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SUFFICIENT CONDITIONS FOR CERTAIN SUBCLASSES OF MEROMORPHIC MULTIVALENT FUNCTIONS

(COMMUNICATED BY R.K RAINA)

J. K. PRAJAPAT, AMBUJ KUMAR MISHRA

Abstract. In the present paper we derive various useful properties and characteristics for certain class of meromorphic multivalent function involving a linear operator in the punctured unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$, and derive various useful properties and characteristics of this function class. Several results are presented exhibiting relevant connections to some of the results presented here and those obtained in earlier works.

1. Introduction

For any integer $m > -p$, let $\Sigma_{p,m}$ denote the class of all meromorphic functions $f(z)$ normalized by

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \qquad (p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the punctured unit disk $\mathbb{U}^*$. Also let $\mathbb{U} = \mathbb{U}^* \cup \{0\}$. We denote by $\Sigma^*_{p,m}(\alpha)$, $\Sigma_{p,m,k}(\alpha)$ and $\Sigma_{p,m,c}(\alpha)$, the three subclasses of the class $f \in \Sigma_{p,m}$, which are defined as follows:

$$\Sigma^*_{p,m}(\alpha) = \left\{ f : \ f \in \Sigma_{p,m} \ \text{ and } \ \Re\left(-\frac{zf'(z)}{f(z)} \right) > \alpha \ \ (z \in \mathbb{U}; 0 \le \alpha < p) \right\} \tag{1.2}$$

$$\Sigma_{p,m,k}(\alpha) = \left\{ f : \ f \in \Sigma_{p,m} \ \text{ and } \ \Re\left(-\left(1 + \frac{zf'(z)}{f(z)}\right) \right) > \alpha \ \ (z \in \mathbb{U}; 0 \le \alpha < p) \right\} \tag{1.3}$$

$$\Sigma_{p,m,c}(\alpha) = \left\{ f : \ f \in \Sigma_{p,m} \ \text{ and } \ \Re\left(-z^{p+1} f'(z) \right) > \alpha \ \ (z \in \mathbb{U}; 0 \le \alpha < p) \right\} \tag{1.4}$$

where, for $0 \le \alpha < p$, the classes $\Sigma^*_{p,m}(\alpha)$, $\Sigma_{p,m,k}(\alpha)$ and $\Sigma_{p,m,c}(\alpha)$ denotes, respectively the subclass of meromorphic p-valently starlike functions of order α in $\mathbb{U}$, meromorphic p-valently convex functions of order α in $\mathbb{U}$ and meromorphic p-valently close-to-convex functions of order α in $\mathbb{U}$. In particular, we have

$$\Sigma^*_{1,1}(\alpha) = \Sigma^*(\alpha), \quad \Sigma_{1,1,k}(\alpha) = \Sigma_k(\alpha) \qquad \text{and} \qquad \Sigma_{1,1,c}(\alpha) = \Sigma_c(\alpha),$$

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where $\Sigma^*(\alpha)$, $\Sigma_k(\alpha)$ and $\Sigma_c(\alpha)$ are, respectively, subclass of meromorphic starlike function of order $\alpha(0 \leq \alpha < 1)$ in $\mathbb{U}$, subclass of meromorphic convex function of order $\alpha(0 \leq \alpha < 1)$ in $\mathbb{U}$ and subclass of meromorphic close-to-convex function of order $\alpha(0 \leq \alpha < 1)$ in $\mathbb{U}$.

If

$$h_i(z) = \frac{a_i}{z^p} + \sum_{k=m}^{\infty} a_{k,i}\, z^k \qquad (i = 1, 2;\ a_i \in \mathbb{R} - \{0\}; m > -p; p \in \mathbb{N}),$$

are analytic in $\mathbb{U}^*$, the their Hadamard product(or convolution) is defined by

$$(h_1 * h_2)(z) = \frac{a_1\, a_2}{z^p} + \sum_{k=m}^{\infty} a_{k,1}\, a_{k,2}\, z^k, \qquad z \in \mathbb{U}^*, \tag{1.5}$$

We now introduce a subclass $\Sigma_{p,m}(\phi, \psi; \mu, \alpha)$ of the function class $\Sigma_{p,m}$ which is defined as follows:

Definition 1.1. suppose the function $\phi(z)$ and $\psi(z)$ are given by

$$\phi(z) = \frac{c_1}{z^p} + \sum_{k=m}^{\infty} \lambda_k\, z^k \qquad (c_1 \in \mathbb{R} - \{0\};\ m > -p;\ p \in \mathbb{N}), \tag{1.6}$$

and

$$\psi(z) = \frac{c_2}{z^p} + \sum_{k=m}^{\infty} \mu_k\, z^k \qquad (c_2 \in \mathbb{R} - \{0\};\ m > -p;\ p \in \mathbb{N}), \tag{1.7}$$

then we say that $f(z) \in \Sigma_{p,m}$, is in the class $\Sigma_{p,m}(\phi, \psi; \mu, \alpha)$ if it satisfies the inequality

$$\Re\left(-\frac{z^{(1-\mu)p}\ (f * \phi)(z)}{((f * \psi)(z))^{\mu}}\right) > \alpha \quad (z \in \mathbb{U}; p \in \mathbb{N}; \mu \geq 0; 0 \leq \alpha < p). \tag{1.8}$$

provided that $(f * \psi)(z) \neq 0$; $< \lambda_k >_{k=1}^{\infty}$ and $< \mu_k >_{k=1}^{\infty}$ are increasing sequence such that $\lambda_k \geq \mu_k \geq 0$ (λ_k and μ_k are not both simultaneously equal to zero).

If the coefficients λ_k and μ_k and constants c_1 and c_2, in (1.8) and (1.9) are, chosen as follows:

$$\lambda_k = (p + n + 1)^n k \ \ (n \in \mathbb{N}), \quad \mu_k = \frac{\lambda_k}{k}, \quad c_1 = -p, \quad c_2 = 1, \tag{1.9}$$

then the function class $\Sigma_{p,m}(\phi, \psi; 1, \alpha)$ reduces to the class $\Sigma^n_{p,m}(\alpha)$, studied recently by Srivastava and Patel [4]. Furthermore we have the following relationships:

$$\Sigma_{1,1}\left(\frac{2z-1}{z(1-z)^2},\ \frac{z^2-z+1}{z(1-z)};\ 1, \alpha\right) = \Sigma^*(\alpha), \tag{1.10}$$

$$\Sigma_{1,1}\left(\frac{1-3z+4z^2}{z(1-z)^3},\ \frac{2z-1}{z(1-z)^2};\ 1, \alpha\right) = \Sigma_k(\alpha), \tag{1.11}$$

$$\Sigma_{1,1}\left(\frac{2z-1}{z(1-z)^2},\ \frac{1}{z};\ 1, \alpha\right) = \Sigma_c(\alpha), \tag{1.12}$$

Some other interesting developments involving convolution of meromorphic functions were considered in [3]. The object in the present paper is to obtain some sufficient conditions of functions belonging to the above defined subclass $\Sigma_{p,m}(\phi, \psi; \mu, \alpha)$

2. Results

In our present investigation of the function class $\Sigma_{p,m}(\phi,\psi;\mu,\alpha)$, we shall require the following Lemmas.

Lemma 2.1. *(see, [1]) Let the nonconstant function $w(z)$ be analytic in $\mathbb{U}$, with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathbb{U}$, then*

$$z_0w'(z_0)=kw(z_0),$$

where $k\geq 1$ is a real number.

Lemma 2.2. *(see, [2]) Let S be a set in the complex plane $\mathbb{C}$ and suppose that $\phi(z)$ is a mapping from $\mathbb{C}^2\times\mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z)\notin S$ for all $z\in\mathbb{U}$, and for all real x,y such that $y\leq-(1+x^2)/2$. If the function $q(z)=1+q_1z+q_2z^2+\cdots$ is analytic in $\mathbb{U}$ such that $\phi(q(z),\ zq'(z);\ z)\in S$ for all $z\in\mathbb{U}$, then $\Re\,(q(z))>0$.*

Making use of Lemma 2.1, we first prove

Theorem 2.3. *If $f(z)\in\Sigma_{p,m}$ satisfies the following inequality*

$$\left|(1-\mu)p+\frac{z(f*\phi)'(z)}{(f*\psi)(z)}-\mu\frac{z(f*\psi)'(z)}{(f*\psi)(z)}-\gamma\left(\frac{z^{(1-\mu)p}(f*\phi)(z)}{((f*\psi)(z))^{\mu}}+p\right)\right| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \qquad (2.1)$$

then $f(z)\in\Sigma_{p,m}(\phi,\psi;\mu,\alpha)$.

Proof. Define the function $w(z)$ by

$$\frac{z^{(1-\mu)p}(f*\phi)(z)}{((f*\psi)(z))^{\mu}}=-p+(\alpha-p)w(z), \qquad (2.2)$$

then $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. Differentiating logarithmically both sides of (2.2) with respect to z, we get

$$(1-\mu)p+\frac{z(f*\phi)'(z)}{(f*\psi)(z)}-\mu\frac{z(f*\psi)'(z)}{(f*\psi)(z)}=\frac{(p-\alpha)zw'(z)}{p+(p-\alpha)w(z)}. \qquad (2.3)$$

Now using (2.2) in (2.3), we find that

$$(1-\mu)p+\frac{z(f*\phi)'(z)}{(f*\psi)(z)}-\mu\frac{z(f*\psi)'(z)}{(f*\psi)(z)}-\gamma\left(\frac{z^{(1-\mu)p}\ (f*\phi)(z)}{((f*\psi)(z))^{\mu}}+p\right) =\gamma(p-\alpha)w(z)+\frac{(p-\alpha)zw'(z)}{p+(p-\alpha)w(z)}. \qquad (2.4)$$

Let us suppose that there exist $z_0\in\mathbb{U}$ such that

$$max_{|z|<|z_0|}\ |w(z)|=|w(z_0)|=1,$$

and apply Lemma 2.1, we find that

$$z_0w'(z_0)=kw(z_0) \qquad (k\geq 1) \qquad (2.5)$$

writing $w(z)=e^{i\theta}$ $(0\leq\theta<2\pi)$ and setting $z=z_0$ in (2.4), we get

$$\left|(1-\mu)p+\frac{z_0(f*\phi)'(z_0)}{(f*\psi)(z_0)}-\mu\frac{z_0(f*\psi)'(z_0)}{(f*\psi)(z_0)}-\gamma\left(\frac{z_0^{(1-\mu)p}(f*\phi)(z_0)}{(f*\psi)(z_0))^{\mu}}+p\right)\right|$$

$$\begin{aligned}
&= \left|\gamma(p-\alpha)e^{i\theta} + \frac{(p-\alpha)ke^{i\theta}}{p+(p-\alpha)e^{i\theta}}\right| \\
&\geq \Re\left(\gamma(p-\alpha) + \frac{(p-\alpha)k}{p+(p-\alpha)e^{i\theta}}\right) \\
&> \gamma(p-\alpha) + \frac{(p-\alpha)}{2p-\alpha} \\
&= \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha},
\end{aligned}$$

which contradicts our assumption (2.1). Therefore, we have $|w(z)| < 1$ in $\mathbb{U}$. Finally, we have

$$\left|\frac{z^{(1-\mu)p}\,(f*\phi)(z)}{((f*\psi)(z))^{\mu}} + p\right| = |(p-\alpha)w(z)| = (p-\alpha)|w(z)| < p-\alpha \qquad (z \in \mathbb{U}), \tag{2.6}$$

that is $f(z) \in \Sigma_{p,m}(\phi,\psi;\mu,\alpha)$. This proves the Theorem 2.3. □

Theorem 2.4. *If $f(z) \in \Sigma_{p,m}$ satisfies the following inequality*

$$\Re\left\{\left(\frac{z^{(1-\mu)p}(f*\phi)(z)}{((f*\psi)(z))^{\mu}}\right)^2 - \frac{z^{(1-\mu)p}(f*\phi)(z)}{((f*\psi)(z))^{\mu}}\left(\frac{z(f*\phi)'(z)}{(f*\phi)(z)} - \mu\frac{z(f*\psi)'(z))}{(f*\psi)(z)}\right)\right\} > \delta\left(\delta+\frac{1}{2}\right) + p\left(\delta(\mu-1)-\frac{1}{2}\right), \tag{2.7}$$

then $f(z) \in \Sigma_{p,m}(\phi,\psi;\mu,\delta)$.

Proof. Define the functions $q(z)$ by

$$\frac{z^{(1-\mu)p}(f*\phi)(z)}{((f*\psi)(z))^{\mu}} = -\delta + (\delta-p)q(z), \tag{2.8}$$

then we see that $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ is analytic in $\mathbb{U}$. Now differentiating both sides of (2.8) with respect to z logarithmically, we get

$$(\delta+(p-\delta)q(z))\left(\frac{z(f*\phi)'(z)}{(f*\phi)(z)} - \mu\frac{z(f*\psi)'(z))}{(f*\psi)(z)}\right) = (p-\delta)zq'(z) + p(\mu-1)(\delta+(p-\delta)q(z)). \tag{2.9}$$

Again using (2.8) in (2.9), we find that

$$\begin{aligned}
&\left(\frac{z^{(1-\mu)p}\,(f*\phi)(z)}{((f*\psi)(z))^{\mu}}\right)^2 - \frac{z^{(1-\mu)p}\,(f*\phi)(z)}{((f*\psi)(z))^{\mu}}\left(\frac{z(f*\phi)'(z)}{(f*\phi)(z)} - \mu\frac{z(f*\psi)'(z))}{(f*\psi)(z)}\right) \\
&= (p-\delta)zq'(z) + (p-\delta)^2q^2(z) + (p-\delta)(2\delta+p(\mu-1))q(z) + p\delta(\mu-1) + \delta^2 \\
&= \phi(q(z), zq'(z); z),
\end{aligned}$$

where

$$\phi(r,s,z) = (p-\delta)s + (p-\delta)^2r^2 + (p-\delta)(2\delta+p(\mu-1))r + p\delta(\mu-1) + \delta^2. \tag{2.10}$$

For all real x, y satisfying $y \leq -(1+x^2)/2$, we have

$$\begin{aligned}
\Re\ (\phi(ix,y,z)) &= (p-\delta)y + (p-\delta)^2x^2 + p\,\delta(\mu-1) + \delta^2 \\
&\leq -\frac{1}{2}(p-\delta)(1+x^2) - (p-\delta)^2x^2 + \delta p(\mu-1) + \delta^2 \\
&= -\frac{1}{2}(p-\delta) - (p-\delta)\left(\frac{1}{2} + p - \delta\right)x^2 + \delta p(\mu-1) + \delta^2 \\
&\leq \delta p(\mu-1) + \delta^2 - \frac{1}{2}(p-\delta) \\
&= \delta\left(\delta + \frac{1}{2}\right) + p\left(\delta(\mu-1) - \frac{1}{2}\right).
\end{aligned}$$

Let

$$S = \left\{w: \ \Re(w) > \delta\left(\delta + \frac{1}{2}\right) + p\left(\delta(\mu-1) - \frac{1}{2}\right)\right\},$$

then $\phi(q(z), zq'(z); z) \in S$ and $\phi(ix, y; z) \notin S$ for all real x and $y < -(1+x^2)/2,\ z \in \mathbb{U}$. By using Lemma 2.2, we have $\Re\ (q(z)) > 0$, that is $f(z) \in \Sigma_{p,m}(\phi, \psi; \mu, \delta)$. This proves the Theorem 2.4. □

3. Some Consequences Of Main Results

In this concluding section, we consider some Corollaries and Consequences of our main results (Theorem 2.3 and Theorem 2.4) established in section 2.

By setting $m = \mu = 1,\ \gamma = 0$ and parametric substitution as (1.10) in Theorem 2.3, we get

Corollary 3.1. *If $f(z) \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right| < \frac{(p-\alpha)}{2p-\alpha},$$

then $f(z) \in \Sigma^*(\alpha)$.

Corollary 3.1, further on setting $p = 1$ and $\alpha = 1$ gives

Corollary 3.2. *If $f(z) \in \Sigma$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} + 1\right| < \frac{1}{2},$$

then $f(z) \in \Sigma^*$.

Taking $p = m = \mu = 1,\ \gamma = 0$ and parametric substitution as (1.11) in Theorem 2.3, we get

Corollary 3.3. *If $f(z) \in \Sigma$ satisfies*

$$\left|\frac{z^3f'''(z) + 3z^2f''(z) + zf'(z)}{z^2f''(z) + zf'(z)} - \frac{z^2f'(z) + 2}{zf(z) - 2}\right| < \frac{(1-\alpha)}{2-\alpha},$$

then $f(z) \in \Sigma_k(\alpha)$.

Setting $p = m = \mu = 1,\ \gamma = 0$ and parametric substitution as (1.12) in Theorem 2.3, we get

Corollary 3.4. *If* $f(z) \in \Sigma$ *satisfies*

$$\left|\frac{2}{z^2 f'(z)} + 2\right| < \frac{(1-\alpha)}{2-\alpha},$$

then $f(z) \in \Sigma_c(\alpha)$.

Taking $\mu = 1$ and parametric substitution as (1.11) in Theorem 2.4, we have

Corollary 3.5. *If* $f(z) \in \Sigma$ *satisfies*

$$\Re\left\{2\left(\frac{zf'(z)}{f(z)}\right)^2 - \frac{z^2 f''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right\} > \delta\left(\delta + \frac{1}{2}\right) - \frac{p}{2},$$

then $f(z) \in \Sigma^*(\delta)$.

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J. K. Prajapat
Department of Mathematics, Central University of Rajasthan, Kishangarh, Distt.-Ajmer, Rajasthan, India.
E-mail address: jkp_0007@rediffmail.com

Ambuj Kumar Mishra
Department of Mathematics, Institute of Engineering and Technology,, G. L. A. University, Mathura, U. P., India.
E-mail address: ambuj_math@rediffmail.com