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ON (ϵ)-TRANS-SASAKIAN MANIFOLDS

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RAJENDRA PRASAD[1] AND JAI PRAKASH[2]

Abstract. In this paper we study (ϵ)−trans-Sasakian manifolds and show their existence by an example. Some basic results regarding to such manifolds have been obtained in this context. Conformally flat and quasi-conformally flat (ϵ)−trans-Sasakian manifolds are also studied. It is proved that in a conformally flat (ϵ)−trans-Sasakian manifold, ξ is an eigen vector of Ricci operator Q. We also obtain some expressions, lemmas and theorems for (ϵ)−trans-Sasakian manifolds. Amang the lemmas of section 3, it is proved that Conformally flat and quasi-conformally flat (ϵ)−trans-Sasakian manifolds are η−Einstein.

1. INTRODUCTION

It is well known that the intrinsic properties of a manifold depends on the nature of metric defined on it and the geometric properties of the manifold does not depend on the choice of local co-ordinates defined on it.In Riemannian geometry,we study manifolds with metric which is positive definite.Since manifolds with indefinite metric have significant use in Physics, it is interesting to study such manifolds equipped with different structures.In [1], A.Benjancu and K.L. Duggal introduced the notion of (ϵ)-Sasakian manifolds with indefinite metric. In [20], Xu Xufeng and Chao Xixaoli proved that (ϵ)-Sasakian manifold is a hypersurface of an indifinite Kählerian manifold. Further in [10] R. Kumar, R. Rani and R. Nagaich studied (ϵ)-Sasakian manifolds. Since Sasakian manifolds with indifinite metric play significant role in Physics [9], so it is important to study them. Recently, in 2009, U.C. De and Avijit Sarkar [5] introduced and studied the notion of (ϵ)-Kenmotsu manifolds with indefinite metric. Oubina [13] studied a new class of almost contact metric manifolds known as trans-Sasakian manifolds which generalizes both α−Sasakian and β−Kenmotsu manifolds. Sasakian, α−Sasakian, Kenmotsu, β−Kenmotsu manifolds are particular cases of trans-Sasakian manifolds of type (α, β).Nearly trans-Sasakian manifolds was introduced by C.Gherghe [4]. In [15], Prasad, Shukla and Tripathi have studied some special type of trans-Sasakian manifolds.

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A quasi-Conformal curvature tensor was introduced by Yano and Sawaki [21]. A $(2n+1)-$dimensional Riemannian manifold (M,g) is quasi-Conformally flat if $\check{C}=0$,where $\check{C}$ is the quasi-Conformal curvature tensor, defined as

(1.1)

$$\begin{aligned}\check{C}(X,Y)Z &= aR(X,Y)Z+b[g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y]\\ &-\frac{r}{(2n-1)}\{\frac{a}{2n}+2b\}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

where a,b are constants and R,S,Q and r are the Riemannian curvature tensor, the Ricci-tensor, the Ricci operator and the scalar curvature tensor of the manifold respectively. If $a=1$ and $b=-\frac{1}{2n-1}$, then $\check{C}$ becomes a conformal curvature C, given by

(1.2)

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z-\frac{1}{(2n-1)}[g(Y,Z)QX-g(X,Z)QY-S(Y,Z)X+S(X,Z)Y]\\ &+\frac{r}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

If M is conformally flat and of dimension $2n+1$, $n>1$ then $C=0$ and we have from (1.2)

(1.3)

$$\begin{aligned}R(X,Y)Z &= \frac{1}{(2n-1)}[g(Y,Z)QX-g(X,Z)QY-S(Y,Z)X+S(X,Z)Y]\\ &-\frac{r}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

for all vector fields X,Y,Z on M.

In this paper, we study $(\epsilon)-$trans-Sasakian manifolds with indefinite metric, which appear as a natural generalization of both $(\epsilon)-$Sasakian and $(\epsilon)-$Kenmostsu manifolds.This paper is organised as follows:

Section 1,is introductry. Section 2 contains necessary details about $(\epsilon)-$trans-Sasakian manifold. Further in section 2, existance of $(\epsilon)-$trans-Sasakian manifold is shown by an example. Some basic results regarding to such type of manifolds are also given in section 2 .In Section 3, we introduce the conformally flat and quasi-conformally flat $(\epsilon)-$trans-Sasakian manifolds and show that in a conformally flat $(\epsilon)-$trans-Sasakian manifold, ξ is an eigen vector of Ricci operator Q. We also obtained some expressions, lemmas and theorems for $(\epsilon)-$trans-Sasakian manifolds in section 3. Amang the lemmas of section 3, it is proved that Conformally flat and quasi-conformally flat $(\epsilon)-$trans-Sasakian manifolds are $\eta-$Einstein.

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact metric manifold equipped with almost contact metric struture (ϕ,ξ,η,g), where ϕ is (1,1) tensor field, ξ is a vector field , η is 1-form and g is indefinite metric such that

(2.1)

$$\phi^2=-I+\eta\otimes\xi, \eta(\xi)=1, \phi\xi=0$$

(2.2)
$$g(\xi,\xi)=\epsilon, \qquad \eta(X)=\epsilon g(X,\xi)$$
(2.3)
$$g(\phi X,\phi Y)=g(X,Y)-\epsilon\eta(X)\eta(Y)$$
for all vector fields X,Y on M,where ϵ is 1 or -1 according as ξ is space like or light like vector fields and rank ϕ is $2n$. if
(2.4)
$$d\eta(X,Y)=g(X,\phi Y), \quad \text{for all} \quad X,Y\in\Gamma(TM),$$
then M (ϕ,ξ,η,g) is called an (ϵ)-almost contact metric manifold. An (ϵ)-almost cantact metric manifold is called an (ϵ)-trans-Sasakian manifold if
(2.5)
$$(\nabla_X\phi)Y=\alpha\{g(X,Y)\xi-\epsilon\eta(Y)X\}+\beta\{g(\phi X,Y)\xi-\epsilon\eta(Y)\phi X\}$$
for any $X,Y\in\Gamma(TM)$, where ∇ is Levi-Civita connection of semi-Riemannian metric g and α and β are smooth functions on M.
From equations (2.1), (2.2), (2.3) and (2.5), we have
(2.6)
$$\nabla_X\xi=\epsilon\{-\alpha\phi X+\beta(X-\eta(X)\xi)\}$$
(2.7)
$$(\nabla_X\eta)Y=-\alpha g(\phi X,Y)+\beta\{g(X,Y)-\epsilon\eta(X)\eta(Y)\}$$
(2.8)
$$\nabla_\xi\phi=0$$
Let us define the tensor h by $2h=\pounds_\xi\phi$, where £ is the Lie differentiation operator.
Further, on such a$(\epsilon)-$trans Sasakian manifold M of dimension $(2n+1)$ with structure (ϕ,ξ,η,g) the following relations holds
(2.9)
$$\begin{aligned}R(X,Y)\xi &= (\alpha^2-\beta^2)\{\eta(Y)X-\eta(X)Y\}+2\alpha\beta\{\eta(Y)\phi X-\eta(X)\phi Y\}\\ &+\epsilon\{(Y\alpha)\phi X-(X\alpha)\phi Y+(Y\beta)\phi^2X-(X\beta)\phi^2Y\}\end{aligned}$$
(2.10)
$$\begin{aligned}R(\xi,Y)X &= (\alpha^2-\beta^2)\{\epsilon g(X,Y)\xi-\eta(X)Y\}+2\alpha\beta\{\epsilon g(\phi X,Y)\xi+\eta(X)\phi Y\}+\epsilon(X\alpha)\phi Y\\ &+\epsilon g(\phi X,Y)(\operatorname{grad}\alpha)-\epsilon g(\phi X,\phi Y)(\operatorname{grad}\beta)+\epsilon(X\beta)\{Y-\eta(Y)\xi\}\end{aligned}$$
(2.11)
$$R(\xi,Y)\xi=\{\alpha^2-\beta^2-\epsilon(\xi\beta\}\{-Y+\eta(Y)\xi\}+\{2\alpha\beta+\epsilon(\xi\alpha)\}(\phi Y)$$
(2.12)
$$2\alpha\beta+\epsilon(\xi\alpha)=0$$
(2.13)
$$S(X,\xi)=\{2n(\alpha^2-\beta^2)-\epsilon(\xi\beta)\}\eta(X)-\epsilon(\phi X)\alpha-\epsilon(2n-1)(X\beta)$$
(2.14)
$$Q\xi=\epsilon[\{2n(\alpha^2-\beta^2)-\epsilon\xi\beta\}\xi+\phi(\operatorname{grad}\alpha)-(2n-1)(\operatorname{grad}\beta)]$$

If

$$(2n-1)(\operatorname{grad}\beta) - \phi(\operatorname{grad}\alpha) = (2n-1)(\xi\beta)\xi$$

then (2.13)and (2.14) respectively reduce to

(2.15)

$$S(X,\xi) = \{2n(\alpha^2-\beta^2) - \epsilon(\xi\beta)\}\eta(X)$$

(2.16)

$$Q\xi = 2n\epsilon\{(\alpha^2-\beta^2) - \epsilon(\xi\beta)\}\xi$$

Now we shall give an example of (ϵ)-trans-Sasakian manifold.

Example (2.1) Let us consider a 3-dimensinal manifold $M = \{(x,y.z) \in \mathbb{R}^3 : z \neq 0\}$,where (x,y,z) are the standard co-ordinates in $\mathbb{R}^3$

Let $e_1 = (e^z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z})$ $e_2 = e^z\frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$, which are linearly independent vector fields at each point of M. Define a semi-Riemannian metric g on M as

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0 \ \ , g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = \epsilon,$$

where $\epsilon = \pm 1$

Let η be the 1-form defined by $\eta(Z) = \epsilon g(Z,e_3)$, for any Z$\in \Gamma(TM)$ Let ϕ be a tensor field of type $(1,1)$ defined by $\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$. Then by using linearity of ϕ and g , we have

$$\eta(e_3) = 1 \ , \ \phi^2 Z = -Z + \eta(Z)e_3 \ \ , \ g(\phi Z, \phi U) = g(Z,U) - \epsilon\eta(Z)\eta(U) \ \ \text{,for any} \ \ Z,U \in \Gamma(TM)$$

If we take $e_3 = \xi$, $(\phi,\xi,\eta,g,\epsilon)$ defines an $(\epsilon)-$almost contact metric structure on M.

Let ∇ be the Levi-Citvita connection with respect to g and R be the curvature tensor of type (1,3), then we have

$$[e_1,e_2] = \epsilon(ye^z e_2 - e^{2z}e_3) \quad , [e_1,e_2] = -\epsilon e_1 \ , [e_1,e_3] = \epsilon e_2$$

By using Koszul,s formula for the Levi-Civita connection with respect to g ,we obtain

$$\nabla_{e_1}e_3 = -\epsilon e_1 + \frac{1}{2}\epsilon e^{2z}e_2 \ ,\nabla_{e_2}e_3 = -\epsilon e_2 - \frac{1}{2}\epsilon e^{2z}e_1 \ ,\nabla_{e_3}e_3 = 0$$

$$\nabla_{e_1}e_2 = -\frac{1}{2}\epsilon e^{2z}e_3 \ ,\nabla_{e_2}e_2 = \epsilon e_3 + \epsilon y e^z e_1, \nabla_{e_3}e_2 = -\frac{1}{2}\epsilon e^{2z}e_1 \ ,$$

$$\nabla_{e_1}e_1 = \epsilon e_3, \quad \nabla_{e_2}e_1 = -\epsilon y e^z e_2 + \frac{1}{2}\epsilon e^{2z}e_3 \, , \qquad \nabla_{e_3}e_1 = \frac{1}{2}\epsilon e^{2z}e_2 \ ,$$

Now for $\xi = e_3$, above results satisfy

$$\nabla_X\xi = \epsilon\{-\alpha\phi X + \beta(X - \eta(X)\xi)\},$$

with $\alpha = \frac{1}{2}e^{2z}$ and $\beta = -1$ Consequently M$(\phi,\xi,\eta,g,\epsilon)$ is a 3-dimensional (ϵ)-trans-Sasakian manifold.

3. CONFORMALLY FLAT AND QUASI-CONFORMALLY FLAT (ϵ)-TRANS-SASAKIAN MANIFOLDS

Definition: A semi-Riemannian manifold M with semi-Riemannian metric g is called quasi-conformally flat if $\check{C}=0$,where $\check{C}$ is quasi-conformal curvature tensor defined as

(3.1)

$$\begin{aligned}\check{C}(X,Y)Z &= aR(X,Y)Z+b[g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y]\\ &-\frac{r}{(2n-1)}\{\frac{a}{2n}+2b\}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

where a,b are costants R,S,Q and r are the curvature tensor,the Ricci-tensor, the Ricci operator and the scalar curvature tensor of the semi-Riemannian manifold respectively. If $a=1$ and $b=-\frac{1}{2n-1}$, then $\check{C}$ becomes a conformal curvature C, given by

$(3.1a)$

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z-\frac{1}{(2n-1)}[g(Y,Z)QX-g(X,Z)QY-S(Y,Z)X+S(X,Z)Y]\\ &+\frac{r}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

Let M be a $(2n+1)$-dimnsonal quasi-conformally flat manifold, then from equation (3.1),we have

(3.2)

$$\begin{aligned}aR(X,Y)Z &= b[-g(Y,Z)QX+g(X,Z)QY-S(Y,Z)X+S(X,Z)Y]\\ &+\frac{r}{(2n-1)}\{\frac{a}{2n}+2b\}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

(3.3)

$$\begin{aligned}ag(R(X,Y)Z,W) &= b[-g(Y,Z)g(QX,W)+g(X,Z)g(QY,W)-S(Y,Z)g(X,W)+S(X,Z)g(Y,W)]\\ &+\frac{r}{(2n-1)}\{\frac{a}{2n}+2b\}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\end{aligned}$$

Let $\{e_1,e_2,...,e_{2n},e_{2n+1}=\xi\}$ is orthonormal base field.

Putting $Y=Z=e_i$ in equation (3.3) we get

(3.4)

$$S(X,W)=\frac{r}{2n+1}g(X,W)$$

if $a+(2n-1)b\neq 0$

Hence we have the following lemma.

Lemma 3.1: A $(2n+1)-$dimensional quasi-conformally flat semi-Riemannian manifold M is an Einstein manifold if $a+(2n-1)b\neq 0$.

if $a+(2n-1)b=0$, then form equation (3.1) and $(3.1a)$, we have

(3.5)

$$\check{C}(X,Y)Z=a(C(X,Y)Z)$$

if $a = 1, b = -\frac{1}{2n+1}$ then $a + (2n+1)b = 1 + (2n-1)(-\frac{1}{2n-1}) = 0$

So condition $a + (2n-1) = 0$ is satisfied for conformal curvature tensor of semi-Riemannin manifold

From equation (3.5) we have following Lemma.

Lemma 3.2: A$(2n+1)$-dimensional quasi-conformally flat semi-Riemannian manifold M is conformally flat if $a + (2n-1)b = 0$ and $a \neq 0$.

from equation (3.3), (3.4),we have

(3.6)

$$R(X,Y)Z = \frac{r(1 + \frac{4nb}{a})}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\},$$

if $a \neq 0$, $a + (2n+1)b \neq 0$

Hence we have following theorem

Theorem 3.3: A quasi-conformally flat manifold is a manifold of constant curvature $\frac{r(1+\frac{4nb}{a})}{2n(2n+1)}$ if $a \neq 0, a + (2n-1)b \neq 0$.

Lemma 3.4: A quasi-conformally flat $(\epsilon)-$trans-Sasakian manifold is of constant curvature $(\alpha^2 - \beta^2 - \epsilon(\xi\beta))$, if $a \neq 0, a + (2n-1)b \neq 0$.

Lemma 3.5: In a $(\epsilon)-$trans-Sasakian manifold, $h = 0$ [15]

Proof: Using definition of h

(3.7)

$$\begin{aligned}
h &= \frac{1}{2}\pounds_\xi \phi \\
hX &= \frac{1}{2}[\pounds_\xi \phi]X \\
&= \frac{1}{2}[\pounds_\xi(\phi X) - \phi(\pounds_\xi X)] \\
&= \frac{1}{2}\{[\xi, \phi X] - \phi[\xi, X]\} \\
&= \frac{1}{2}\{\nabla_\xi \phi X - \nabla_{\phi X}\xi - \phi(\nabla_\xi X - \nabla_X \xi)\} \\
&= \frac{1}{2}\{(\nabla_\xi \phi)X - \nabla_{\phi X}\xi - \phi(\nabla_\xi X - \nabla_X \xi)\} \\
&= \frac{1}{2}\{(\nabla_\xi \phi)X - \nabla_{\phi X}\xi - \phi(\nabla_\xi X) + (\nabla_X \xi)\} \\
&= \frac{1}{2}\{(\nabla_\xi \phi)X - \nabla_{\phi X}\xi + (\nabla_X \xi)\}
\end{aligned}$$

using equations (2.1), (2.2) and (2.6),we have

$$\begin{aligned}
hX &= 0 \qquad \forall X \\
h &= 0 \\
So\ \ h\phi &= \phi h
\end{aligned}$$

Lemma:3.6 In a (ϵ)-trans-Sasakian manifold M^{2n+1}, the following relation holds.

(3.8)

$$R(X,Y)\phi Z-\phi R(X,Y)Z = (\alpha^2-\beta^2)\epsilon[g(X,Z)\phi Y-g(Y,Z)\phi X+g(\phi X,Z)Y-g(\phi Y,Z)X] +2\epsilon\alpha\beta[g(Y,Z)X-g(X,Z)Y-g(\phi X,Z)\phi Y-g(\phi Y,Z)\phi X] +(X\alpha)[g(Y,Z)\xi-\epsilon\eta(Z)Y]-(Y\alpha)[g(X,Z)\xi-\epsilon\eta(Z)X] +(X\beta)[g(\phi Y,Z)\xi-\epsilon\eta(Z)\phi Y]-(Y\beta)[g(\phi X,Z)\xi-\epsilon\eta(Z)\phi X]$$

for all vector fields X, Y and Z on M.

Proof: we know that

(3.9)

$$R(X,Y)\phi Z-\phi R(X,Y)Z=(\nabla_X\nabla_Y\phi)Z-(\nabla_Y\nabla_X\phi)Z-(\nabla_{[X,Y]}\phi)Z$$

(3.10)

$$(\nabla_X\nabla_Y\phi)Z = (X\alpha)[g(Y,Z)\xi-\epsilon\eta(Z)Y]+(X\beta)[g(\phi Y,Z)\xi-\epsilon\eta(Z)\phi Y] +\alpha g(Y,Z)[\epsilon\{-\alpha\phi X+\beta(X-\eta(X)\xi)\}]-\epsilon\alpha X\{\eta(Z)\}Y-\epsilon\alpha\eta(Z)(\nabla_X Y) +\alpha[g(\nabla_X Y,Z)+g(Y,\nabla_X Z)]\xi+\beta g(\phi Y,Z)[\epsilon\{-\alpha\phi X+\beta(X-\eta(X)\xi)\}] -\epsilon\beta X(\eta(Z))\phi Y-\epsilon\beta\eta(Z)(\nabla_X(\phi Y))+\beta[\{g(\nabla_X\phi Y,Z)+g(\phi Y,\nabla_X Z)\}\xi] -\alpha[g(Y,\nabla_X Z)\xi-\epsilon\eta(\nabla_X Z)Y]-\beta[g(\phi Y,\nabla_X Z)\xi-\epsilon\eta(\nabla_X Z)\phi Y]$$

(3.11)

$$(\nabla_Y\nabla_X\phi)Z = (Y\alpha)[g(X,Z)\xi-\epsilon\eta(Z)X]+(Y\beta)[g(\phi X,Z)\xi-\epsilon\eta(Z)\phi X] +\alpha g(X,Z)[\epsilon\{-\alpha\phi Y+\beta(Y-\eta(Y)\xi)\}]-\epsilon\alpha Y\{\eta(Z)\}X-\epsilon\alpha\eta(Z)(\nabla_Y X) +\alpha[g(\nabla_Y X,Z)+g(X,\nabla_Y Z)]\xi+\beta g(\phi X,Z)[\epsilon\{-\alpha\phi Y+\beta(Y-\eta(Y)\xi)\}] -\epsilon\beta Y(\eta(Z))\phi X-\epsilon\beta\eta(Z)(\nabla_Y(\phi X))+\beta[\{g(\nabla_Y\phi X,Z)+g(\phi X,\nabla_Y Z)\}\xi] -\alpha[g(X,\nabla_Y Z)\xi-\epsilon\eta(\nabla_Y Z)X]-\beta[g(\phi X,\nabla_Y Z)\xi-\epsilon\eta(\nabla_Y Z)\phi X]$$

(3.12)

$$(\nabla_{[X,Y]}\phi)Z=\alpha\{g([X,Y],Z)\xi-\epsilon\eta(Z)[X,Y]\}+\beta\{g(\phi[X,Y],Z)\xi-\epsilon\eta(Z)\phi[X,Y]\}$$

From the equations (2.1), (2.2), (2.5), (2.6), (3.9), (3.10), (3.11) and (3.12), we lemma (3.6).

Putting $X=\xi$ in equation (3.8), we get

(3.13)

$$R(\xi,Y)\phi Z-\phi R(\xi,Y)Z=-\{\epsilon(\alpha^2-\beta^2)-(\xi\beta)[g(\phi Y,Z)\xi-\epsilon\eta(Z)\phi Y]$$

Let$\{e_1,e_2,...,e_{2n},e_{2n+1}=\xi\}$be orthonormal basis. Putting $Y=Z=e_i$ in equation(3.16) and taking summation , we get.

(3.14)

$$\sum_{i=1}^{2n+1}R(\xi,e_i)\phi e_i-\sum_{i=1}^{2n+1}\phi R(\xi,e_i)e_i=-\{\epsilon(\alpha^2-\beta^2)-(\xi\beta)\sum_{i=1}^{2n+1}[g(\phi e_i,e_i)\xi-\epsilon\eta(e_i)\phi e_i]$$

(3.15)

$$\sum_{i=1}^{2n+1} R(\xi, e_i)\phi e_i - \sum_{i=1}^{2n+1} \phi R(\xi, e_i)e_i = 0$$

(3.16)

$$\sum_{i=1}^{2n+1} R(\xi, e_i)\phi e_i = \phi Q\xi$$

If the manifold is conformally flat , $C(X,Y)Z = 0$
Then
(3.17)

$$\begin{aligned} R(X,Y)Z &= \frac{1}{(2n-1)}[g(Y,Z)QX - g(X,Z)QY - S(Y,Z)X + S(X,Z)Y] \\ &+ \frac{r}{2n(2n-1)}\{\frac{a}{2n} + 2b\}[g(Y,Z)X - g(X,Z)Y] \end{aligned}$$

Putting $X = \xi, Y = e_i, Z = \phi e_i$ in equation (3.17) and taking summation , we get
(3.18)

$$\begin{aligned} \sum_{i=1}^{2n+1} R(\xi, e_i)\phi e_i &= \frac{1}{(2n-1)} \sum_{i=1}^{2n+1} [g(e_i, \phi e_i)Q\xi - g(\xi, \phi e_i)Qe_i - S(e_i, \phi e_i)\xi + S(\xi, \phi e_i)e_i] \\ &+ \frac{r}{2n(2n-1)}\{\frac{a}{2n} + 2b\} \sum_{i=1}^{2n+1} [g(e_i, \phi e_i)\xi - g(\xi, \phi e_i)e_i] \end{aligned}$$

(3.19)

$$\sum_{i=1}^{2n+1} R(\xi, e_i)\phi e_i = \frac{1}{(2n-1)} \sum_{i=1}^{2n+1} [S(e_i, \phi e_i)\xi + S(\xi, \phi e_i)e_i]$$

using equation(3.16) and (3.19), we get

$$\phi Q\xi = \frac{1}{2n-1}\{(trQ\phi)\xi + \phi Q\xi\}$$

if $n > 1$,then $trQ\phi = 0$, therefore $\phi Q\xi = 0$ which implies that $Q\xi = (trl)\xi$. Hence we have a Theorem.

Theorem:3.7 In a conformally flat (ϵ)−trans-Sasakian manifold ξ is an eigen vector of Ricci operator Q.

For conformally flat manifolds
(3.20)

$$\begin{aligned} R(X,Y)Z &= \frac{1}{(2n-1)}[g(Y,Z)QX - g(X,Z)QY - S(Y,Z)X + S(X,Z)Y] \\ &- \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y] \end{aligned}$$

Putting $Y = Z = \xi$, and using the hypothesis $Q\xi = (trl)\xi$, $C(X,Y)Z = 0$
(3.21)

$$R(X,\xi)\xi = \frac{1}{(2n-1)}[g(\xi,\xi)QX - g(X,\xi)Q\xi - S(\xi,\xi)X + S(X,\xi)\xi] - \frac{r}{2n(2n-1)}[g(\xi,\xi)X - g(X,\xi)\xi]$$

From equations $(2.1), (2.2), (2.9), (2.11), (2.12), (2.15)$ and (2.16) we get
(3.22)

$$(2n-1)\{(\alpha^2-\beta^2)-\epsilon(\xi\beta)\}\{X-\eta(X)\xi\} = [\epsilon QX - \epsilon\eta(X)Q\xi - 2n\{(\alpha^2-\beta^2)-\epsilon(\xi\beta)\}X + \{2n(\alpha^2-\beta^2-\epsilon(\xi\beta))\}\eta(X)\xi - \epsilon(\phi X)\alpha\xi - \epsilon(2n-1)(X\beta)\xi] + \frac{r}{2n}[\epsilon X - \epsilon\eta(X)\xi]$$

(3.23)

$$QX = \{\frac{r}{2n} - \epsilon(\alpha^2-\beta^2-\epsilon(\xi\beta))\}X + (2n+1)\{\epsilon(\alpha^2-\beta^2-\epsilon(\xi\beta)) - \frac{r}{2n}\}\eta(X)\xi$$

Hence we have a lemma.

Lemma:3.8 A conformally flat $(\epsilon)-$trans-Sasakian manifold is $\eta-$Einstein.

Note:- In a $\eta-$Einstein manifold the condition $\phi Q = Q\phi$ always satisfied.

Proof:

If manifold is quasi-conformally flat, $\check{C}(X,Y)Z = 0$
(3.24)

$$R(X,Y)Z = -\frac{b}{a}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] - \frac{r}{(2n-1)}\{\frac{a}{2n}+2b\}[g(Y,Z)X - g(X,Z)Y]$$

Putting $X=\xi, Y=e_i$, $Z=Qe_i$ in equation (3.24) and taking summation

$$\sum_{i=1}^{2n+1} R(\xi,e_i)Qe_i = -\frac{b}{a}\sum_{i=1}^{2n+1}[g(e_i,\phi e_i)Q\xi - g(\xi,Qe_i)Qe_i - S(e_i,Qe_i)\xi + S(\xi,Qe_i)e_i] + \frac{r}{(2n-1)}\{\frac{a}{2n}+2b\}\sum_{i=1}^{2n+1}[-g(e_i,Qe_i)\xi + g(\xi,Qe_i)e_i]$$

(3.25)

$$\begin{aligned}
\sum_{i=1}^{2n+1} R(\xi,e_i)Qe_i &= -\frac{b}{a}\sum_{i=1}^{2n+1}[(trQ\phi)\xi + \phi Q\xi \\
trQ\phi &= 0 \\
(a+b)\phi Q\xi &= 0 \\
\phi Q\xi &= 0\ , \text{ if } (a+b)\neq 0 \\
\phi Q\xi &= 0 \Rightarrow \xi \quad \text{is an eigen vector of Ricci operator Q.}
\end{aligned}$$

Hence we have following lemma.

Lemma :3.9 If an (ϵ)-trans-Sasakian manifold is quasi-conformally flat, then ξ is an eigen vector of Ricci operator Q, if $(a+b)\neq 0$

Puting $Y=\xi,\ Z=\xi$ in equation (3.24)

(3.26)

$$\begin{aligned} R(X,\xi)\xi &= -\frac{b}{a}[g(\xi,\xi)QX-g(X,\xi)Q\xi+S(\xi,\xi)X-S(X,\xi)\xi] \\ &\quad -\frac{r}{(2n-1)}\{\frac{a}{2n}+2b\}[g(\xi,\xi)X-g(X,\xi)\xi], \end{aligned}$$

From equation (2.2), (2.9), (2.13), (2.14), (2.15), (2.16) and (3.26), we get

(3.27)

$$\begin{aligned} \{(\alpha^2-\beta^2)-\epsilon(\xi\beta)\}\{X-\eta(X)\xi\} &= -\frac{b}{a}[\epsilon QX-\eta(X)\{(2n(\alpha^2-\beta^2)-\epsilon\xi\beta)\xi \\ &\quad +\phi(\mathrm{grad}\,\alpha)-(2n-1)(\mathrm{grad}\,\beta)\} \\ &\quad +2n\{(\alpha^2-\beta^2)-\epsilon(\xi\beta)\}X \\ &\quad -\{2n(\alpha^2-\beta^2-\epsilon\xi\beta)\}\eta(X)\xi \\ &\quad +\epsilon(\phi X)\alpha\xi+\epsilon(2n-1)(X\beta)\xi] \\ &\quad +\frac{r}{(2n-1)}\left(\frac{a}{2n}+2b\right)\{\epsilon X-\epsilon\eta(X)\xi\} \end{aligned}$$

(3.28)

$$\begin{aligned} -\frac{b}{a}\epsilon QX &= \{(\alpha^2-\beta^2)-\epsilon(\xi\beta)\left(1+\frac{2nb}{a}\right)+\frac{r\epsilon}{2n-1}\left(\frac{a}{2n}+2b\right)\}X \\ &\quad -((\alpha^2-\beta^2)-\epsilon\xi\beta)\left(1-\frac{2nb\epsilon}{a}-\frac{r}{a(2n-1)}(\frac{a}{2n}+2b)-\frac{2nb}{a}\right)\eta(X)\xi \\ &\quad +\frac{b}{a}\{\phi(\mathrm{grad}\,\alpha)-(2n-1)(\mathrm{grad}\,\beta)\}\eta(X) \\ &\quad -\frac{b}{a}\epsilon\{(\phi X)\alpha\xi+(2n-1)(X\beta)\xi\} \end{aligned}$$

(3.29)

$$\begin{aligned} QX &= -\frac{a}{b}\epsilon[\{(\alpha^2-\beta^2)-\epsilon(\xi\beta)\left(1+\frac{2nb}{a}\right)+\frac{r\epsilon}{2n-1}\left(\frac{a}{2n}+2b\right)\}X \\ &\quad -((\alpha^2-\beta^2)-\epsilon\xi\beta)\left(1-\frac{2nb\epsilon}{a}-\frac{r}{a(2n-1)}(\frac{a}{2n}+2b)-\frac{2nb}{a}\right)\eta(X)\xi] \end{aligned}$$

Hence we have

Lemma :3.10 A quasi-conformally flat$(\epsilon)-$trans-Sasakian manifold is $\eta-$Einstein.

Lemma :3.11 In a three-dimensional, $(\epsilon)-$trans-Sasakian manifold, the Ricci-operator is given by

$$\begin{aligned} QX &= \epsilon[\frac{r\epsilon}{2}-(2n-1)\{(\alpha^2-\beta^2)-\epsilon(\xi\beta\}]X+\epsilon[4n(\alpha^2-\beta^2)-\epsilon(\xi\beta)-\frac{r\epsilon}{2}]\eta(X)\xi \\ &\quad +\epsilon\{\phi(\mathrm{grad}\,\alpha)-(2n-1)(\mathrm{grad}\,\beta)\}\eta(X)-\{(\phi X)\alpha-(2n-1)(X\beta)\}\xi \end{aligned}$$

Proof: We know that Weyl conformal curvature tensor C of type (1,3) vanishes in three- dimensional manifold (M, g) is defined by

(3.30)

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}$$

Put $Z = \xi$ in equation (3.30)

(3.31)

$$R(X,Y)\xi = g(Y,\xi)QX - g(X,\xi)QY + S(Y,\xi)X - S(X,\xi)Y - \frac{r}{2}\{g(Y,\xi)X - g(X,\xi)Y\}$$

from equations $(2.2), (2.9), (2.12), (2.13)$ and (3.31) we get

(3.32)

$$\begin{aligned}
& (\alpha^2-\beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\
& +\epsilon\{(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y\} \\
= \; & \epsilon\eta(Y)QX - \epsilon\eta(X)QY \\
& +[\{2n(\alpha^2-\beta^2) - \epsilon(\xi\beta)\}\eta(Y) - \epsilon(\phi Y)\alpha - \epsilon(2n-1)(Y\beta)]X \\
& -[\{2n(\alpha^2-\beta^2) - \epsilon(\xi\beta)\}\eta(X) - \epsilon(\phi X)\alpha - \epsilon(2n-1)(X\beta)]Y \\
& -\frac{r}{2}\{\epsilon\eta(Y)X - \epsilon\eta(X)Y\}
\end{aligned}$$

Putting $Y = \xi$ in equation (3.32) and using equation (2.1), we get

(3.33)

$$\begin{aligned}
& (\alpha^2-\beta^2)\{X - \eta(X)\xi\} + 2\alpha\beta(\phi X) + \epsilon\{(\xi\alpha)\phi X + (\xi\beta)\phi^2 X \\
= \; & \epsilon QX - \epsilon\eta(X)Q\xi + [\{2n(\alpha^2-\beta^2) - \epsilon(\xi\beta)\} - \epsilon(2n-1)(\xi\beta)]X \\
& -[\{2n(\alpha^2-\beta^2) - \epsilon(\xi\beta)\}\eta(X) - \epsilon(\phi X)\alpha - \epsilon(2n-1)(X\beta)]\xi \\
& -\frac{r}{2}\{\epsilon X - \epsilon\eta(X)\xi\}
\end{aligned}$$

From equations $(2.1), (2.14)$ and (3.33), we get

(3.34)

$$\begin{aligned}
& (\alpha^2-\beta^2)\{X - \eta(X)\xi\} + 2\alpha\beta(\phi X) + \epsilon\{(\xi\alpha)\phi X + (\xi\beta)\phi^2 X \\
= \; & \epsilon QX - \eta(X)[\{2n(\alpha^2-\beta^2) - \epsilon\xi\beta\}\xi + \phi(\operatorname{grad}\alpha) - (2n-1)(\operatorname{grad}\beta)] \\
& +[\{2n(\alpha^2-\beta^2) - \epsilon(\xi\beta)\} - \epsilon(2n-1)(\xi\beta)]X \\
& -[\{2n(\alpha^2-\beta^2) - \epsilon(\xi\beta)\}\eta(X) - \epsilon(\phi X)\alpha - \epsilon(2n-1)(X\beta)]\xi \\
& -\frac{r}{2}\{\epsilon X - \epsilon\eta(X)\xi\}
\end{aligned}$$

(3.35)

$$\begin{aligned}
QX \;=\; & \epsilon[\frac{r\epsilon}{2} - (2n-1)\{(\alpha^2-\beta^2) - \epsilon(\xi\beta\}]X + \epsilon[4n(\alpha^2-\beta^2) - \epsilon(\xi\beta) - \frac{r\epsilon}{2}]\eta(X)\xi \\
& +\epsilon\{\phi(\operatorname{grad}\alpha) - (2n-1)(\operatorname{grad}\beta)\}\eta(X) - \{(\phi X)\alpha - (2n-1)(X\beta)\}\xi
\end{aligned}$$

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DEPARTMENT OF MATHEMATICS AND ASTRONOMY, UNIVERSITY OF LUCKNOW,LUCKNOW-226007(INDIA),
E-mail address: rp.manpur@rediffmail.com

Current address: DEPARTMENT OF MATHEMATICS AND ASTRONOMY, UNIVERSITY OF LUCKNOW,LUCKNOW-226007(INDIA),
E-mail address: dr.jp0000@gmail.com