defined by

$$c_{jn}=\begin{cases} \frac{1}{j}, & if\ 1\le n\le j \\ 0, & \text{otherwise,}\end{cases}$$

is a regular matrix. Assume now that A is a nonnegative regular summability matrix. Freedman and Sember [22] introduced the notion of A-statistical convergence as the following way, which is a more general method of statistical convergence. The sequence $(x_n)_{n\in N}$ is said to be A-statistically convergent to M if, for every $\varepsilon>0$,

$$\lim_{j\to\infty} \sum_{n:|x_n-M|\ge\varepsilon} a_{jn}=0$$

holds. This limit is denoted by

$$st_A-\lim_{n\to\infty} x_n = M.$$

Replacing the matrix A by the identity matrix, A-statistical convergence reduces to the ordinary convergence. And it is not hard to see that if we take $A=C_1$, then C_1-statistical convergence also coincides with the statistical convergence mentioned above, i.e.,

$$st_{C_1}-\lim_{n\to\infty} x_n = st-\lim_{n\to\infty} x_n.$$

Here we should remark that every convergent sequence is A-statistically convergent to the same value for any nonnegative regular matrix A, but its converse is not valid. In particular, Kolk proved that A-statistical convergence is stronger than ordinary convergence whenever the non-negative regular matrix $A=(a_{jn})$ satisfies $\lim_n \max\{a_{jn}\}=0$ [23].

Recently, statistical convergence of functions by means of linear operators were introduced [16, 17, 18, 19]. Using the concept of statistical convergence in the approximation theory provides us with many advantages. In particular, the matrix summability methods of Cesàro type are strong enough to correct the lack of convergence of various sequences of linear operators such as the interpolation operator of Hermite-Fejér, because these types of operators do not converge at points of simple discontinuity [20, 21]. A-statistical convergence has been shown to be quite effective in summing non-convergent sequences of positive linear operators [17, 18, 19].

In this paper, a new modification of q-Bernstein operators will be defined as follows.

For $0<q\le 1, f\in C[0,1]$,

$$L_n(f;q;x)=\sum_{k=0}^{n} f\left(q^n\frac{[k]_q}{[n]_q}\right)\left[\begin{array}{c} n\\ k\end{array}\right]_q x^k(x;q)_{n-k}. \tag{1.3}$$

And we will study the A-statistical convergence for this operators. Furthermore, the rate of A-statistical convergence of the $L_n(f,q;x)$ by means of the modulus continuity was computed, and we prove that this modification provides a better estimation than the operators $B_n(f;q;x)$ on the interval $[0,\gamma_n]$, where $\gamma_n=\frac{1}{1+[n]_q}$.

2. Main results

In this section, we will state and prove our main results in this paper. In order to prove our main results, we need the following two lemmas.

Lemma 2.1. *For all $n\in N$ and $0<q\le 1$, $f\in C[0,1]$,*

$$L_n(e_0;q;x)=1, \tag{2.1}$$

$$L_n(e_1;q;x)=q^n x, \tag{2.2}$$

$$L_n(e_2;q;x)=q^{2n}x^2+\frac{x(1-x)}{[n]_q}q^{2n}, \tag{2.3}$$

where $e_i(t)=t^i$ for $i=0,1,2$.

Proof. It is proved in [1, 4] that $B_n(f;q;x)$ reproduce linear functions, that is,

$$B_n(at+b;q;x)=ax+b. \tag{2.4}$$

It follows from (2.4) that (2.1) and (2.2) are correct. Using the following formula

$$[k]_q=q[k-1]_q+1,$$

we get

$$\begin{aligned}
L_n(e_2;q;x) &= \sum_{k=0}^{n} q^{2n}\frac{[k]_q^2}{[n]_q^2}x^k\prod_{s=0}^{n-k-1}\frac{[n]_q!}{[k]_q![n-k]_q!}(1-q^s x)\\
&= \frac{q^{2n}}{[n]_q}\sum_{k=1}^{n}(q[k-1]_q+1)\frac{[n-1]_q!}{[k-1]_q![n-k]_q!}x^k\prod_{s=0}^{n-k-1}(1-q^s x)\\
&= \frac{q^{2n}}{[n]_q}\sum_{k=0}^{n-1}\frac{[n-1]_q!}{[k-1]_q![n-k]_q!}x^{k+1}\prod_{s=0}^{n-k-2}(1-q^s x)\\
&\quad +\frac{q^{2n+1}}{[n]_q}[n-1]_q\sum_{k=2}^{n}\frac{[n-2]_q!}{[k-2]_q![n-k]_q!}x^k\prod_{s=0}^{n-k-1}(1-q^s x)\\
&= \frac{q^{2n}}{[n]_q}x+\frac{q^{2n+1}[n-1]_q}{[n]_q}\sum_{k=0}^{n-2}\frac{[n-2]_q!}{[k]_q![n-k-2]_q!}x^{k+2}\prod_{s=0}^{(n-2)-k-1}(1-q^s x)\\
&= q^{2n}x^2+\frac{x(1-x)}{[n]_q}q^{2n},
\end{aligned}$$

as desired.

Lemma 2.2. [17] *Let $A = (a_{jn})$ be a non-negative regular summability matrix. If the sequence of positive linear operators L_n from $C[a,b]$ into $C[a,b]$ satisfies the conditions*

$$st_A - \lim_{n\to\infty} \| L_n(e_i;x) - e_i \| = 0 \quad \textit{with} \quad e_i(t^i) = t^i \textit{ for } i = 0,1,2$$

then, for all $f \in C[a,b]$, we have

$$st_A - \lim_{n\to\infty} \| L_n(f;x) - f \| = 0.$$

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{q_n\}$ be a sequence in the interval $(0,1]$ which satisfies*

$$st_A - \lim_{n\to\infty} q_n^n = 1 \quad \textit{and} \quad st_A - \lim_{n\to\infty} \frac{1}{[n]_{q_n}} = 0. \tag{2.5}$$

Then for all $f \in C[0,1]$,

$$st_A - \lim_{n\to\infty} \| L_n(f;q_n;x) - f \| = 0.$$

Proof. By (2.1) and (2.2), it is clear that

$$st_A - \lim_{n\to\infty} \| L_n(e_0;q_n;x) - e_0 \| = 0, \tag{2.6}$$

$$\| L_n(e_1;q_n;x) - e_1 \| \leq 1 - q_n. \tag{2.7}$$

For a given $\varepsilon > 0$, we define the following sets:

$$S = \{n : \| L_n(e_1;q_n;x) - e_1 \| \geq \varepsilon\} \quad \textit{and} \quad S^* = \{n : \| 1 - q_n \| \geq \varepsilon\}.$$

From (2.7) we can see that $S \subseteq S^*$ and then, for each $j \in N$, that

$$0 \leq \sum_{n\in S} a_{jn} \leq \sum_{n\in S^*} a_{jn}. \tag{2.8}$$

Letting $j \to \infty$ in (2.8) and using (2.5) we conclude that

$$\lim_{j\to\infty} \sum_{n\in S} a_{jn} = 0.$$

We obtain

$$st_A - \lim_{n\to\infty} \| L_n(e_1;q_n;x) - e_1 \| = 0. \tag{2.9}$$

Finally, by (2.3), we get

$$\|L_n(e_2;q_n;x) - e_2\| \leq 1 - q_n^{2n} + \frac{q^{2n}}{4[n]_{q_n}}.$$

Using the same method as in the proof of (2.9), we can obtain

$$st_A - \lim_{n\to\infty} \|L_n(e_2;q_n;x) - e_2\| = 0. \tag{2.10}$$

By lemma 2.2, and using (2.7), (2.9) and (2.10), we get the desired result. The proof is completed.

Remark 1. In fact, we can construct a sequence q_n satisfying (2.5). For example, take

$$a_{jn} = \begin{cases} 1, & n = j \text{ and } j \neq m^2 \quad (m = 1, 2, 3 \ldots) \\ \frac{1}{j}, & j = n = m^2 \\ \frac{j-1}{j}, & j = m^2 \text{ and } n = (m-1)^2 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $A = (a_{jn})$ is a regular matrix. For $\alpha > 1$, we define the sequence $\{q_n\}$ by

$$q_n = \begin{cases} 0, & n = m^2 \quad (m = 1, 2, 3 \ldots) \\ 1 - (\frac{1}{n})^\alpha, & n \neq m^2. \end{cases}$$

We can say that $st_A - \lim_n q_n^n = 1$, but that the sequence $\{q_n^n\}$ is non-convergence in the ordinary sense. On the other hand, if $n \neq m^2$, then it is not hard to obtain

$$st_A - \lim_{n\to\infty} \frac{1}{[n]_{q_n}} = 0.$$

Now, we are ready to compute the rate of A-statistical convergence of the operators $L_n(f; q_n; x)$ by means of the modulus of continuity. Let $f \in C[0, 1]$. The modulus of continuity of f, denoted as $\omega(f, \delta)$, is defined to be

$$\omega(f, \delta) = \sup_{|t-x|\leq\delta} |f(t) - f(x)|.$$

Then it is known that $\lim_{\delta\to 0} \omega(f, \delta) = 0$. Also, for any $\delta > 0$ and each $t, x \in [0, 1]$, using the property of modulus of continuity $\omega(f, \lambda t) \leq (1 + \lambda)\omega(f, t), \quad \lambda > 0$, we obtain

$$|f(t) - f(x)| \leq \omega(f, \delta) \left(\frac{|t - x|}{\delta} + 1 \right). \tag{2.11}$$

Theorem 2.2. *Let $n \in N$, $f \in C[0, 1]$ and $\{q_n\}$ be a sequence such that $0 < q_n \leq 1$. Then*

$$| L_n(f; q_n; x) - f(x) | \leq 2\omega(f, \delta_n^*),$$

where

$$\delta_n^* = \sqrt{x^2 \left(1 - 2q_n^n + q_n^{2n} - \frac{q_n^{2n}}{[n]_{q_n}}\right) + \frac{xq_n^{2n}}{[n]_{q_n}}}. \tag{2.12}$$

Proof. Since the operators $L_n(f; q_n; x)$ are linear and positive, for all $f \in [0, 1]$, we get

$$| L_n(f; q_n; x) - f(x) | \leq L_n(| f(t) - f(x) |; q_n; x).$$

Using (2.11), we have, for any $\delta > 0$,

$$\begin{aligned} \mid L_n(f;q_n;x) - f(x) \mid &\leq L_n(\mid f(t) - f(x) \mid; q_n; x) \\ &\leq L_n(\omega(f,\delta)\left(\frac{|t-x|}{\delta} + 1); q_n; x\right) \\ &= \omega(f,\delta) L_n\left((\frac{|t-x|}{\delta} + 1); q_n; x\right) \\ &= \omega(f,\delta)\left(1 + \frac{1}{\delta} L_n(|t-x|; q_n; x)\right). \end{aligned}$$

Using the Cauchy-Schwarz inequality for positive linear operators, we get

$$\begin{aligned} & |L_n(f;q_n;x) - f(x)| \\ \leq\ & \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{L_n(|t-x|^2; q_n; x)}\ \right) \\ =\ & \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{x^2 L_n(e_0;q_n;x) - 2x L_n(e_1;q_n;x) + L_n(e_2;q_n;x)}\ \right) \\ \leq\ & \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{q_n^{2n}x^2 + \frac{x(1-x)}{[n]_{q_n}} q_n^{2n} - 2x^2 q_n^n + x^2}\ \right). \end{aligned}$$

Choosing $\delta = \delta_n^*$ as in (2.12) it follows that the proof is completed.

Remark 2. If $\{q_n\}$ satisfies(2.5), it yields that $st_A - \lim_{n\to\infty} \omega(f, \delta_n^*) = 0$. So Theorem 4 shows the rate of A-statistical approximation of the operators $L_n(f;q_n;x)$ to $f(x)$.

Remark 3. Let $\gamma_n = \frac{1}{1+[n]_{q_n}}$, for $x \in [0,\gamma_n]$, theorem 2.2 show that this modification provides a better estimation than the operators $B_n(f;q_n;x)$ on the interval $[0,\gamma_n]$.

Indeed, in (1.2), let

$$\delta_n = \sqrt{\frac{x(1-x)}{[n]_{q_n}}}.$$

Then we get

$$\begin{aligned} (\delta_n)^2 - (\delta_n^*)^2 &= \frac{x(1-x)}{[n]_{q_n}}(1 - q_n^{2n}) - q_n^{2n}x^2 + 2x^2 q_n^n - x^2 \\ &= x(1-q_n^n)\left(\frac{(1-x)(1+q_n^n)}{[n]_{q_n}} - x(1-q_n^n)\right) \\ &\geq x(1-q_n^n)\left(\frac{1-x}{[n]_{q_n}} - x\right) \\ &\geq 0, \end{aligned}$$

as desired.

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SHENGGUI LIU
Department of Mathematics,
Jiaying University,
Meizhou 514015,
China.

E-mail address: lsg55@163.com