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REPRESENTATIONS FOR THE GENERALIZED DRAZIN INVERSE IN A BANACH ALGEBRA

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Abstract. In this paper, we investigate additive properties for the generalized Drazin inverse in a Banach algebra $\mathscr{A}$. We give some representations for the generalized Drazin inverse of $a+b$, where a and b are elements of $\mathscr{A}$ under some new conditions, extending some known results.

1. Introduction

The Drazin inverse has important applications in matrix theory and fields such as statistics, probability, linear systems theory, differential and difference equations, Markov chains, and control theory ([1, 2, 11]). In [9], Koliha extended the Drazin invertibility in the setting of Banach algebras with applications to bounded linear operators on a Banach space. In this paper, Koliha was able to deduce a formula for the generalized Drazin inverse of $a+b$ when $ab = ba = 0$. The general question of how to express the generalized Drazin inverse of $a+b$ as a function of a, b, and the generalized Drazin inverses of a and b without side conditions, is very difficult and remains open. R.E. Hartwig, G.R. Wang, and Y. Wei studied in [8] the Drazin inverse of a sum of two matrices A and B when $AB = 0$. In the papers [3, 4, 5, 7], some new conditions under which the generalized Drazin inverse of the sum $a+b$ in a Banach algebra is explicitly expressed in terms of a, b, and the generalized Drazin inverses of a and b.

In this paper we introduce some new conditions and we extend some known expressions for the generalized Drazin inverse of $a+b$, where a and b are generalized Drazin invertible in a unital Banach algebra.

Throughout this paper we will denote by $\mathscr{A}$ a unital Banach algebra with unity 1. Let $\mathscr{A}^{-1}$ and $\mathscr{A}^{\mathsf{qnil}}$ denote the sets of all invertible and quasinilpotent elements in $\mathscr{A}$, respectively. Explicitly,

$$\mathscr{A}^{-1} = \{a \in \mathscr{A} : \exists\, x \in \mathscr{A} : ax = xa = 1\},$$
$$\mathscr{A}^{\mathsf{qnil}} = \{a \in \mathscr{A} : \lim_{n\to+\infty} \|a^n\|^{1/n} = 0\}.$$

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If $\mathscr{B}$ is a subalgebra of the unital algebra $\mathscr{A}$, for an element $b \in \mathscr{B}^{-1}$, we shall denote by $[b^{-1}]_{\mathscr{B}}$ the inverse of b in $\mathscr{B}$. Let us observe that in general $\mathscr{B}^{-1} \not\subset \mathscr{A}^{-1}$ (for example, if $p \in \mathscr{A}$ is a nontrivial idempotent and $\mathscr{B}$ is the subalgebra $p\mathscr{A}p$, then the unity of $\mathscr{B}$ is p, and therefore, $p \in \mathscr{B}^{-1} \setminus \mathscr{A}^{-1}$).

Let $a \in \mathscr{A}$, if there exists $b \in \mathscr{A}$ such that

$$bab = b, \qquad ab = ba, \qquad a(1-ab) \text{ is nilpotent}, \tag{1.1}$$

then b is the *Drazin inverse* of a, denoted by a^{D} and it is unique. If the last condition in (1.1) is replaced by $a(1-ab)$ is quasinilpotent, then b is the *generalized Drazin inverse*, denoted by a^{d} and is also unique. Evidently aa^{d} is an idempotent, and it is customary to denote $a^{\pi} = 1 - aa^{\mathsf{d}}$. We shall denote

$$\mathscr{A}^{\mathsf{d}} = \{a \in \mathscr{A} : \exists\ a^{\mathsf{d}}\}.$$

In particular, if $a(1-ab) = 0$ then b is called the *group inverse* of a. It was proved in [9, Lemma 2.4] that a^{d} exists if and only if and only if exists an idempotent $q \in \mathscr{A}$ such that $aq = qa$, aq is quasinilpotent, and $a+q$ is invertible. The following simple remark will be useful.

Remark 1.1. *If the subalgebra $\mathscr{B} \subset \mathscr{A}$ has unity, then $\mathscr{B}^{-1} \subset \mathscr{A}^{\mathsf{d}}$ and if $b \in \mathscr{B}^{-1}$, then $b^{\mathsf{d}} = [b^{-1}]_{\mathscr{B}}$. In fact, let e be the unity of $\mathscr{B}$, since $b[b^{-1}]_{\mathscr{B}} = [b^{-1}]_{\mathscr{B}}b = e$, it is easy to see $b[b^{-1}]_{\mathscr{B}}b = b$, $[b^{-1}]_{\mathscr{B}}b[b^{-1}]_{\mathscr{B}} = [b^{-1}]_{\mathscr{B}}$, and $[b^{-1}]_{\mathscr{B}}b = b[b^{-1}]_{\mathscr{B}}$.*

Following [4], we say that $\mathscr{P} = \{p_1, p_2, \ldots, p_n\}$ is a *total system of idempotents* in $\mathscr{A}$ if $p_i^2 = p_i$ for all i, $p_ip_j = 0$ if $i \neq j$, and $p_1 + \cdots + p_n = 1$. Given a total system $\mathscr{P}$ of idempotents in $\mathscr{A}$, we consider the set $\mathscr{M}_n(\mathscr{A}, \mathscr{P})$ consisting of all matrices $A = [a_{ij}]_{i,j=1}^n$ with elements in $\mathscr{A}$ such that $a_{ij} \in p_i\mathscr{A}p_j$ for all $i, j \in \{1, \ldots, n\}$. Let us recall that $p_i\mathscr{A}p_i$ is a subalgebra of $\mathscr{A}$ with unity p_i. In [4, Lemma 2.1] it was proved that $\phi : \mathscr{A} \to \mathscr{M}_n(\mathscr{A}, \mathscr{P})$ given by

$$\phi(x) = \begin{bmatrix} p_1xp_1 & p_1xp_2 & \cdots & p_1xp_n \\ p_2xp_1 & p_2xp_2 & \cdots & p_2xp_n \\ \vdots & \vdots & \ddots & \vdots \\ p_nxp_1 & p_nxp_2 & \cdots & p_nxp_n \end{bmatrix}_{\mathscr{P}}$$

is an isometric algebra isomorphism. In the sequel, we shall identify $x = \phi(x)$ for $x \in \mathscr{A}$. Another useful (although trivial) identity is

$$x = \sum_{i,j=1}^{n} p_ixp_j \qquad \forall\ x \in \mathscr{A}.$$

If $a \in \mathscr{A}$ is generalized Drazin invertible, then we have the following matrix representations:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}}, \qquad a^{\mathsf{d}} = \begin{bmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{bmatrix}_{\mathscr{P}}, \qquad a^{\pi} = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}}, \tag{1.2}$$

where $p = aa^{\mathsf{d}}$, $\mathscr{P} = \{p, 1-p\}$, $a_1 \in [p\mathscr{A}p]^{-1}$, and $a_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{qnil}}$. Let us remark that if a has the above representation, then $a^{\mathsf{d}} = [a_1{}^{-1}]_{p\mathscr{A}p} = a_1^{\mathsf{d}}$.

The following lemmas are needed in what follows.

Lemma 1.1. *Let $\mathscr{P} = \{p, 1-p\}$ be a total system of idempotents in $\mathscr{A}$ and let $a, b \in \mathscr{A}$ have the following representation*

$$a = \begin{bmatrix} x & 0 \\ z & y \end{bmatrix}_{\mathscr{P}}, \qquad b = \begin{bmatrix} x & t \\ 0 & y \end{bmatrix}_{\mathscr{P}}.$$

Then there exist $(z_n)_{n=0}^{\infty} \subset (1-p)\mathscr{A}p$ and $(t_n)_{n=0}^{\infty} \subset p\mathscr{A}(1-p)$ such that

$$a^n = \begin{bmatrix} x^n & 0 \\ z_n & y^n \end{bmatrix}_{\mathscr{P}} \qquad \text{and} \qquad b^n = \begin{bmatrix} x^n & t_n \\ 0 & y^n \end{bmatrix}_{\mathscr{P}} \qquad \forall\, n \in \mathbb{N}.$$

The proof of this lemma is trivial by induction and we will not give it.

Lemma 1.2. [4, Theorem 3.3] *Let $b \in \mathscr{A}^{\mathsf{d}}$, $a \in \mathscr{A}^{\mathsf{qnil}}$, and let $ab^{\pi} = a$ and $b^{\pi}ab = 0$. Then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a+b)^n.$$

The following Lemma is a generalization of Theorem 1 in [6]. Although it was stated for bounded linear operators in a Banach space, its proof remains valid for Banach algebras.

Lemma 1.3. *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ such that $ab = ba$. Then $a + b \in \mathscr{A}^{\mathsf{d}}$ if and only if $1 + a^{\mathsf{d}}b \in \mathscr{A}^{\mathsf{d}}$. In this case we have*

$$(a+b)^{\mathsf{d}} = a^{\mathsf{d}}(1 + a^{\mathsf{d}}b)bb^{\mathsf{d}} + b^{\pi} \sum_{n=0}^{\infty} (-b)^n (a^{\mathsf{d}})^{n+1} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+1} (-a)^n a^{\pi}.$$

Lemma 1.4. [4, Example 4.5] *Let $a, b \in \mathscr{A}$ be generalized Drazin invertible and $ab = 0$, then $a + b$ is generalized Drazin invertible and*

$$(a+b)^{\mathsf{d}} = b^{\pi} \sum_{n=0}^{\infty} b^n (a^{\mathsf{d}})^{n+1} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+1} a^n a^{\pi}.$$

Lemma 1.5. [4, Theorem 2.3] *Let $x, y \in \mathscr{A}$, p an idempotent of $\mathscr{A}$ and let x and y have the representation*

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}_{\{p,1-p\}}, \qquad y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}_{\{1-p,p\}}. \tag{1.3}$$

(i) *If $a \in [p\mathscr{A}p]^{\mathsf{d}}$ and $b \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$, then $x, y \in \mathscr{A}^{\mathsf{d}}$ and*

$$x^{\mathsf{d}} = \begin{bmatrix} a^{\mathsf{d}} & 0 \\ u & b^{\mathsf{d}} \end{bmatrix}_{\{p,1-p\}}, \qquad y^{\mathsf{d}} = \begin{bmatrix} b^{\mathsf{d}} & u \\ 0 & a^{\mathsf{d}} \end{bmatrix}_{\{1-p,p\}} \tag{1.4}$$

where

$$u = \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} c a^n a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} b^n c (a^{\mathsf{d}})^{n+2} - b^{\mathsf{d}} c a^{\mathsf{d}}. \tag{1.5}$$

(ii) *If $x \in \mathscr{A}^{\mathsf{d}}$ and $a \in [p\mathscr{A}p]^{\mathsf{d}}$, then $b \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$, and x^{d} and y^{d} are given by* (1.4) and (1.5).

Lemma 1.6. [5, Lemma 2.1] *Let $a, b \in \mathscr{A}^{\mathsf{qnil}}$ and let $ab = ba$ or $ab = 0$, then $a + b \in \mathscr{A}^{\mathsf{qnil}}$.*

2. Main results

In this section, for $a, b \in \mathscr{A}$, we will investigate some formulas of $(a+b)^{\mathsf{d}}$ in terms of a, b, a^{d}, and b^{d}.

Theorem 2.1. *Let $a, b \in \mathscr{A}$ be generalized Drazin invertible and satisfying $b^\pi a^\pi ba = 0$, $b^\pi aa^{\mathsf{d}}baa^{\mathsf{d}} = 0$, $ab^\pi = a$. Then*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + u + b^\pi v,$$

where

$$v = a^{\mathsf{d}} + \sum_{n=0}^{\infty} (a^{\mathsf{d}})^{n+2} b (a+b)^n, \quad u = \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a+b)^n (a+b)^\pi - b^{\mathsf{d}} av.$$

Proof. Let $p = bb^{\mathsf{d}}$ and $\mathscr{P} = \{p, 1-p\}$. Let a and b have the following representation

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}}, \qquad a = \begin{bmatrix} a_3 & a_1 \\ a_4 & a_2 \end{bmatrix}_{\mathscr{P}}, \tag{2.1}$$

where b_1 is invertible in $p\mathscr{A}p$ and b_2 is quasinilpotent in $(1-p)\mathscr{A}(1-p)$. Since $ab^\pi = a$ and

$$ab^\pi = \begin{bmatrix} a_3 & a_1 \\ a_4 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}},$$

we have $a_3 = a_4 = 0$. Hence

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}}, \qquad a = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}}, \qquad a+b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2+b_2 \end{bmatrix}_{\mathscr{P}}. \tag{2.2}$$

By observing the representation of a given in (2.2) and a simple appealing of Lemma 1.5 yield

$$a^{\mathsf{d}} = \begin{bmatrix} 0 & a_1(a_2^{\mathsf{d}})^2 \\ 0 & a_2^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}} \tag{2.3}$$

and

$$a^\pi = 1 - aa^{\mathsf{d}} = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} - \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1(a_2^{\mathsf{d}})^2 \\ 0 & a_2^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} p & -a_1 a_2^{\mathsf{d}} \\ 0 & a_2^\pi - p \end{bmatrix}_{\mathscr{P}}. \tag{2.4}$$

To explain better the "southeast" block of a^π in the above relation, let us permit say that a_2^π is defined as $1 - a_2 a_2^{\mathsf{d}}$, the element $1 - p - a_2 a_2^{\mathsf{d}} = a_2^\pi - p$ belongs to $(1-p)\mathscr{A}(1-p)$, but a_2^π does not need belong to $(1-p)\mathscr{A}(1-p)$. In fact, since $a_2^\pi - p \in (1-p)\mathscr{A}(1-p)$, one has $(a_2^\pi - p)p = p(a_2^\pi - p) = 0$, or equivalently, $a_2^\pi p = pa_2^\pi = p$.

In view of the last representation in (2.2), we shall apply Lemma 1.5 to find an expression of $(a+b)^{\mathsf{d}}$. To this end, we need prove $b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and $a_2 + b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$. The fact $b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ follows from $b \in \mathscr{A}^{\mathsf{d}}$ and the representation of b in (2.2), in fact, we have $b^{\mathsf{d}} = [{b_1}^{-1}]_{p\mathscr{A}p} = b_1^{\mathsf{d}}$. We shall study $(a_2+b_2)^{\mathsf{d}}$ in the following lines. Let us represent a_2 and b_2 as follows:

$$a_2 = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_{\mathscr{Q}}, \qquad b_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{\mathscr{Q}}, \tag{2.5}$$

where $q = a_2^{\mathsf{d}} a_2$ and $\mathscr{Q} = \{q, 1-p-q\}$ (which is a total system of idempotents in the subalgebra $(1-p)\mathscr{A}(1-p)$). Observe that since $q \in (1-p)\mathscr{A}(1-p)$ and $1-p$ is the unity of $(1-p)\mathscr{A}(1-p)$, then $q(1-p) = (1-p)q = q$, or equivalently,

$qp = pq = 0$. Recall that in the above representation of a_2 in (2.5), the element a_{11} is invertible in $q\mathscr{A}q$ and a_{22} is quasinilpotent.

Since $b^\pi a^\pi ba = 0$ and by (2.2), (2.4), we have

$$\begin{aligned} 0 &= b^\pi a^\pi ba \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} p & -a_1 a_2^{\mathsf{d}} \\ 0 & a_2^\pi - p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (a_2^\pi - p)b_2 a_2 \end{bmatrix}_{\mathscr{P}} \\ &= (a_2^\pi - p)b_2 a_2. \end{aligned}$$

But observe that $b_2 \in (1-p)\mathscr{A}(1-p)$, and thus, $pb_2 = 0$. Therefore, $0 = a_2^\pi b_2 a_2$ holds.

It seems that we can use (2.5) and $0 = a_2^\pi b_2 a_2$ to get some information on b_{ij}, but observe that we cannot represent a_2^π in the total system of idempotents $\mathscr{Q}$ since in general $a_2^\pi \notin (1-p)\mathscr{A}(1-p)$. To avoid this situation, let us define $\mathscr{R} = \{p, q, 1-p-q\}$, which in view of $pq = qp = 0$, it is trivial to see that $\mathscr{R}$ is a total system of idempotents in $\mathscr{A}$. Since $a_2^\pi = 1 - a_2 a_2^{\mathsf{d}} = 1 - q$,

$$\begin{aligned} 0 &= a_2^\pi b_2 a_2 = \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-p-q \end{bmatrix}_{\mathscr{R}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{bmatrix}_{\mathscr{R}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix}_{\mathscr{R}} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{21}a_{11} & b_{22}a_{22} \end{bmatrix}_{\mathscr{R}}. \end{aligned}$$

Thus, $b_{21}a_{11} = 0$. Since a_{11} is invertible in $q\mathscr{A}q$ and $b_{21} \in (1-p-q)\mathscr{A}q$ (this last assertion follows from the representation of b_2 given in (2.5)), we get

$$b_{21} = 0. \tag{2.6}$$

Let us calculate $b^\pi aa^{\mathsf{d}}baa^{\mathsf{d}}$.

$$\begin{aligned} b^\pi aa^{\mathsf{d}}baa^{\mathsf{d}} &= \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 a_2^{\mathsf{d}} \\ 0 & a_2 a_2^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 a_2^{\mathsf{d}} \\ 0 & a_2 a_2^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & a_2 a_2^{\mathsf{d}} b_2 a_2 a_2^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}}. \end{aligned}$$

Thus, $b^\pi aa^{\mathsf{d}}baa^{\mathsf{d}} = a_2 a_2^{\mathsf{d}} b_2 a_2 a_2^{\mathsf{d}} = qb_2 q$; hence representation (2.5) entails $b_{11} = 0$. Since (2.6) holds, then

$$b_2 a_2^\pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{12} \\ 0 & 0 & b_{22} \end{bmatrix}_{\mathscr{R}} \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-p-q \end{bmatrix}_{\mathscr{R}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{12} \\ 0 & 0 & b_{22} \end{bmatrix}_{\mathscr{R}} = b_2.$$

Thus, the following conditions

(i) $a_2 \in \mathscr{A}^{\mathsf{d}}$ (ii) b_2 is quasinilpotent (iii) $b_2 a_2^\pi = b_2$ (iv) $a_2^\pi b_2 a_2 = 0$.

are satisfied. Hence, we can apply Lemma 1.2 to get an expression of $(b_2 + a_2)^d$ obtaining

$$(a_2 + b_2)^{\mathsf{d}} = a_2^{\mathsf{d}} + \sum_{n=0}^{\infty} (a_2^{\mathsf{d}})^{n+2} b_2 (a_2 + b_2)^n.$$

By Lemma 1.5 applied to the representation of $a+b$ given in (2.2) we obtain

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} b_1^{\mathsf{d}} & u \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}}, \tag{2.7}$$

where

$$u = \sum_{n=0}^{\infty} (b_1^{\mathsf{d}})^{n+2} a_1 (a_2+b_2)^n (a_2+b_2)^{\pi} + \sum_{n=0}^{\infty} b_1^{\pi} b_1^n a_1 [(a_2+b_2)^{\mathsf{d}}]^{n+2} - b_1^{\mathsf{d}} a_1 (a_2+b_2)^{\mathsf{d}}. \tag{2.8}$$

Recall that $b_1^{\mathsf{d}} = b^{\mathsf{d}}$. Easily we have $bb^{\mathsf{d}}a = a_1$, $bb^{\mathsf{d}}b = b_1$, and $b^{\pi}b = b_2$. From (2.3) we get $b^{\pi}a^{\mathsf{d}} = a_2^{\mathsf{d}}$. By Lemma 1.1 and the representations of $a+b$ and a^{d} given in (2.2) and (2.3), respectively, we have $b^{\pi}(a+b)^k = (a_2+b_2)^k$, and $b^{\pi}(a^{\mathsf{d}})^k = (a_2^{\mathsf{d}})^k$ for any positive integer k, Hence

$$(a_2+b_2)^{\mathsf{d}} = b^{\pi}a^{\mathsf{d}} + \sum_{n=0}^{\infty} b^{\pi}(a^{\mathsf{d}})^{n+2} b^{\pi} b b^{\pi} (a+b)^n.$$

This last expression can be simplified by observing that $b^{\pi}bb^{\pi} = b^{\pi}b$ and that by Lemma 1.1, there exists a sequence $z_n \in \mathscr{A}$ such that

$$(a^{\mathsf{d}})^{n+2} b^{\pi} = \begin{bmatrix} 0 & z_n \\ 0 & (a_2^{\mathsf{d}})^{n+2} \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & z_n \\ 0 & (a_2^{\mathsf{d}})^{n+2} \end{bmatrix}_{\mathscr{P}} = (a^{\mathsf{d}})^{n+2}.$$

Therefore,

$$(a_2+b_2)^{\mathsf{d}} = b^{\pi} \left(a^{\mathsf{d}} + \sum_{n=0}^{\infty} (a^{\mathsf{d}})^{n+2} b (a+b)^n \right). \tag{2.9}$$

Now we will simplify the expression of u given in (2.8). Observe that for any $n \geq 0$, one has

$$n \geq 0 \qquad \Rightarrow \qquad (b_1^{\mathsf{d}})^{n+2} a_1 = (b^{\mathsf{d}})^{n+2} b b^{\mathsf{d}} a = (b^{\mathsf{d}})^{n+2} a. \tag{2.10}$$

Moreover, by (2.2) we have

$$\begin{aligned} (a+b)^{\pi} &= 1-(a+b)(a+b)^{\mathsf{d}} \\ &= \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} - \begin{bmatrix} b_1 & a_1 \\ 0 & a_2+b_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1^{\mathsf{d}} & u \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}} \\ &= \begin{bmatrix} p - b_1 b_1^{\mathsf{d}} & -b_1 u - a_1 (a_2+b_2)^{\mathsf{d}} \\ 0 & 1-p-(a_2+b_2)(a_2+b_2)^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}}. \end{aligned}$$

Thus, and by using Lemma 1.1 and (2.7), there exists a sequence $(x_n)_{n=1}^{\infty}$ in $\mathscr{A}$ such that

$$\begin{aligned} & b^{\pi}(a+b)^n (a+b)^{\pi} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1^n & x_n \\ 0 & (a_2+b_2)^n \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} p - b_1 b_1^{\mathsf{d}} & -b_1 u - a_1 (a_2+b_2)^{\mathsf{d}} \\ 0 & (a_2+b_2)^{\pi} - p \end{bmatrix}_{\mathscr{P}} \\ &= (a_2+b_2)^n \left[(a_2+b_2)^{\pi} - p \right], \end{aligned}$$

but recall that $a_2 + b_2 \in (1-p)\mathscr{A}(1-p)$, and thus, if $n > 0$, then $(a_2 + b_2)^n \left[(a_2+b_2)^{\pi} - p \right] = (a_2+b_2)^n (a_2+b_2)^{\pi}$. Thus

$$n > 0 \qquad \Rightarrow \qquad b^{\pi}(a+b)^n (a+b)^{\pi} = (a_2+b_2)^n (a_2+b_2)^{\pi}. \tag{2.11}$$

Now we can prove that

$$(b_1^{\mathsf{d}})^{n+2} a_1 (a_2+b_2)^n (a_2+b_2)^{\pi} = (b^{\mathsf{d}})^{n+2} a (a+b)^n (a+b)^{\pi} \tag{2.12}$$

holds for any $n \in \mathbb{N}$. Since, as is easy to see, $ab^\pi = a$, then we have for any $n > 0$ that (2.10) and (2.11) lead to

$$(b_1^{\mathsf{d}})^{n+2} a_1 (a_2+b_2)^n (a_2+b_2)^\pi = (b^{\mathsf{d}})^{n+2} ab^\pi (a+b)^n (a+b)^\pi = (b^{\mathsf{d}})^{n+2} a(a+b)^n (a+b)^\pi.$$

Now, we will prove that (2.12) holds for $n = 0$:

$$\begin{aligned}(b^{\mathsf{d}})^2 a(a+b)^\pi &= \begin{bmatrix} (b_1^{\mathsf{d}})^2 & 0 \\ 0 & 0 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} p - b_1 b_1^{\mathsf{d}} & -b_1 u - a_1 (a_2+b_2)^{\mathsf{d}} \\ 0 & (a_2+b_2)^\pi - p \end{bmatrix}_{\mathscr{P}} \\ &= \begin{bmatrix} 0 & (b_1^{\mathsf{d}})^2 a_1 \left[(a_2+b_2)^\pi - p\right] \\ 0 & 0 \end{bmatrix}_{\mathscr{P}} = (b_1^{\mathsf{d}})^2 a_1 \left[(a_2+b_2)^\pi - p\right].\end{aligned}$$

But observe that $a_1 \in p\mathscr{A}(1-p)$, and hence $a_1 p = 0$. Thus, we have proved

$$(b^{\mathsf{d}})^2 a(a+b)^\pi = (b_1^{\mathsf{d}})^2 a_1 (a_2+b_2)^\pi.$$

And thus, (2.12) holds for any $n \in \mathbb{N}$.

Now, we are going to simplify the expression $\sum_{n=0}^\infty b_1^\pi b_1^n a_1 (a_2+b_2)^{n+2}$ appearing in (2.8). Recall that $b_1^\pi = b^\pi$ and $a_1 = bb^{\mathsf{d}}a$ were obatined. Observe that if $n > 0$, then $b_1^\pi b_1^n = b^\pi b_1^n = (1-p)b_1^n = 0$ since $b_1 \in p\mathscr{A}p$. Furthermore, $b_1^\pi a_1 = b^\pi bb^{\mathsf{d}} a = 0$. Thus

$$\sum_{n=0}^\infty b_1^\pi b_1^n a_1 [(a_2+b_2)^{\mathsf{d}}]^{n+2} = 0. \tag{2.13}$$

From (2.9), $b_1^{\mathsf{d}} = b^{\mathsf{d}}$, $a_1 = bb^{\mathsf{d}}a$, and $ab^\pi = a$ we get

$$\begin{aligned} &b_1^{\mathsf{d}} a_1 (a_2+b_2)^{\mathsf{d}} = \\ &= b^{\mathsf{d}} bb^{\mathsf{d}} ab^\pi \left(a^{\mathsf{d}} + \sum_{n=0}^\infty (a^{\mathsf{d}})^{n+2} b(a+b)^n \right) = b^{\mathsf{d}} a \left(a^{\mathsf{d}} + \sum_{n=0}^\infty (a^{\mathsf{d}})^{n+2} b(a+b)^n \right) \end{aligned} \tag{2.14}$$

Now, (2.7), (2.8), (2.9), (2.12), (2.13), and (2.14) prove the theorem. □

Remark 2.1. *If b is group invertible, then the condition $b^\pi a^\pi ba = 0$ implies $b^\pi aa^{\mathsf{d}} baa^{\mathsf{d}} = 0$. In fact, since $bb^\pi = 0$, then $b^\pi aa^{\mathsf{d}} baa^{\mathsf{d}} = b^\pi (1 - a^\pi) baa^{\mathsf{d}} = -b^\pi a^\pi baa^{\mathsf{d}} = 0$.*

Remark 2.2. *Theorem 2.1 extends Lemma 1.2. If b is quasinilpotent, then $b^{\mathsf{d}} = 0$ and $b^\pi = 1$. Notice that $ba^\pi = b$ clearly implies $aa^{\mathsf{d}} baa^{\mathsf{d}} = 0$.*

Theorem 2.2. *Let $a, b \in \mathscr{A}$ be generalized Drazin invertible. Assume that $b^\pi ab = b^\pi ba$ and $ab^\pi = a$. Then*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= b^{\mathsf{d}} + \\ &+ b^\pi \sum_{n=0}^\infty (-1)^n (a^{\mathsf{d}})^{n+1} b^n + \sum_{n=0}^\infty (b^{\mathsf{d}})^{n+2} a(a+b)^n (a+b)^\pi - b^{\mathsf{d}} a \sum_{n=0}^\infty (-1)^n (a^{\mathsf{d}})^{n+1} b^n.\end{aligned}$$

Proof. As in the proof of Theorem 2.1, if we set $p = bb^{\mathsf{d}}$ and by using $ab^\pi = a$, then the representations given in (2.2) are valid, where $\mathscr{P} = \{p, 1-p\}$, b_1 is invertible in $p\mathscr{A}p$ and b_2 is quasinilpotent. Since

$$b^\pi ab = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & a_2 b_2 \end{bmatrix}_{\mathscr{P}},$$

$$b^\pi ba = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & b_2 a_2 \end{bmatrix}_{\mathscr{P}},$$

and $b^\pi ab = b^\pi ba$, then $a_2b_2 = b_2a_2$. Lemma 1.3 guarantees that b_2+a_2 is generalized Drazin invertible if and only if $1+b_2^{\mathsf{d}}a_2$ is generalized Drazin invertible; but observe that b_2 is quasinilpotent, and therefore $b_2^{\mathsf{d}} = 0$. So, $b_2 + a_2$ is generalized Drazin invertible. Also, $b_2^\pi = 1 - b_2b_2^{\mathsf{d}} = 1$ and Lemma 1.3 lead to

$$(b_2 + a_2)^{\mathsf{d}} = \sum_{n=0}^{\infty}(a_2^{\mathsf{d}})^{n+1}(-b_2)^n.$$

By (2.3) and Lemma 1.1, there exist a sequence $(x_n)_{n=1}^{\infty}$ in $p\mathscr{A}(1-p)$ such that $(a^{\mathsf{d}})^n = x_n + (a_2^{\mathsf{d}})^n$ for any $n \in \mathbb{N}$, thus

$$\begin{aligned} b^\pi(a^{\mathsf{d}})^{n+1}b^n &= \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & x_n \\ 0 & (a_2^{\mathsf{d}})^{n+1} \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1^n & 0 \\ 0 & b_2^n \end{bmatrix}_{\mathscr{P}} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (a_2^{\mathsf{d}})^{n+1}b_2^n \end{bmatrix}_{\mathscr{P}} = (a_2^{\mathsf{d}})^{n+1}b_2^n. \end{aligned}$$

Thus,

$$(a_2 + b_2)^{\mathsf{d}} = b^\pi \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{d}})^{n+1}b^n. \tag{2.15}$$

By employing Lemma 1.5 for the representation of $a + b$ given in (2.2) we get

$$(a + b)^{\mathsf{d}} = b_1^{\mathsf{d}} + (b_2 + a_2)^{\mathsf{d}} + u, \tag{2.16}$$

where

$$u = \sum_{n=0}^{\infty}(b_1^{\mathsf{d}})^{n+2}a_1(a_2 + b_2)^n(a_2 + b_2)^\pi + \sum_{n=0}^{\infty} b_1^\pi b_1^n a_1[(a_2 + b_2)^{\mathsf{d}}]^{n+2} - b_1^{\mathsf{d}}a_1(a_2 + b_2)^{\mathsf{d}}.$$

As in the proof of Theorem 2.1, we have that (2.12) and $b_1^\pi b_1^n a_1 = 0$ for any $n \geq 0$ hold. Furthermore, since $a_1 = bb^{\mathsf{d}}a$ and $ab^\pi = a$, then

$$b_1^{\mathsf{d}}a_1(a_2 + b_2)^{\mathsf{d}} = b^{\mathsf{d}}bb^{\mathsf{d}}ab^\pi \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{d}})^{n+1}b^n = b^{\mathsf{d}}a\sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{d}})^{n+1}b^n.$$

Therefore,

$$u = \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a + b)^n(a + b)^\pi - b^{\mathsf{d}}a\sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{d}})^{n+1}b^n. \tag{2.17}$$

Expressions (2.15), (2.16), and (2.17) permit finish the proof. □

Theorem 2.3. *Let $a, b \in \mathscr{A}$ be generalized Drazin invertible. Assume that they satisfy $aba = 0$ and $ab^2 = 0$. Then*

$$a + b \in \mathscr{A}^{\mathsf{d}} \iff a^\pi(a + b) \in \mathscr{A}^{\mathsf{d}} \iff b^\pi a^\pi(a + b) \in \mathscr{A}^{\mathsf{d}} \iff a^\pi b^\pi(a + b) \in \mathscr{A}^{\mathsf{d}}.$$

Furthermore, if $b^\pi ab = 0$ or $b^\pi ba = 0$, or $b^\pi ab = b^\pi ba$, then $a + b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a + b)^{\mathsf{d}} = b^{\mathsf{d}} + v + b^\pi a^{\mathsf{d}} + u + b^\pi(a^{\mathsf{d}})^2b + ua^{\mathsf{d}}b,$$

where

$$u = b^\pi a^\pi b^\pi \sum_{n=0}^{\infty}(a + ba^\pi)^n bb^\pi(a^{\mathsf{d}})^{n+2},$$

$$v = \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a + b)^n[1 - s(a + b)] - b^{\mathsf{d}}aa^{\mathsf{d}},$$

where $s = b^\pi a^{\mathsf{d}} + u + b^\pi(a^{\mathsf{d}})^2b + ua^{\mathsf{d}}b$.

Proof. Since $ab^2 = 0$, then a and b have the matrix representation given in (2.2). Now, we use $0 = aba$.

$$aba = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & a_1b_2a_2 \\ 0 & a_2b_2a_2 \end{bmatrix}_{\mathscr{P}}.$$

Thus $a_1b_2a_2 = a_2b_2a_2 = 0$. Let $q = a_2a_2^{\mathsf{d}}$ and $\mathscr{Q} = \{q, 1-p-q\}$ (a total system of idempotents in the algebra $(1-p)\mathscr{A}(1-p)$). We represent

$$a_2 = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_{\mathscr{Q}}, \qquad b_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{\mathscr{Q}}, \tag{2.18}$$

where a_{11} is invertible in the subalgebra $q\mathscr{A}q$ and a_{22} is quasinilpotent. We use $a_2b_2a_2 = 0$

$$a_2b_2a_2 = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_{\mathscr{Q}} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{\mathscr{Q}} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_{\mathscr{Q}} = \begin{bmatrix} a_{11}b_{11}a_{11} & a_{11}b_{12}a_{22} \\ a_{22}b_{21}a_{11} & a_{22}b_{22}a_{22} \end{bmatrix}_{\mathscr{Q}}.$$

Thus, $0 = a_{11}b_{11}a_{11}$ and $0 = a_{11}b_{12}a_{22}$. The invertibility of a_{11} in the subalgebra $q\mathscr{A}q$ and $b_{11} \in q\mathscr{A}q$ ensure $b_{11} = 0$. In a similar way we have $b_{12}a_{22} = 0$. Using $ab^2 = 0$ leads to $a_2b_2^2 = 0$. Hence $a_{11}b_{12}b_{21} = 0$ and $a_{11}b_{12}b_{22} = 0$. The invertibility of a_{11} (in $q\mathscr{A}q$) leads to $b_{12}b_{21} = 0$ and $b_{12}b_{22} = 0$. Now let us define

$$x = \begin{bmatrix} 0 & b_{12} \\ 0 & 0 \end{bmatrix}_{\mathscr{Q}} \qquad \text{and} \qquad y = \begin{bmatrix} a_{11} & 0 \\ b_{21} & a_{22} + b_{22} \end{bmatrix}_{\mathscr{Q}} \tag{2.19}$$

From (2.18) and $b_{11} = 0$ one trivially gets $a_2 + b_2 = x + y$. From $b_{12}b_{21} = 0$, $b_{12}a_{22} = 0$, and $b_{12}b_{22} = 0$ we have $xy = 0$.

Let us prove

$$a + b \in \mathscr{A}^{\mathsf{d}} \qquad \Longleftrightarrow \qquad a_{22} + b_{22} \in [(1-p-q)\mathscr{A}(1-p-q)]^{\mathsf{d}}. \tag{2.20}$$

$\Rightarrow$: Assume that $a + b \in \mathscr{A}^{\mathsf{d}}$, then by the representations given in (2.2) we have that $a_2 + b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$, i.e., $a_2 + b_2 = x + y \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$. We can apply Lemma 1.4 to $y = -x + (x+y)$ because $-x \in \mathscr{A}^{\mathsf{d}}$ (since $(-x)^2 = 0$) and $-x(x+y) = 0$ obtaining $y \in \mathscr{A}^{\mathsf{d}}$. Lemma 1.5 and the representation of y in (2.19) ensure that $a_{22} + b_{22}$ is generalized Drazin invertible.

$\Leftarrow$: Assume in this paragraph that $a_{22} + b_{22}$ is generalized Drazin invertible. By recalling that a_{11} is invertible in the subalgebra $q\mathscr{A}q$, the representation of y in (2.19) leads to $y \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$. Since $x^2 = 0$, $xy = 0$, and $x + y = a_2 + b_2$, Lemma 1.4 yields $a_2 + b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$. Now, Lemma 1.5 and the representation of $a + b$ in (2.2) ensure that $a + b \in \mathscr{A}^{\mathsf{d}}$.

Our next goal is to express the right side of the equivalence (2.20) in terms of a and b. To this end, let us define $\mathscr{R} = \{p, q, 1-q-p\}$, which is easy to see that it is a total system of idempotents in $\mathscr{A}$. First we notice that

$$\begin{aligned} a_2^{\pi}(a_2 + b_2) &= \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-p-q \end{bmatrix}_{\mathscr{R}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{11} + b_{11} & b_{12} \\ 0 & b_{21} & a_{22} + b_{22} \end{bmatrix}_{\mathscr{R}} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (1-p-q)a_{21} & a_{22} + b_{22} \end{bmatrix}_{\mathscr{R}}, \end{aligned} \tag{2.21}$$

which, in view of Lemma 1.5, ensures that $a_{22} + b_{22}$ is generalized Drazin invertible if and only if $a_2^{\pi}(a_2 + b_2)$ is generalized Drazin invertible. Now, we shall use (2.4),

$b_1 \in p\mathscr{A}p$, $a_1 \in p\mathscr{A}(1-p)$, and $a_2, b_2 \in (1-p)\mathscr{A}(1-p)$,

$$a^\pi(a+b) = \begin{bmatrix} p & -a_1a_2^{\mathsf{d}} \\ 0 & a_2^\pi - p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & a_1 \\ 0 & a_2+b_2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} b_1 & a_1 - a_1a_2^{\mathsf{d}}(a_2+b_2) \\ 0 & a_2^\pi(a_2+b_2) \end{bmatrix}_{\mathscr{P}},$$

$$b^\pi a^\pi(a+b) = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & a_1 - a_1a_2^{\mathsf{d}}(a_2+b_2) \\ 0 & a_2^\pi(a_2+b_2) \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & a_2^\pi(a_2+b_2) \end{bmatrix}_{\mathscr{P}},$$

and

$$\begin{aligned} a^\pi b^\pi(a+b) &= \begin{bmatrix} p & -a_1a_2^{\mathsf{d}} \\ 0 & a_2^\pi - p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & a_1 \\ 0 & a_2+b_2 \end{bmatrix}_{\mathscr{P}} \\ &= \begin{bmatrix} 0 & -a_1a_2^{\mathsf{d}}(a_2+b_2) \\ 0 & a_2^\pi(a_2+b_2) \end{bmatrix}_{\mathscr{P}}, \end{aligned}$$

an appealing to Lemma 1.5, leads to

$$a_2^\pi(a_2+b_2) \in \mathscr{A}^{\mathsf{d}} \iff a^\pi(a+b) \in \mathscr{A}^{\mathsf{d}} \iff b^\pi a^\pi(a+b) \in \mathscr{A}^{\mathsf{d}} \iff a^\pi b^\pi(a+b) \in \mathscr{A}^{\mathsf{d}}.$$

We shall prove the second part of the Theorem. Since

$$b^\pi ab = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & a_2b_2 \end{bmatrix}_{\mathscr{P}} = a_2b_2$$

and

$$b^\pi ba = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & b_2a_2 \end{bmatrix}_{\mathscr{P}} = b_2a_2,$$

then

$$b^\pi ab = 0 \text{ or } b^\pi ba = 0 \text{ or } b^\pi ab = b^\pi ba \ \Rightarrow \ a_2b_2 = 0 \text{ or } b_2a_2 = 0 \text{ or } a_2b_2 = b_2a_2.$$

The representations given in (2.18) lead to

$$a_2b_2 = 0 \text{ or } b_2a_2 = 0 \text{ or } a_2b_2 = b_2a_2 \ \Rightarrow \ a_{22}b_{22} = 0 \text{ or } b_{22}a_{22} = 0 \text{ or } a_{22}b_{22} = b_{22}a_{22}.$$

Since a_{22} and b_{22} are quasinilpotent, then the above implications and Lemma 1.6 lead to

$$b^\pi ab = 0 \text{ or } b^\pi ba = 0 \text{ or } b^\pi ab = b^\pi ba \ \Rightarrow \ a_{22}+b_{22} \text{ is quasinilpotent.}$$

In particular, by employing equivalence (2.20) we get that $a+b \in \mathscr{A}^{\mathsf{d}}$. Furthermore, by using Lemma 1.4, $x^2 = 0$, and $xy = 0$, one gets

$$(a_2+b_2)^{\mathsf{d}} = (x+y)^{\mathsf{d}} = y^\pi \sum_{n=0}^{\infty} y^n (x^{\mathsf{d}})^{n+1} + \sum_{n=0}^{\infty} (y^{\mathsf{d}})^{n+1} x^n x^\pi = y^{\mathsf{d}} + (y^{\mathsf{d}})^2 x.$$

From (2.19) and Lemma 1.5 we get

$$y^{\mathsf{d}} = \begin{bmatrix} a_{11}^{\mathsf{d}} & 0 \\ u & (a_{22}+b_{22})^{\mathsf{d}} \end{bmatrix}_{\mathscr{Q}} = \begin{bmatrix} a_{11}^{\mathsf{d}} & 0 \\ u & 0 \end{bmatrix}_{\mathscr{Q}},$$

where

$$u = \sum_{n=0}^{\infty} (a_{22}+b_{22})^n b_{21} (a_{11}^{\mathsf{d}})^{n+2}. \tag{2.22}$$

So

$$(y^{\mathsf{d}})^2 x = \begin{bmatrix} a_{11}^{\mathsf{d}} & 0 \\ u & 0 \end{bmatrix}_{\mathscr{Q}} \begin{bmatrix} a_{11}^{\mathsf{d}} & 0 \\ u & 0 \end{bmatrix}_{\mathscr{Q}} \begin{bmatrix} 0 & b_{12} \\ 0 & 0 \end{bmatrix}_{\mathscr{Q}} = \begin{bmatrix} 0 & (a_{11}^{\mathsf{d}})^2 b_{12} \\ 0 & u a_{11}^{\mathsf{d}} b_{12} \end{bmatrix}_{\mathscr{Q}}. \tag{2.23}$$

In view of (2.2) and (2.18) it is simple to obtain $b^\pi a^{\mathsf{d}} = a_2^{\mathsf{d}} = a_{11}^{\mathsf{d}}$ and $b^\pi (a^{\mathsf{d}})^2 = (a_2^{\mathsf{d}})^2 = (a_{11}^{\mathsf{d}})^2$. We have

$$a + ba^\pi = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} p & -a_1 a_2^{\mathsf{d}} \\ 0 & a_2^\pi - p \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} b_1 & a_1 - b_1 a_1 a_2^{\mathsf{d}} \\ 0 & a_2 + b_2 a_2^\pi \end{bmatrix}_{\mathscr{P}}.$$

By Lemma 1.1, there exists a sequence $(w_n)_{n=0}^\infty$ in $\mathscr{A}$ such that

$$(a + ba^\pi)^n = \begin{bmatrix} b_1^n & w_n \\ 0 & (a_2 + b_2 a_2^\pi)^n \end{bmatrix}_{\mathscr{P}}$$

and thus, $b^\pi (a + ba^\pi)^n = (a_2 + b_2 a_2^\pi)^n$. But another appealing to Lemma 1.1 and some computations as in (2.21) lead to $(1 - p - q)(a_2 + b_2 a_2^\pi)^n = (a_{22} + b_{22})^n$. Observe that (2.4) yields $b^\pi a^\pi = a_2^\pi - p = 1 - q - p$. Thus $b^\pi a^\pi b^\pi (a + ba^\pi)^n = (a_{22} + b_{22})^n$. In view of (2.18) and $b_{11} = 0$ we get $b_{21} = b_2 q$. But, it is simple to prove $b^\pi a a^{\mathsf{d}} = a_2 a_2^{\mathsf{d}} = q$ and $b_2 = bb^\pi$. Hence $b_{21} = bb^\pi b^\pi a a^{\mathsf{d}} = bb^\pi a a^{\mathsf{d}}$. Moreover, $(a_{11}^{\mathsf{d}})^k = (a_2^{\mathsf{d}})^k = b^\pi (a^{\mathsf{d}})^k$ holds for any $k \in \mathbb{N}$ in view of Lemma 1.1. If we take into account that $a^{\mathsf{d}} b^\pi = a^{\mathsf{d}}$ holds, then (2.22) becomes

$$u = b^\pi a^\pi b^\pi \sum_{n=0}^\infty (a + ba^\pi)^n bb^\pi a (a^{\mathsf{d}})^{n+3} = b^\pi a^\pi b^\pi \sum_{n=0}^\infty (a + ba^\pi)^n bb^\pi (a^{\mathsf{d}})^{n+2}. \quad (2.24)$$

From $b_{11} = 0$, (2.18), and $a^{\mathsf{d}} b^\pi = a^{\mathsf{d}}$ we have $b_{12} = qb_2 = b^\pi a a^{\mathsf{d}} b^\pi b = b^\pi a a^{\mathsf{d}} b$. This observation allows us to simplify the entries of $(y^{\mathsf{d}})^2 x$ given in (2.23):

$$(a_{11}^{\mathsf{d}})^2 b_{12} = \left[b^\pi (a^{\mathsf{d}})^2\right] \left[b^\pi a a^{\mathsf{d}} b\right] = b^\pi (a^{\mathsf{d}})^2 b$$

and

$$u a_{11}^{\mathsf{d}} b_{12} = \left[b^\pi a^\pi b^\pi \sum_{n=0}^\infty (a + ba^\pi)^n bb^\pi (a^{\mathsf{d}})^{n+2} \right] \left[b^\pi a^{\mathsf{d}}\right] \left[b^\pi a a^{\mathsf{d}} b\right] = u a^{\mathsf{d}} b.$$

Therefore,

$$(a_2 + b_2)^{\mathsf{d}} = y^{\mathsf{d}} + (y^{\mathsf{d}})^2 x = a_{11}^{\mathsf{d}} + u + (a_{11}^{\mathsf{d}})^2 b_{12} + u a_{11}^{\mathsf{d}} b_{12} = b^\pi a^{\mathsf{d}} + u + b^\pi (a^{\mathsf{d}})^2 b + u a^{\mathsf{d}} b. \quad (2.25)$$

By Lemma 1.5,

$$(a + b)^{\mathsf{d}} = \begin{bmatrix} b_1^{\mathsf{d}} & v \\ 0 & (a_2 + b_2)^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}}, \quad (2.26)$$

where

$$v = \sum_{n=0}^\infty (b_1^{\mathsf{d}})^{n+2} a_1 (a_2 + b_2)^n a_2^\pi + \sum_{n=0}^\infty b_1^\pi b_1^n a_1 [(a_2 + b_2)^{\mathsf{d}}]^{n+2} - b_1^{\mathsf{d}} a_1 (a_2 + b_2)^{\mathsf{d}}.$$

Since $b_1^{\mathsf{d}} = b^{\mathsf{d}}$, $a_1 = bb^{\mathsf{d}} a$, $(a_2 + b_2)^n = b^\pi (a + b)^n$, $a_2, b_2 \in (1 - p)\mathscr{A}(1 - p)$, $a_2^\pi = p + b^\pi a^\pi$, $ab^\pi = a$, $a^{\mathsf{d}} b^\pi = a^{\mathsf{d}}$ and $ub^\pi = u$ (this last equality is obtained from (2.24)) we have

$$\begin{aligned} (a_2 + b_2)^\pi &= 1 - (a_2 + b_2)^{\mathsf{d}} (a_2 + b_2) \\ &= 1 - (b^\pi a^{\mathsf{d}} + u + b^\pi (a^{\mathsf{d}})^2 b + u a^{\mathsf{d}} b) b^\pi (a + b) \\ &= 1 - (b^\pi a^{\mathsf{d}} + u + b^\pi (a^{\mathsf{d}})^2 b + u a^{\mathsf{d}} b)(a + b) \end{aligned}$$

and

$$\begin{aligned} (b_1^{\mathsf{d}})^{n+2} a_1 (a_2 + b_2)^n (a_2 + b_2)^\pi &= (b^{\mathsf{d}})^{n+2} bb^{\mathsf{d}} ab^\pi (a + b)^n (a_2 + b_2)^\pi \\ &= (b^{\mathsf{d}})^{n+2} a (a + b)^n (a_2 + b_2)^\pi. \end{aligned}$$

As is easy to see, $b_1^\pi b_1^n = 0$ for any $n \geq 1$. Moreover, $b_1^\pi a_1 = b^\pi(bb^{\mathsf{d}}a) = b^\pi(1-b^\pi)a = 0$, and $b_1^{\mathsf{d}} a_1 a_2^{\mathsf{d}} = (b^{\mathsf{d}})(bb^{\mathsf{d}}a)(b^\pi a^{\mathsf{d}}) = b^{\mathsf{d}}aa^{\mathsf{d}}$. Thus, v reduces to

$$v = \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n[1 - s(a+b)] - b^{\mathsf{d}}aa^{\mathsf{d}}, \tag{2.27}$$

where $s = b^\pi a^{\mathsf{d}} + u + b^\pi(a^{\mathsf{d}})^2 b + ua^{\mathsf{d}}b$. Expressions (2.25)–(2.27) allow finish the proof. □

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