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BOEHMIANS, ULTRADIFFERENTIAL OPERATORS AND ABELIAN TYPE THEOREMS

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Abstract. Spaces of generalized functions known as Boehmians contain all Schwartz distributions as well as all ultradistributions of Beurling and Roumieu type. Even though these spaces are quite large, it is still possible to investigate local behavior. In this note, ultradifferential operators are introduced in the context of Boehmians. As an application, some Abelian type theorems are established for the unilateral Laplace transform.

1. Introduction

The space of generalized functions known as Boehmians has an algebraic construction which utilizes convolution and approximate identities (see [12]). The space of Boehmians is quite large. It contains a proper subspace which can be identified with the space of distributions. There are Boehmians which are not hyperfunctions and hyperfunctions which are not Boehmians.

A differential operator (possibly of infinite order) having the form $P(D) = \sum_{n=0}^{\infty} c_n D^n$, where D is the differentiation operator and $\{c_n\}$ is a sequence of complex numbers satisfying a growth condition, is called an ultradifferential operator. This type of operator is important in the theory of ultradistributions of Beurling and Roumieu type (see [9], [18], [19]).

In this note, we introduce and investigate the notion of an ultradifferential operator defined on the space of Boehmians. As an application, by using the notion of an ultradiffential operator, we establish an initial value theorem as well as a final value theorem for the Laplace transform.

2. Preliminaries

Let $C(\mathbb{R})$ denote the space of all continuous functions on the real line $\mathbb{R}$, and let $\mathcal{D}(\mathbb{R})$ denote the subspace of $C(\mathbb{R})$ of all infinitely differentiable functions with compact supports.

A sequence $\varphi_n \in \mathcal{D}(\mathbb{R})$ is called a *delta sequence* provided:

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(i) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,
(ii) $\int |\varphi_n| \leq M$ for some constant $M > 0$ and all $n \in \mathbb{N}$,
(iii) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\varphi_n(x) = 0$ for $|x| > \varepsilon$ and $n > n_\varepsilon$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences* if $f_n \in C(\mathbb{R})$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_k * \varphi_m = f_m * \varphi_k$ for all $k, m \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * \varphi)(x) = \int_{-\infty}^{\infty} f(x-t)\varphi(t)dt. \tag{2.1}$$

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_k * \psi_m = g_m * \varphi_k$ for all $k, m \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called Boehmians. The space of all Boehmians will be denoted by $\beta(\mathbb{R})$ and a typical element of $\beta(\mathbb{R})$ will be written as $W = \left[\frac{f_n}{\varphi_n}\right]$.

The operations of addition, scalar multiplication, and differentiation are defined as follows: $\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n*\psi_n + g_n*\varphi_n}{\varphi_n*\psi_n}\right], \gamma\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\gamma f_n}{\varphi_n}\right]$, where $\gamma \in \mathbb{C}$, $D^m\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n*\varphi_n^{(m)}}{\varphi_n*\varphi_n}\right]$.

Define the map $\iota : C(\mathbb{R}) \to \beta(\mathbb{R})$ by

$$\iota(f) = \left[\frac{f * \varphi_n}{\varphi_n}\right], \tag{2.2}$$

where $\{\varphi_n\}$ is any fixed delta sequence.

It is not difficult to show that the mapping ι is an injection which preserves the algebraic properties of $C(\mathbb{R})$. Thus, $C(\mathbb{R})$ can be identified with a proper subspace of $\beta(\mathbb{R})$. Likewise, the space of Schwartz distributions $\mathcal{D}'(\mathbb{R})$ [25] can be identified with a proper subspace of $\beta(\mathbb{R})$. Using this identification, the Dirac measure δ corresponds to the Boehmian $\left[\frac{\varphi_n}{\varphi_n}\right]$, where $\{\varphi_n\}$ is any delta sequence.

A Boehmian W is said to vanish on an open set $\Omega \subset \mathbb{R}$, denoted $W(x) = 0$ on Ω, provided that there exists a delta sequence $\{\varphi_n\}$ such that $W * \varphi_n \in C(\mathbb{R})$, $n \in \mathbb{N}$, and $W * \varphi_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$. The support of a Boehmian W is the complement of the largest open set on which W vanishes. $\beta_c(\mathbb{R})$ denotes the space of all Boehmians with compact supports. A Boehmian $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta_c(\mathbb{R})$ provided there exists a compact set $K \subset \mathbb{R}$ such that $\text{supp} f_n \subset K$, for all $n \in \mathbb{N}$.

Convolution can be extended to $\beta_c(\mathbb{R}) \times \beta(\mathbb{R})$. For $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta_c(\mathbb{R})$ and $V = \left[\frac{g_n}{\psi_n}\right] \in \beta(\mathbb{R}), W * V = \left[\frac{f_n*g_n}{\varphi_n*\psi_n}\right]$. Let $W, V \in \beta_c(\mathbb{R})$ and $U \in \beta(\mathbb{R})$. Then,

$$W * (V * U) = (W * V) * U. \tag{2.3}$$

For $f \in C(\mathbb{R})$, let $\tau_a f(x) = f(x-a)$, $a \in \mathbb{R}$. The translation operator τ_a can be extended to the space $\beta(\mathbb{R})$. For $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, define $\tau_a W = \left[\frac{\tau_a f_n}{\varphi_n}\right], a \in \mathbb{R}$. It is routine to show that $\left[\frac{\tau_a f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$.

For $\psi \in \mathcal{D}(\mathbb{R})$ and $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, $\psi * W$ is defined as

$$\psi * \left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\psi * f_n}{\varphi_n}\right] \tag{2.4}$$

3. Ultradifferential Operators

Ultradifferential operators have been found to be important tools in the theory and applications of ultradistributions of Beurling and Roumieu type (see [9], [18], [19]). By using the mapping (2.2) with a delta sequence consisting of ultradifferentiable functions, spaces of ultradistributions can be viewed as proper subspaces of $\beta(\mathbb{R})$. For any sequence $\{\alpha_n\}$, the series $\sum_{-\infty}^{\infty} \alpha_n e^{i(2n)!x}$ can be shown to converge in $\beta(\mathbb{R})$. Thus, by picking the sequence $\{\alpha_n\}$ appropriately, $\sum_{-\infty}^{\infty} \alpha_n e^{i(2n)!x}$ is an example of a Boehmian which is not an ultradistribution.

In this section, we introduce ultradifferential operators and establish some useful properties in the context of Boehmians.

Let $\{M_n\}$ be a sequence of positive numbers which satisfies the following conditions.

(i) $M_0 = 1$,
(ii) $M_n^2 \leq M_{n-1}M_{n+1}$, for all $n \in \mathbb{N}$,
(iii) $\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} < \infty$.

An operator of the form

$$P(D) = \sum_{n=0}^{\infty} c_n D^n, \tag{3.1}$$

where $c_n \in \mathbb{C}$, is called an *ultradifferential operator* provided there exist constants $A > 0, B > 0$ such that $|c_n| \leq \frac{AB^n}{M_n}, n \in \mathbb{N}$.

Let $P(D) = \sum_{n=0}^{\infty} c_n D^n$ be an ultradifferential operator. Then, $P(z) = \sum_{n=0}^{\infty} c_n z^n$ ($z \in \mathbb{C}$) is an entire function of exponential type with Hadamard's factorization

$$P(z) = az^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

where $a, z_n \in \mathbb{C}$ ($n \in \mathbb{N}$), and m is a nonnegative integer (see [9]).

Throughout the sequel, $P(D) = \sum_{n=0}^{\infty} c_n D^n$ (or just $P(D)$) will denote an ultradifferential operator.

Definition 3.1. *Let $W_n, W \in \beta(\mathbb{R})$ for $n = 1, 2, \ldots$. We say that the sequence $\{W_n\}$ is δ-convergent to W if there exists a delta sequence $\{\varphi_n\}$ such that for each n and k, $W_n * \varphi_k, W * \varphi_k \in C(\mathbb{R})$, and for each k, $W_n * \varphi_k \to W * \varphi_k$ uniformly on compact sets as $n \to \infty$. This will be denoted by $\delta\text{-}\lim_{n\to\infty} W_n = W$.*

Theorem 3.1. *The series $P(D) = \sum_{n=0}^{\infty} c_n D^n W$ converges for all $W \in \beta(\mathbb{R})$. (That is, there exists $V \in \beta(\mathbb{R})$ such that $\delta\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} c_k D^k W = V$.)*

Proof. Since the series $\sum_{n=0}^{\infty} c_n D^n \delta$ is δ-convergent [16], the proof follows by observing that the sequence $\sum_{k=0}^{n} c_k D^k \delta$ is δ-convergent, supp $\sum_{k=0}^{n} c_k D^k \delta = \{0\}$, for all $n \in \mathbb{N}$, and that $\sum_{k=0}^{n} c_k D^k W = (\sum_{k=0}^{n} c_k D^k \delta) * W$, for all $n \in \mathbb{N}$. $\square$

Define $P(D) : \beta(\mathbb{R}) \to \beta(\mathbb{R})$ by $P(D)W = \sum_{n=0}^{\infty} c_n D^n W$.

The next corollary follows from the proof of the previous theorem.

Corollary 3.1. *$P(D)W = P(D)\delta * W$, for all $W \in \beta(\mathbb{R})$.*

Lemma 3.1. *Let $V \in \beta_c(\mathbb{R})$ such that supp $V = \{0\}$. If $W \in \beta(\mathbb{R})$ such that $W(x) = 0$ on (a,b), then $(V * W)(x) = 0$ on (a,b).*

Proof. Let $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$ and $V = \left[\frac{g_n}{\psi_n}\right] \in \beta_c(\mathbb{R})$ such that supp $g_n \to \{0\}$ as $n \to \infty$. Let $[c,d] \subset (a,b)$. Choose $\varepsilon > 0$ such that $[c-\varepsilon, d+\varepsilon] \subset (a,b)$. Now, $W(x) = 0$ on (a,b) gives $f_n \to 0$ uniformly on $[c-\varepsilon, d+\varepsilon]$ as $n \to \infty$.

Pick $n_0 > 0$ such that for all $n \geq n_0$, supp $(g_n * \varphi_n) \subset (-\varepsilon, \varepsilon)$. Now, let n be fixed such that $n \geq n_0$.

Then, for all $k \in \mathbb{N}$,

$$f_n * g_n = ((f_n * g_n) - (f_n * g_n) * \varphi_k) + (f_n * g_n) * \varphi_k \tag{3.2}$$

Now,

$$(f_n * g_n) * \varphi_k \to f_n * g_n \text{ uniformly on } [c-\varepsilon, d+\varepsilon] \text{ as } k \to \infty, \tag{3.3}$$

and, for all $x \in [c,d]$,

$$\begin{aligned} |((f_n * g_n) * \varphi_k)(x)| &= |(f_k * (g_n * \varphi_n))(x)| \\ &\leq \int_{-\varepsilon}^{\varepsilon} |f_k(x-t)||(g_n * \varphi_n)(t)|dt \\ &\leq M \sup_{y \in [c-\varepsilon, d+\varepsilon]} |f_k(y)| \int_{-\infty}^{\infty} |g_n(t)|dt, \end{aligned} \tag{3.4}$$

for some constant $M > 0$ independent of k and n.

So, by (3.2), (3.3), and (3.4), we see that $(f_n * g_n)(x) = 0$, for all $x \in [c,d]$. Thus, $(V * W)(x) = 0$ on (a,b). □

The next theorem follows directly from Corollary 3.1 and the previous lemma.

Theorem 3.2. *Let $W \in \beta(\mathbb{R})$. If $W(x) = 0$ on (a,b), then $P(D)W(x) = 0$ on (a,b).*

Theorem 3.3. *Let $W \in \beta_c(\mathbb{R})$ and $V \in \beta(\mathbb{R})$. Then,*

$$P(D)(W * V) = P(D)W * V = W * P(D)V.$$

Proof. Since $W, P(D)\delta \in \beta_c(\mathbb{R})$, we obtain

$$P(D)\delta * (W * V) = (P(D)\delta * W) * V = W * (P(D)\delta * V).$$

Thus, by Corollary 3.1, we see that $P(D)(W * V) = P(D)W * V = W * P(D)V$. □

A mapping $\Lambda : \beta(\mathbb{R}) \to \beta(\mathbb{R})$ is continuous provided $\delta\text{-}\lim_{n\to\infty} \Lambda(W_n) = \Lambda(W)$ whenever $\delta\text{-}\lim_{n\to\infty} W_n = W$.

Theorem 3.4. *$P(D)$ is a continuous linear operator on $\beta(\mathbb{R})$. Moreover, $P(D)$ is injective if and only if $P(D)$ is a nonzero multiple of the identity.*

Proof. Clearly, $P(D)$ is linear. Let $V \in \beta_c(\mathbb{R})$. Observe that the mapping $\beta(\mathbb{R}) \to \beta(\mathbb{R})$ given by $W \to W * V$ is continuous. Applying the above to Corollary 3.1, the continuity follows.

Clearly if $P(D) = c_0 I$ $(c_0 \neq 0)$, then $P(D)$ is injective.
Now suppose $P(D)$ is not a nonzero multiple of the identity. Then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. Then,

$$\begin{aligned} P(D)e^{z_0 x} &= \delta\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} c_k D^k (e^{z_0 x}) \\ &= \delta\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} c_k z_0^k e^{z_0 x} \\ &= P(z_0)e^{z_0 x} = 0. \end{aligned}$$

Thus, since $P(D)$ is linear, $P(D)$ is not injective. □

Remark 3.1. *With the obvious modifications, the results in this section are also valid for ultradifferential operators defined on* $\beta(\mathbb{R}^n)$.

4. Transformable Boehmians

Boehmian spaces have been found favorable by several authors for extending classical integral transforms. For example, see [1, 2, 3, 4, 6, 7, 8, 10, 11, 13, 14, 15, 17, 21, 22, 23, 24].

The space $\beta_{\mathcal{L}}(\mathbb{R})$ of Laplace transformable Boehmians was introduced in [14]. It was shown, among other things, that the space of Laplace transformable distributions of Zemanian [25] supported on $[0, \infty)$ can be identified with a proper subspace of $\beta_{\mathcal{L}}(\mathbb{R})$. Subsequently, some Abelian theorems were established [15].

Abelian theorems for the Laplace transform can be found in many areas of mathematics; control theory, probability theory, and signal analysis to name a few.

In this section, as an application of ultradifferential operators on Boehmians, we present an initial value theorem and a final value theorem for the Laplace transform. The initial value theorem, which extends Theorem 3.5 in [15], relates the behavior of a transformable Boehmian at zero to the behavior of its transform at infinity. While the final value theorem, which extends Theorem 2.7 in [14], relates the behavior of the transformable Boehmian at infinity to the behavior of its transform at a singularity.

The space of all functions $f \in C(\mathbb{R})$ such that $f(t) = 0$ for $t < 0$ will be denoted by $C_+(\mathbb{R})$.

The space of transformable Boehmians $\beta_{\mathcal{L}}(\mathbb{R})$ is defined as

$$\{ \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R}) : \varphi_n \geq 0, f_n, \varphi_n \in C_+(\mathbb{R}), f_n(t) = O(e^{\gamma t}) \text{ as } t \to \infty \, (\exists \gamma \in \mathbb{R}), n \in \mathbb{N}\}.$$

Let $f \in C_+(\mathbb{R})$ such that $f(t) = O(e^{\gamma t})$ as $t \to \infty$ (some $\gamma \in \mathbb{R}$). The Laplace transform of f, denoted $\mathcal{L}[f]$, is given by

$$\mathcal{L}[f](z) = \int_0^\infty e^{-zt} f(t)dt, \text{ for } \operatorname{Re} z > \gamma. \tag{4.1}$$

The Laplace transform for $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta_{\mathcal{L}}(\mathbb{R})$,where $f_n(t) = O(e^{\gamma t})$ as $t \to \infty$, is given by

$$\mathcal{L}[W](z) = \mathcal{W}(z) = \lim_{n\to\infty} \mathcal{L}[f_n](z), \text{ for Re } z > \gamma. \tag{4.2}$$

Remark 4.1.

(1) *The above limit exists and is independent of the representative.*
(2) *The convergence is uniform on compact subsets in the half-plane Re $z > \gamma$.*
(3) $\mathcal{W}$ *is an analytic function in the half-plane Re $z > \gamma$.*
(4) *If W has bounded support, then $\mathcal{W}$ is entire.*

Theorem 4.1. *Let $W \in B_{\mathcal{L}}(\mathbb{R})$. Then, $P(D)W \in B_{\mathcal{L}}(\mathbb{R})$.*

The proof of the previous theorem follows directly from Corollary 3.1 and observing that the Boehmian $P(D)\delta$ has bounded support and has a representation in which the delta sequence $\{\varphi_n\}$ is nonnegative and for each $n \in \mathbb{N}$, $\operatorname{supp}\varphi_n \subseteq \left[\frac{1}{2n}, \frac{1}{n}\right]$(see the proof of Theorem 4 in [16]).

Theorem 4.2. *Let $W \in B_{\mathcal{L}}(\mathbb{R})$ such that $\mathcal{W}(z)$ exists in the half-plane Re $z > \gamma$. Then, $\mathcal{L}[P(D)W](z) = P(z)\mathcal{W}(z)$, Re $z > \gamma$.*

Proof. Supp $\displaystyle\sum_{k=0}^{\infty} c_k D^k\delta = \{0\}$. Thus,

$$\begin{aligned}
\mathcal{L}[P(D)\delta](z) &= \mathcal{L}[\delta\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} c_k D^k\delta](z) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \mathcal{L}[c_k D^k\delta](z) \quad \text{(see [14])} \\
&= \sum_{k=0}^{\infty} c_k z^k = P(z), z \in \mathbb{C}.
\end{aligned}$$

So, $\mathcal{L}[P(D)W](z) = \mathcal{L}[P(D)\delta * W](z) = \mathcal{L}[(P(D)\delta)(z)]\mathcal{W}(z) = P(z)\mathcal{W}(z)$, Re$z > \gamma$. □

Lemma 4.1. *Let $W \in B_{\mathcal{L}}(\mathbb{R})$ and $f \in L^1(a,b)\,(a < 0, b > 0)$ such that supp $f \subseteq [0,\infty)$, $\frac{f(t)}{t^\lambda} \to \xi$ as $t \to 0^+\,(Re\,\lambda > -1)$, and $W(t) = P(D)f(t)$ on (a,b). Then there exists $f^\# \in L^1(a,b)$ such that supp $f^\# \subseteq [0,b]$, $\frac{f^\#(t)}{t^\lambda} \to \xi$ as $t \to 0^+$, and $W(t) = P(D)f^\#(t)$ on (a,b).*

Definition 4.1. *Let $\lambda, \xi, z_0 \in \mathbb{C}, W \in B_{\mathcal{L}}(\mathbb{R})$, and $g_{\lambda,z_0}(t) = t^\lambda e^{z_0 t}$. W is said to be equivalent at the origin (infinity) to $\xi P(D)g_{\lambda,z_0}$, denoted $W(t) \overset{e}{\sim} \xi P(D)g_{\lambda,z_0}(t)$ as $t \to 0^+\,(t \to \infty)$, provided there exist an interval (a,b) with $a < 0$ and $b > 0$ $(a > 0$ and $b = \infty)$ and a locally integrable function f such that $W(t) = P(D)f(t)$ on (a,b) and $\frac{f(t)}{g_{\lambda,z_0}(t)} \to \xi$ as $t \to 0^+\,(t \to \infty)$.*

Theorem 4.3. Initial Value Theorem. *Let $W \in \beta_{\mathcal{L}}(\mathbb{R})$ and $\lambda, \xi \in \mathbb{C}$ such that Re $\lambda > -1$. Suppose that $W(t) \overset{e}{\sim} \xi P(D)g_{\lambda,0}(t)$ as $t \to 0^+$. If $e^{\gamma t}|P(t)| \to \infty$ as $t \to \infty$ (for some $0 < \gamma < b$, where b is as in Definition 4.1), then $\frac{t^{\lambda+1}\mathcal{W}(t)}{\Gamma(\lambda+1)} \sim \xi P(t)$ as $t \to \infty$, where Γ is the well known gamma function. That is, $\displaystyle\lim_{t\to\infty} \frac{t^{\lambda+1}\mathcal{W}(t)}{\Gamma(\lambda+1)P(t)} = \xi$.*

Proof. $W = W_1 + W_2$, where $W_1 = W - P(D)f$ and $W_2 = P(D)f$ with supp $W_1 \subseteq [b, \infty)$. By Lemma 4.1, we may assume that supp $f \subseteq [0, b]$. Thus,

$$\begin{aligned} \frac{t^{\lambda+1}\mathcal{W}_2(t)}{\Gamma(\lambda+1)P(t)} &= \frac{t^{\lambda+1}\mathcal{L}[P(D)f](t)}{\Gamma(\lambda+1)P(t)} \\ &= \frac{t^{\lambda+1}\mathcal{L}[f](t)}{\Gamma(\lambda+1)} \to \xi \text{ as } t \to \infty. \end{aligned}$$

This follows by a classical Abelian theorem (see [5]).

Now, to complete the proof it suffices to show that

$$\frac{t^{\lambda+1}\mathcal{W}_1(t)}{\Gamma(\lambda+1)P(t)} \to 0 \text{ as } t \to \infty. \tag{4.3}$$

Let $V \in B_{\mathcal{L}}(\mathbb{R})$ such that $\tau_{b-\varepsilon}V = W_1$, where $\varepsilon = \frac{b-\gamma}{4}$.

Now, by Theorem 2.2 in [15], there exist $M > 0$ and $t_0 > 0$ such that $|\mathcal{V}(t)| \leq Me^{\varepsilon t}$ for all $t \geq t_0$.

Therefore for $t \geq t_0$,

$$\begin{aligned} \left|\frac{t^{\lambda+1}\mathcal{W}_1(t)}{\Gamma(\lambda+1)P(t)}\right| &= \left|\frac{t^{\lambda+1}\mathcal{L}[\tau_{b-\varepsilon}V](t)}{\Gamma(\lambda+1)P(t)}\right| \\ &\leq M\left|\frac{t^{\lambda+1}e^{-(b-2\varepsilon)t}}{\Gamma(\lambda+1)P(t)}\right| \\ &= M\left|\frac{t^{\lambda+1}e^{-(\gamma+2\varepsilon)t}}{\Gamma(\lambda+1)P(t)}\right| \\ &= M\left|\left(\frac{t^{\lambda+1}}{\Gamma(\lambda+1)e^{2\varepsilon t}}\right)\left(\frac{1}{e^{\gamma t}P(t)}\right)\right| \end{aligned}$$

Thus, $\frac{t^{\lambda+1}\mathcal{W}_1(t)}{\Gamma(\lambda+1)P(t)} \to 0$ as $t \to \infty$. □

Remark 4.2. *There exist many ultradifferential operators which satisfy* $e^{\gamma t}|P(t)| \to \infty$ *as* $t \to \infty$. *For example, let* $\alpha > 0$, *then* $P_\alpha(D) = \prod_{n=1}^{\infty}\left(1 + \frac{\alpha D}{m_n}\right)$, *where* $m_n = \frac{M_n}{M_{n-1}}$ $(n \in \mathbb{N})$, *is an ultradifferential operator (see* [9], *p.60). Moreover, for* $\gamma > 0$, $e^{\gamma t}|P_\alpha(t)| \to \infty$ *as* $t \to \infty$, *for all* $\alpha > 0$.

Example 4.1.

(1) *Let* $W = \sum_{k=0}^{\infty}\left[\frac{\tau_k\varphi_n}{\varphi_n}\right]$. *It is routine to show that* $W \in B_{\mathcal{L}}(\mathbb{R})$. *Now,* $W(t) = DH(t)$ *on* $(-1, 1)$, *where* H *is the Heaviside function. This follows from*

$$W - \delta = \sum_{k=1}^{\infty}\left[\frac{\tau_k\varphi_n}{\varphi_n}\right], \text{ and } \sum_{k=1}^{\infty}\tau_k\varphi_n \to 0$$

uniformly on compact subsets of $(-1, 1)$ *as* $n \to \infty$. *Thus,* $W(t) \overset{e}{\sim} P(D)g_{0,0}(t)$ *as* $t \to 0^+$, *where* $P(D) = D$. *Since, for any* $\gamma > 0$, $e^{\gamma t}|P(t)| = e^{\gamma t}t \to \infty$ *as* $t \to \infty$, *Theorem 4.3 yields*

$$\lim_{t\to\infty}\mathcal{W}(t) = 1.$$

(2) *Let* $f(t) = \frac{e^{ct}-e^{dt}}{\sqrt{t}} H(t) \quad (c, d \in \mathbb{R})$ *and* $P(D) = \sum_{n=0}^{\infty} \frac{D^n}{(2n)!}$. *Suppose* $W \in B_{\mathcal{L}}(\mathbb{R})$ *such that* $W(t) = P(D)f(t)$ *on* (a, b) *for some* $a < 0$ *and* $b > 0$. *Then,* $W(t) \overset{e}{\sim} (c-d)P(D)g_{\frac{1}{2},0}(t)$ *as* $t \to 0^+$.
Notice that for any $\gamma > 0$,

$$e^{\gamma t}|P(t)| = e^{\gamma t}|\cosh \sqrt{t}| \to \infty \text{ as } t \to \infty.$$

Thus, by Theorem 4.3,

$$\lim_{t\to\infty} \frac{t^{\frac{3}{2}}\mathcal{W}(t)}{\cosh \sqrt{t}} = \frac{\sqrt{\pi}}{2}(c-d).$$

Notice any $W \in \beta_{\mathcal{L}}(\mathbb{R})$ *having the form* $W = V + P(D)f$, *where* $V \in \beta_c(\mathbb{R})$ *with* $supp\, V \subseteq [a, \infty)$ $(a > 0)$, *will work (for example* $V = \sum_{n=0}^{\infty} \frac{D^n \tau_\pi \delta}{n^{2n}}$*).*

(3) *Some condition like* $e^{\gamma t}|P(t)| \to \infty$ *as* $t \to \infty$ *in Theorem 4.3 is needed.* *Let* $f \in L^1(-1, 1)$ *such that* $supp\, f \subseteq [0, \infty)$ *and* $\frac{f(t)}{t^\lambda} \to \xi$ *as* $t \to 0^+$ $(\lambda > -1)$ *and* $P(D) = \sum_{n=0}^{\infty} \frac{(-1)^n D^n}{(2n)!}$. *Notice that for any* $\gamma > 0$, $e^{\gamma t}|P(t)| = e^{\gamma t} \cos \sqrt{t}$ *does not converge to infinity as* $t \to \infty$. *Let* $W = P(D)f + \delta(t-1)$. *Since* $W(t) = P(D)f(t)$ *on* $(-1, 1)$, $W(t) \overset{e}{\sim} P(D)g_{\lambda,0}(t)$ *as* $t \to 0^+$. *Now,*

$$\begin{aligned} \frac{t^{\lambda+1}\mathcal{W}(t)}{\Gamma(\lambda+1)P(t)} &= \left(\frac{t^{\lambda+1}}{\Gamma(\lambda+1)P(t)}\right)\mathcal{L}[P(D)f + \delta(t-1)](t) \\ &= \frac{t^{\lambda+1}\mathcal{L}[f](t)}{\Gamma(\lambda+1)} + \frac{t^{\lambda+1}}{\Gamma(\lambda+1)e^t \cos \sqrt{t}}. \end{aligned}$$

By a classical Abelian theorem, the first term converges to ξ *as* $t \to \infty$. *However, the second term does not converge as* $t \to \infty$. *Therefore,* $\frac{t^{\lambda+1}\mathcal{W}(t)}{\Gamma(\lambda+1)P(t)}$ *does not converge to* ξ *as* $t \to \infty$.

Theorem 4.4. Final Value Theorem. *Let* $W \in B_{\mathcal{L}}(\mathbb{R})$ *and* $\lambda, z_0, \xi \in \mathbb{C}$ *such that* $Re\ \lambda > -1$. *If* $W(t) \overset{e}{\sim} \xi P(D)g_{\lambda,z_0}(t)$ *as* $t \to \infty$, *then* $\mathcal{W}(z)$ *exists for* $Re\ z > Re\ z_0$. *Moreover,* $\mathcal{W}(z)$ *has the following asymptotic behavior .*

(I) *When* $P(z_0) \neq 0$,
$\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)} \sim \xi P(z)$ *as* $z \to z_0$ *in* $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$.
That is,
$$\lim_{\substack{z\to z_0 \\ |arg(z-z_0)|\leq\psi<\frac{\pi}{2}}} \frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)P(z)} = \xi.$$

(II) *If* $P^{(k)}(z_0) = 0$, *for* $0 \leq k \leq n-1$ *and* $P^{(n)}(z_0) \neq 0$, *for some* $n \in \mathbb{N}$, *then:*
  (i) *For* $n < Re\ \lambda + 1$,
  $\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)} \sim \xi P(z)$ *as* $z \to z_0$ *in* $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$.

  (ii) *For* $n \geq Re\ \lambda + 1$,
  $\frac{(z-z_0)^{\lambda+1}\left(\mathcal{W}(z) - \sum_{k=0}^{m}(\frac{\mathcal{U}^{(k)}(z_0)}{k!})(z-z_0)^k\right)}{\Gamma(\lambda+1)} \sim \xi P(z)$ *as* $z \to z_0$ *in* $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$ *(where* $U = W - P(D)f$, $m = n - [Re\,\lambda] - 1$, *and* $[\cdot]$ *is the greatest integer function).*

Proof. Since, $W = U + P(D)f$ (where U has compact support), $\mathcal{W}(z)$ exists for $\operatorname{Re} z > \operatorname{Re} z_0$. Now,

$$\begin{aligned}\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)P(z)} &= \frac{(z-z_0)^{\lambda+1}(\mathcal{U}(z)+P(z)\mathcal{L}[f](z))}{\Gamma(\lambda+1)P(z)} \\ &= \frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)P(z)} + \frac{(z-z_0)^{\lambda+1}\mathcal{L}[f](z)}{\Gamma(\lambda+1)}.\end{aligned}$$

By a classical Abelian theorem [5], the second term converges to ξ as $z \to z_0$ in $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$.

(I) Assume $P(z_0) \neq 0$.
Then, $\frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)P(z)} \to 0$ as $z \to z_0$ in $|\arg(z-z_0)| \leq \psi < \frac{\pi}{2}$. This proves (I).

(II) Assume $P^{(k)}(z_0) = 0$, $0 \leq k \leq n-1$ and $P^{(n)}(z_0) \neq 0$.

(i) Suppose $n < \operatorname{Re}\,\lambda + 1$. Then,

$$\begin{aligned}\frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)P(z)} &= \frac{(z-z_0)^{\lambda+1}\mathcal{U}(z)}{\Gamma(\lambda+1)(z-z_0)^n Q(z)} \\ &= \frac{(z-z_0)^{\lambda+1-n}\mathcal{U}(z)}{\Gamma(\lambda+1)Q(z)},\end{aligned}$$

where Q is an entire function and $Q(z_0) \neq 0$. This term converges to zero as $z \to z_0$ in $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$. Thus, the proof of part (i) is complete.

(ii) Suppose $n \geq \operatorname{Re}\,\lambda + 1$. Then,

$$\frac{(z-z_0)^{\lambda+1}\left(\mathcal{W}(z) - \sum_{k=0}^{m}\left(\frac{\mathcal{U}^{(k)}(z_0)}{k!}\right)(z-z_0)^k\right)}{\Gamma(\lambda+1)P(z)}$$
$$= \frac{(z-z_0)^{\lambda+1}\left(\mathcal{U}(z) - \sum_{k=0}^{m}\left(\frac{\mathcal{U}^{(k)}(z_0)}{k!}\right)(z-z_0)^k\right)}{\Gamma(\lambda+1)P(z)} + \frac{(z-z_0)^{\lambda+1}\mathcal{L}[f](z)}{\Gamma(\lambda+1)}.$$

As before, the second term converges to ξ as $z \to z_0$ in $|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$. And,

$$\frac{(z-z_0)^{\lambda+1}\left(\mathcal{U}(z) - \sum_{k=0}^{m}\left(\frac{\mathcal{U}^{(k)}(z_0)}{k!}\right)(z-z_0)^k\right)}{\Gamma(\lambda+1)P(z)}$$
$$= \frac{(z-z_0)^{\lambda+1}\left(\mathcal{U}(z) - \sum_{k=0}^{m}\left(\frac{\mathcal{U}^{(k)}(z_0)}{k!}\right)(z-z_0)^k\right)}{\Gamma(\lambda+1)(z-z_0)^n Q(z)}$$

(Q is entire and ($Q(z_0) \neq 0$)

$$= \left(\frac{(z-z_0)^{\lambda-[\operatorname{Re}\lambda]}}{\Gamma(\lambda+1)Q(z)}\right)\frac{\left(\mathcal{U}(z) - \sum_{k=0}^{m}\left(\frac{\mathcal{U}^{(k)}(z_0)}{k!}\right)(z-z_0)^k\right)}{(z-z_0)^m} \to 0 \text{ as } z \to z_0 \text{ in}$$

$|arg(z-z_0)| \leq \psi < \frac{\pi}{2}$. This completes the proof of part (ii) and the theorem.

□

For differential equations of fractional order, the Mittag-Leffler functions play a fundamental role.

For $\alpha > 1$, and $n = 0, 1, 2, \ldots$, let $c_{n,\alpha} = \frac{1}{\Gamma(\alpha n+1)}$. Then the ultradifferential operator

$$E_\alpha(D) = \sum_{n=0}^{\infty} c_{n,\alpha} D^n = \sum_{n=0}^{\infty} \frac{D^n}{\Gamma(\alpha n+1)} \tag{4.4}$$

is generated by the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \tag{4.5}$$

Corollary 4.1. *Let $W \in \beta_{\mathcal{L}}(\mathbb{R})$, $\alpha > 1$, and $\lambda, \xi \in \mathbb{C}$ such that $Re\,\lambda > -1$. If $W(t) \overset{e}{\sim} \xi E_\alpha(D) g_{\lambda,0}(t)$ as $t \to \infty$, then $\mathcal{W}(z)$ exists for $Re\,z > 0$ and $\frac{z^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)} \sim \xi E_\alpha(z)$ as $z \to 0$ in $|argz| \le \psi < \frac{\pi}{2}$.*

Since for $\alpha \ge 2$, the zeros of $E_\alpha(z)$ are on the negative real axis [20], the following corollary is immediate from the Final Value Theorem.

Corollary 4.2. *Let $W \in \beta_{\mathcal{L}}(\mathbb{R})$, $\alpha \ge 2$, and $\lambda, \xi, z_0 \in \mathbb{C}$ such that $Re\,\lambda > -1$ and z_0 does not lie on the negative real axis. If $W(t) \overset{e}{\sim} \xi E_\alpha(D) g_{\lambda,z_0}(t)$ as $t \to \infty$, then $\mathcal{W}(z)$ exists for $Re\,z > Re\,z_0$ and $\frac{(z-z_0)^{\lambda+1}\mathcal{W}(z)}{\Gamma(\lambda+1)} \sim \xi E_\alpha(z)$ as $z \to z_0$ in $|arg(z - z_0)| \le \psi < \frac{\pi}{2}$.*

Example 4.2. *Let $f(t) = \sqrt{t}$ and $\varphi \in D(\mathbb{R})$. Notice that $\frac{f(t)}{\sqrt{t}} \to 1$ as $t \to \infty$. Let $W = \varphi\left(pf\frac{1_+(t)}{t}\right) + E_2(D)f$, where $pf\frac{1_+(t)}{t}$ denotes the distributional derivative of $1_+(t)\log t$ (see [25]). Notice that $E_2(z) = \cosh\sqrt{z}$.*

Since the support of $\varphi\left(pf\frac{1_+(t)}{t}\right)$ is bounded, $W(t) \overset{e}{\sim} E_2(D)g_{\frac{1}{2},0}(t)$ as $t \to \infty$. Thus, by Corollary 4.1,

$$z^{\frac{3}{2}}\mathcal{W}(z) \sim \frac{\sqrt{\pi}}{2}\cosh\sqrt{z} \text{ as } z \to 0 \text{ in } |argz| \le \psi < \frac{\pi}{2}.$$

That is,

$$\lim_{\substack{z \to 0 \\ |argz| \le \psi < \frac{\pi}{2}}} \frac{z^{\frac{3}{2}}\mathcal{W}(z)}{\cosh\sqrt{z}} = \frac{\sqrt{\pi}}{2}.$$

References

[1] Arteaga, C. and Marrero, I : *The Hankel Transform of Tempered Boehmians via the Exchange Property*, Appl. Math. Comput. **219** (2012), 810–818.

[2] Anatasiu, D. and Mikusiński, P. : *On the Fourier Transform, Boehmians, and Distributions*, Colloq. Math. **108** (2007), 263–276.

[3] Betancor, J.J., Linares, M., and Mendez, J.M.R. : *Ultraspherical Transform of Summable Boehmians*, Math. Japon. **44** (1996), 81–88.

[4] Betancor, J.J., Linares, M. and Mendez, J.M. R. : *The Hankel Transform of Tempered Boehmians*, Boll. Un. Mat. Ital. B (7) **10** (1996), 325–340.

[5] Doetsch, G. : *Theorie der Laplace Transformation Band I*, Verlag Birkhauser, Basel, 1950.

[6] Kalpakam, N.V. and Ponnusamy, S. : *Convolution Transform for Boehmians*, Rocky Mountain J. Math. **33** (2003), 1353–1378.

[7] Karunakaran, V. and Kalpakam, N.V. : *Weierstrass Transform for Boehmians*, Int. J. Math. Game Theory Algebra **11** (2001), 47–65.

[8] Karunakaran, V. and Roopkumar, R. : *Boehmians and their Hilbert Transforms*, Integral Transforms Spec. Funct. **13** (2002), 131–141.

[9] Komatsu, H. : *Ultradistributions I*, J. Fac. Sci. Univ. Tokyo Sect. IA Mat. **20** (1973), 25–105.

[10] Loonker, D., Banerji, P.K., and Debnath, L. : *On the Hankel Transform for Boehmians, Integral Transforms*, Spec. Funct. **21** (2010), 479–486.

[11] Loonker, D., Banerji, P.K., and Debnath, L. : *Hartley Transform for Integrable Boehmians*, Integral Transforms Spec. Funct. **21** (2010), 459–464.

[12] Mikusiński, P. : *Convergence of Boehmians*, Japan. J. Math. (N.S) **9** (1983), 159–179.

[13] Mikusiński, P. : *Tempered Boehmians and Ultradistributions*, Proc. Amer. Math. Soc. **123** (1995), 813–817.
[14] Nemzer, D. : *The Laplace Transform on a Class of Boehmians*, Bull. Austral. Math. Soc. **146** (1992), 347–352.
[15] Nemzer, D. : *Abelian Theorems for Transformable Boehmians*, Intern. J. Math. Math. Sci. **17** (1994), 489–496.
[16] Nemzer, D. : *A Note on the Convergence of a Series in the Space of Boehmians*, Bull. Pure Appl. Math. **2** (2008), 63–69.
[17] Nemzer, D. : *Asymptotic Behavior of the Laplace Transform Near the Origin*, GFTA 2012 (Ohrid), Adv. Math. Scientific J. **1** (2012), 27–32.
[18] Pilipović, S. : *Characterizations of Bounded Sets in Spaces of Ultradistributions*, Proc. Amer. Math. Soc. **120** (1994), 1191–1206.
[19] Pilipović, S. and Stanković, B. : *Properties of Ultradistributions having S-Asymptotics*, Bull. Cl. Sci. Math. Nat. Sci. Math. **21** (1996), 47–59.
[20] Popov, A. Yu : *On Zeros of Mittag-Leffler Functions with Parameter* $\rho < \frac{1}{2}$, Anal. Math. **32** (2006), 207–246.
[21] Roopkumar, R. : *Stieltjes Transform for Boehmians*, Integral Transforms Spec. Funct. **18** (2007), 819–827.
[22] Roopkumar, R. : *Mellin Transform for Boehmians*, Bull. Inst. Math. Acad. Sin. (N.S.) **4** (2009), 75–96.
[23] Roopkumar, R. and Negrin, E. R. : *Poisson Transform on Boehmians*, Appl. Math. Comput. **216** (2010), 2740–2748.
[24] Roopkumar, R. and Negrin, E.R. : *A Unified Extension of Stieltjes and Poisson Transforms to Boehmians*, Integral Transforms Spec. Funct. **22** (2011), 195–206.
[25] Zemanian, A. : *Distribution Theory and Transform Analysis*, Dover Publications, New York, 1987.

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