THE FUZZY LACUNARY $I-$ CONVERGENT OF Γ^2 SPACE DEFINED BY MODULUS

(COMMUNICATED BY NAIM BRAHA)

N. SUBRAMANIAN[1], P. ANBALAGAN[2], AND P. THIRUNAVUKARASU[3]

ABSTRACT. The aim of this paper is to introduce and study a new concept of the fuzzy $I-$ convergent Γ^2 space defined by modulus and also some topological properties of the resulting sequence spaces of fuzzy numbers were examined.

1. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [30], fuzzy logic has become an important area of research in various branches of Mathematics such as metric and topological spaces, theory of functions, approximation theory etc. Subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. The concept of fuzzyiness has been applied in various fields such as Statistics, Cybernetics, Artificial intelligence, Operation research, Decision making, Agriculture, Weather forecasting, Quantum physics. Similarity relations of fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming etc.

The concept of ideal convergence as a generalization of statistical convergence, and any concept involving statistical convergence plays a vital role in pure mathematics and also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

The notion of $I-$ convergence initially introduced by Kostyrko et al. Later on, it was further investigated from the sequence space point of view and linked with the summability theory by Šalát et al., Tripathy and Hazarika, Kumar, Hazarika and Savas, Khan and Ebdullah, Khan et al, Khan and Tabassum, Das et al., and many other authors.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n = 1, 2, 3, ...) \text{ (see[1])}.$$

We denote w^2 as the class of all complex double sequences (x_{mn}). A sequence $x = (x_{mn})$ is said to be Pringsheim's sense double analytic if

$$sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all Pringsheim's sense double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called Pringsheim's sense double entire sequence if

$$|x_{mn}|^{1/m+n} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all Pringsheim's sense double entire sequences are usually denoted by Γ^2. The spaces Λ^2 and Γ^2 are metric space with the metric

$$d(x, y) = sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, ... \right\}, \qquad (1)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...1, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and zero other wise. An FK-space(or a metric space)X is said to have AK property if (δ_{mn}) is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$ under metric. We need the following inequality in the sequel of the paper:

Lemma 1: For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p$$

Some initial works on double sequence spaces is found in Bromwich[4]. Later on it was investigated by Hardy[9], Moricz[17], Moricz and Rhoades[18], Basarir and Solankan[2], Tripathy[26], Colak and Turkmenoglu[6], Turkmenoglu[28], and many others. Tripathy and Dutta [31], introduced and investigated different types of fuzzy real valued double sequence spaces. Generalizing the concept of ordinary convergence for real sequences Kostyrko et al. [9] introduced the concept of ideal convergence which is a generalization of statistical convergence, by using the ideal I of the subsets of the set of natural numbers.

Throughout the article Λ^2, Γ^2 denote the spaces of Pringsheim's double analytic and Pringsheim's double entire sequences respectively and Λ_F^{2I} and Γ_F^{2I} denote the classes of $I-$ analytic and $I-$ entire fuzzy real valued double sequences respectively.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [14] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_∞ denote the classes of all, convergent,null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2$ and Γ^2. respetively. $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. Further generalized this notion and introduced the following notion. For $m, n \geq 1$,

$$Z\left(\Delta_\gamma^\mu\right) = \left\{ x = (x_{mn} : \left(\Delta_\gamma^\mu x_{mn}\right) \in Z \right\} \text{ for } Z = \Lambda^2 \text{ and } \Gamma^2$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Remark 1: An Modulus function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

In this article are introduce fuzzy $I-$ convergent Γ_Δ^2 space defined by Modulus function.

2. DEFINITIONS AND PRELIMINARIES

Let $\mu = (\lambda_{mn})$ be a sequence of non-zero scalars. Then for a given sequence space E, the multiplier sequence space $E(\mu)$ associated with multiplier sequence μ is defined by

$$E(\mu) = \{ x = (x_{mn}) : (\lambda_{mn} x_{mn}) \in E \}$$

Let X be a non empty set. A non-void class $I \subseteq 2^X$ (power set, of X) is called an ideal if I is additive (i.e $A, B \in I \Rightarrow A \bigcup B \in I$) and hereditary (i.e $A \in I$ and $B \subseteq A \Rightarrow B \in I$). A non-empty family of sets $F \subseteq 2^X$ is said to be a filter on X if $\phi \notin F; A, B \in F \Rightarrow A \bigcap B \in F$ and $A \in F, A \subseteq B \Rightarrow B \in F$. For each ideal I there is a filter $F(I)$ given by $F(I) = \{ K \subseteq N : N \setminus K \in I \}$.
Throughout the ideals of 2^N and $2^{N \times N}$ will be denoted by I and I_2 respectively.
A fuzzy real number X is a fuzzy set on R, a mapping $X : R \to L(=[0,1])$ associating each real number t with its grade of membership $X(t)$. The $\alpha-$ level set of a fuzzy real number $X, 0 < \alpha < 1$ denoted by $[X]^\alpha$ is defined as $[X]^\alpha = \{ t \in R : X(t) \geq \alpha \}$. A fuzzy real number X is called convex if $X(t) \geq X(s) \wedge X(r) = min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called normal. A fuzzy real X is said to be upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a + \epsilon))$, for all $a \in L$ is open in the usual topology of R. The set of all upper semi continuous, normal convex fuzzy number is denoted by $L(R)$.
Throughout a fuzzy real valued double sequence is denoted by (X_{mn}) i.e a double infinite array of fuzzy real number X_{mn} for all $m, n \in \mathbb{N}$.
Every real number r can express as a fuzzy real number $\overline{r}$ as follows:

$$\overline{r} = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{otherwise} \end{cases}$$

Let D be the set of all closed bounded intervals $X = \left[X^L, X^R\right]$. Then $X \leq Y$ if and only if $X^L \leq Y^L$ and $X^R \leq Y^R$.

Also $d(X, Y) = max\left(\left|X^L - Y^L\right|, \left|X^R - Y^R\right|\right)$. Then (D, d) is a complete metric space.

Let $\overline{d} : L(R) \times L(R) \to R$ be defined by

$$\overline{d}(X, Y) = sup_{0 \leq \alpha \leq 1} d\left([X]^\alpha, [Y]^\alpha\right) \text{ for } X, Y \in L(R).$$

Then $\overline{d}$ defined a metric on $L(R)$.

By a lacunary sequence $\theta = (mn)_{rs}$, where

$$(mn)_{00} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix},$$

we shall mean an increasing sequence of non-negative integers with $(mn)_{rs} - (mn)_{r-1,s-1} \to \infty$ as $r, s \to \infty$. The intervals determined by θ will be denoted by $I_{rs} = \left((mn)_{r-1,s-1}, (mn)_{rs}\right]$ and we let $h_{rs} = (mn)_{rs} - (mn)_{r-1,s-1}$. The space of lacunary strongly entire sequences N_θ is defined as follows:

$$N_\theta = \left\{ x = (x_{rs}) : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} |x_{mn} - 0|^{1/m+n} = 0,\, as\, r, s \to \infty \right\}$$

2.1. **Definition.** Let A denote a four dimensional summability method that maps the complex double sequences x into the double sequence. Ax where the $mn - th$ term to Ax is as follows

$$(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}.$$

In [10] Hardy presented the notion of regularity of two dimensional matrix transformations. The definition is as follows: a two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. In addition, to the numerous theorems characterizing regularity. Hardy also presented the Silvermann-Toeplitz characterization of regularity following this work Robison in 1926 presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is $P-$convergent is not necessarily bounded along these same lines, Robison and Hamiltion presented a Silverman-Toeplitz type multidimensional characterization of regularity in [11] and [25]. The definition of regularity for four dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four dimensional matrices.

2.2. **Definition.** A double sequences (X_{mn}) is said to be convergent in Pringsheim's sense to the fuzzy real number X, if for every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon), k_0 = k_0(\epsilon) \in N$ such that $\overline{d}(X_{mn}, X) < \epsilon$ for all $n \geq n_0, k \geq k_0$.

2.3. **Definition.** A double sequence (X_{mn}) is said to be $I-$convergent to the fuzzy number X_0, if for all $\epsilon > 0$, the set $\left\{(n, k) \in N^2 : \overline{d}(X_{mn}, X_0) \geq \epsilon\right\} \in I$. We write $I - lim X_{mn} = X_0$.

2.4. Definition. A double sequence E^F is said to be monotone if E^F contains the canonical pre-image of all its step spaces.

2.5. Definition. A double sequence E^F is said to be symmetric if $\left(X_{\pi(m),\pi(n)}\right) \in E^F$, whenever $(X_{mn}) \in E^F$, where π is a permutation of $N \times N$.

2.6. Definition. A double sequence E^F is said to be sequence algebra if $(X_{mn} \otimes Y_{mn}) \in E^F$, whenever $(X_{mn}), (Y_{mn}) \in E^F$.

2.7. Definition. A double sequence E^F is said to be convergence free if $(Y_{mn}) \in E^F$, whenever $(X_{mn}) \in E^F$ and $X_{mn} = \overline{0}$ implies $Y_{mn} = \overline{0}$.

The notion of the statistical convergence was introduced by H. Fast. Later on it was studied by J.A.Fridy from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát and Wilezyński. Later on it was studied by Šalát, Tripathy and Ziman and Demirci, Das, Kostyrko, Wilczynski, and Malik, Mursaleen and Alotaibi, Mursaleen, Mohiuddine and Edely, Mursaleen and Mohiuddine, Sahiner, Gurdal, Saltan and Gunawan and Kumar, V.A.Khan, Suthep suantai and Khalid Ebadullah. Here we give some preliminaries about the notion of $I-$ convergence.

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$ (power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A\bigcup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\mathcal{L}(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \mathcal{L}(I)$, for $A, B \in \mathcal{L}(I)$ we have $A \bigcap B \in \mathcal{L}(I)$ and for each $A \in \mathcal{L}(I), A \subseteq B \Rightarrow B \in \mathcal{L}(I)$.

An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.

A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter $\mathcal{L}(I)$ corresponding to I. i.e $\mathcal{L}(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

3. DOUBLE ENTIRE SEQUENCE SPACE OF FUZZY NUMERS

This paper to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let I be an admissible ideal of $\mathbb{N}$ and let $p = (p_{mn})$ be a sequence of positive real numbers for all $m, n \in \mathbb{N}$. Let $\theta = (mn)_{rs}$ be a lacunary sequence, f be an modulus function, $\mu = (\lambda_{mn})$ be a sequence of non-zero scalars and $X = (X_{mn})$ be a sequence of fuzzy numbers, we define the following sequence spaces as:

$$\Gamma_{\theta,f,\mu,p}^{2I(F)} = \left\{(r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n}, \overline{0}\right)\right)\right]^{p_{mn}} \geq \epsilon\right\} \in$$

$I,$

$$\Lambda_{\theta,f,\mu,p}^{2I(F)} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq K \right\} \in$$
$I,$

and

$$\Lambda_{\theta,f,\mu,p}^{2F} = \left\{ \sup_{rs} \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} < \infty \right\}.$$

Some classes are obtained by specializing $\theta = (mn)_{rs}, f, \mu = (\lambda_{mn})$ and $p = (p_{mn}):$

(i) If $\theta = (mn)_{rs} = (2^{rs}),$ then we obtain

$$\Gamma_{f,\mu,p}^{2I(F)} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{rs} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[f\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq \epsilon \right\} \in I,$$

$$\Lambda_{f,\mu,p}^{2I(F)} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{rs} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[f\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq K \right\} \in$$
$I,$

and

$$\Lambda_{f,\mu,p}^{2F} = \left\{ \sup_{rs} \frac{1}{rs} \sum_{m=1}^{r} \sum_{n=1}^{s} \left[f\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} < \infty \right\}.$$

(ii) If $f(x) = x,$ then we obtain,

$$\Gamma_{\theta,\mu,p}^{2I(F)} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq \epsilon \right\} \in$$
$I,$

$$\Lambda_{\theta,\mu,p}^{2I(F)} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq K \right\} \in$$
$I,$

and

$$\Lambda_{\theta,\mu,p}^{2F} = \left\{ \sup_{rs} \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[\left(\overline{d}\left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} < \infty \right\}.$$

(iii) If $\mu = (\lambda_{mn}) = \begin{pmatrix} 1, & 1, & ...1, & 1, & ... \\ 1, & 1, & ...1, & 1, & ... \\ . \\ . \\ . \\ 1, & 1, & ...1, & 1, & ... \\ 1, & 1, & ...1, & 1, & ... \end{pmatrix},$ then we obtain

$$\Gamma_{\theta,f,p}^{2I(F)} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f\left(\overline{d}\left((X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq \epsilon \right\} \in I,$$

$$\Lambda_{\theta,f,p}^{2I(F)} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f\left(\overline{d}\left((X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq K \right\} \in$$
$I,$

and

$$\Lambda^{2F}_{\theta,f,p} = \left\{ sup_{rs} \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f \left(\overline{d} \left((X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} < \infty \right\}.$$

(iv) If $p = (p_{mn}) = \begin{pmatrix} 1, & 1, & ...1, & 1, & ... \\ 1, & 1, & ...1, & 1, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 1, & 1, & ...1, & 1, & ... \\ 1, & 1, & ...1, & 1, & ... \end{pmatrix}$, then we obtain

$$\Gamma^{2I(F)}_{\theta,f,\mu} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] \geq \epsilon \right\} \in I,$$

$$\Lambda^{2I(F)}_{\theta,f,\mu} = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] \geq K \right\} \in I,$$

and

$$\Lambda^{2F}_{\theta,f,\mu} = \left\{ sup_{rs} \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] < \infty \right\}.$$

4. Main Results

In this section we examine the basic topological and algebraic properties of these spaces and obtain the inclusion relation between these spaces.

4.1. Theorem. $\Gamma^{2I(F)}_{\theta,f,\mu,p}$ and $\Lambda^{2I(F)}_{\theta,f,\mu,p}$ are linear spaces
Proof: It is routine verification. Therefore we omit the proof.

4.2. Theorem. The space $\Lambda^{2(F)}_{\theta,f,\mu,p}$ is a paranormed space (not totally paranormed) with the paranorm g_μ defined by

$$g_\mu (X) = inf \left\{ sup_{mn} \left[f \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \leq 1 \right\}.$$

Proof:Clearly $g_\mu (-X) = g_\mu (X)$ and $g_\mu (\theta) = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$.

Let $X = (X_{mn})$ and $Y = (Y_{mn})$ be two elements in $\Lambda^{2(F)}_{\theta,f,\mu,p}$. Then

$$A_1 = inf \left\{ sup_{mn} \left[f \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] \leq 1 \right\}$$

and

$$A_2 = inf \left\{ sup_{mn} \left[f \left(\overline{d} \left((\lambda_{mn} Y_{mn})^{1/m+n}, \overline{0} \right) \right) \right] \leq 1 \right\}$$

We obtain the following

$$\left[f \left(\overline{d} \left((\lambda_{mn} (X_{mn} + Y_{mn}))^{1/m+n}, \overline{0} \right) \right) \right] \leq \left[f \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] + \left[f \left(\overline{d} \left((\lambda_{mn} Y_{mn})^{1/m+n}, \overline{0} \right) \right) \right].$$

Thus we have

$$sup_{mn} \left[f \left(\overline{d} \left((\lambda_{mn} (X_{mn} + Y_{mn}))^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \leq 1$$

and

$$g_\mu (X + Y) = inf \{A_1\} + inf \{A_2\} = g_\mu (X) + g_\mu (Y).$$

Let $t_{mn} \to t$ where $t_{mn}, t \in \mathbb{C}$ and let $g_\mu (X_{mn} - X) \to 0$ as $m, n \to \infty$. To prove that $g_\mu (t_{mn} X_{mn} - tX) \to 0$ as $m, n \to \infty$. By the continuity of the function f we observe that

$$\left[f \left(\overline{d} \left((\lambda_{mn} (t_{mn} X_{mn} - tX))^{1/m+n}, \overline{0} \right) \right) \right] \leq \left[f \left(\overline{d} \left((\lambda_{mn} (t_{mn} X_{mn} - tX_{mn}))^{1/m+n}, \overline{0} \right) \right) \right] +$$

$$\left[f \left(\overline{d} \left((\lambda_{mn} (tX_{mn} - tX))^{1/m+n}, \overline{0} \right) \right) \right] \leq \left[f \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] +$$

$$\left[f \left(\overline{d} \left((\lambda_{mn} Y_{mn})^{1/m+n}, \overline{0} \right) \right) \right].$$

From the above inequality it follows that

$$sup_{mn} \left[f \left(\overline{d} \left((\lambda_{mn} (t_{mn} X_{mn} - tX))^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \leq 1$$

and consequently

$$g_\mu (t_{mn} X_{mn} - tX) \leq |t_{mn} - t| \, inf \{A_1\} + |t| \, inf \{A_2\}$$

$$g_\mu (t_{mn} X_{mn} - tX) \leq max \{1, |t_{mn} - t|\} g_\mu (X_{mn}) + max \{1, |t|\} g_\mu (X_{mn} - X). \tag{2}$$

Therefore the above equation (4.1) implies that $g_\mu (X_{mn}) \leq g_\mu (X) + g_\mu (X_{mn} + X)$ for all $m, n \in \mathbb{N}$. Hence by our assumption the right hand side of the relation (4.1) tens to 0 as $m, n \to \infty$. This completes the proof.

4.3. Theorem. Let f and g be modulus functions. Then the following hold:
(i) $\Gamma_{\theta,g,\mu,p}^{2I(F)} \subseteq \Gamma_{\theta,f,\mu,p}^{2I(F)}$, provided $p = (p_{mn})$ be such that $G_0 = inf \, p_{mn} > 0$.
(ii) $\Gamma_{\theta,f,\mu,p}^{2I(F)} \bigcap \Gamma_{\theta,g,\mu,p}^{2I(F)} \subseteq \Gamma_{\theta,f+g,\mu,p}^{2I(F)}$
Proof: (i) Let $\epsilon > 0$ be given. Choose $\epsilon_1 > 0$ such that $max \left\{ \epsilon_1^G, \epsilon_1^{G_0} \right\} < \epsilon$. Choose $0 < \delta < 1$ such that $0 < t < \delta$ implies that $f(t) < \epsilon_1$. Let $X = (X_{mn})$ be any element in $\Gamma_{\theta,g,\mu,p}^{2I(F)}$. Put

$$A_\delta = \left\{ (r,s) \in \mathbb{N} : \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[g \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} \geq \delta^G \right\} \in I.$$

$$\Rightarrow \frac{1}{h_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[g \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} < \delta^G$$

$$\Rightarrow \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[g \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} < h_{rs} \delta^G$$

$$\Rightarrow \left[g \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right]^{p_{mn}} < \delta^G, \forall m, n \in I_{rs}$$

$$\Rightarrow \left[g \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] < \delta^G, \forall m, n \in I_{rs}. \tag{3}$$

Using the continuity of the function f from the relation (4.2) we have

$$f \left(\left[g \left(\overline{d} \left((\lambda_{mn} X_{mn})^{1/m+n}, \overline{0} \right) \right) \right] \right) < \epsilon_1, \forall m, n \in I_{rs}$$

Consequently we get

$$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f\left(\left[g\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n},\overline{0}\right)\right)\right]\right)\right]^{p_{mn}} < h_{rs}\cdot max\left\{\epsilon_1^G,\epsilon_1^{G_0}\right\} < h_{rs}\epsilon$$

$$\Rightarrow \frac{1}{h_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f\left[g\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n},\overline{0}\right)\right)\right]\right]^{p_{mn}} < \epsilon.$$

This implies that

$$\left\{(r,s)\in\mathbb{N}:\frac{1}{h_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f\left[g\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n},\overline{0}\right)\right)\right]\right]^{p_{mn}}\geq\epsilon\right\}\subseteq A_\delta\in I.$$

(ii) Let $X = (X_{mn}) \in \Gamma^{2I(F)}_{\theta,f,\mu,p}\bigcap\Gamma^{2I(F)}_{\theta,g,\mu,p}$. Then by the following inequality the result follows:

$$\frac{1}{h_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[(f+g)\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n},\overline{0}\right)\right)\right]^{p_{mn}}\leq$$
$$H\frac{1}{h_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n},\overline{0}\right)\right)\right]^{p_{mn}}+$$
$$H\frac{1}{h_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[g\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n},\overline{0}\right)\right)\right]^{p_{mn}}. \text{ This completes the proof.}$$

4.4. Theorem. Let $0 < p_{mn} \leq q_{mn}$ and $\left(\frac{q_{mn}}{p_{mn}}\right)$ is positive and bounded sequence, then
$$\Gamma^{2I(F)}_{\theta,q,\mu,p}\subseteq\Gamma^{2I(F)}_{\theta,f,\mu,p}.$$
Proof: It is routine verification.. Therefore omit the proof.

4.5. Theorem. For any two sequences $p = (p_{mn})$ and $q = (q_{mn})$ of positive real numbers, then the following holds:
(i) $\Gamma^{2I(F)}_{\theta,f,\mu,p}\bigcap\Gamma^{2I(F)}_{\theta,f,\mu,q}\neq\phi$;
(ii) $\Lambda^{2I(F)}_{\theta,f,\mu,p}\bigcap\Lambda^{2I(F)}_{\theta,f,\mu,q}\neq\phi$;
(iii) $\Lambda^{2F}_{\theta,f,\mu,p}\bigcap\Lambda^{2F}_{\theta,f,\mu,q}\neq\phi$.

4.6. Lemma. A sequence space E_F is solid implies E_F is monotone

4.7. Lemma. If $I\subset 2^\mathbb{N}$ is a maximal ideal, then for each $A\subset\mathbb{N}$ we have either $A\in I$ or $\mathbb{N}-A\in I$

4.8. Theorem. The sequence spaces $\Gamma^{2I(F)}_{\theta,f,\mu,p}$ and $\Lambda^{2I(F)}_{\theta,f,\mu,p}$ are solid as well as monotone
Proof: Let $X = (X_{mn}) \in \Gamma^{2I(F)}_{\theta,f,\mu,p}$ and $Y = (Y_{mn})$ be such that
$\overline{d}\left(Y_{mn}^{1/m+n},\overline{0}\right)\leq\overline{d}\left(X_{mn}^{1/m+n},\overline{0}\right)$ for all $m,n\in\mathbb{N}$. Then for given $\epsilon>0$ we have

$$\eta = \left\{(r,s)\in\mathbb{N}:\frac{1}{h_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f\left(\overline{d}\left((\lambda_{mn}X_{mn})^{1/m+n},\overline{0}\right)\right)\right]\geq\epsilon\right\}\in I$$

Again the set $\eta = \left\{(r,s)\in\mathbb{N}:\frac{1}{h_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f\left(\overline{d}\left((\lambda_{mn}Y_{mn})^{1/m+n},\overline{0}\right)\right)\right]\geq\epsilon\right\}\subseteq$ $\eta_1\in I$ and so $Y = (Y_{mn})\Gamma^{2I(F)}_{\theta,f,\mu,p}$. Thus the space $\Gamma^{2I(F)}_{\theta,f,\mu,p}$ is solid. From the Lemma 4.6, it follows that $\Gamma^{2I(F)}_{\theta,f,\mu,p}$ is monotone.

4.9. Result. If I is neither maximal nor then the space $\Gamma_{\theta,f,\mu,p}^{2I(F)}$ is not symmetric

Proof: Let us consider a sequence $X = (X_{mn})$ of fuzzy real numbers defined by

$$X_{mn}^{1/m+n}(t) = \begin{cases} 1+t-2t, if\,\{t \in [2\,(m+n)-1, 2\,(m+n)]\}; \\ 1-t+2\,(m+n), if\,\{t \in [2\,(m+n), 2\,(m+n)+1]\}; \\ 0, otherwise. \end{cases}$$

for $(m,n) \subset I$ an infinite set. Then $(X_{mn}) \in \Gamma_{\theta,f,\mu,p}^{2I(F)}$. Let $K \subseteq \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N}-K \notin I$ (the set K exists by the Lemma 4.7, as I is not maximal). Consider the sequence $Y = (Y_{mn})$ a rearrangement of the sequence (X_{mn}) defined as follows:

$$Y_{mn}^{1/m+n} = \begin{cases} X_{mn}^{1/m+n}, if\, m,n \in K; \\ \overline{0}, otherwise. \end{cases}$$

Then $(Y_{mn}) \notin \Gamma_{\theta,f,\mu,p}^{2I(F)}$. Hence $\Gamma_{\theta,f,\mu,p}^{2I(F)}$ is not symmetric.

4.10. Result. If I is neither maximal nor then the space $\Lambda_{\theta,f,\mu,p}^{2I(F)}$ is not symmetric

Proof: Let us consider a sequence $X = (X_{mn})$ of fuzzy real numbers defined by

$$X_{mn}^{1/m+n}(t) = \begin{cases} 1+t-3t, if\,\{t \in [3\,(m+n)-1, 3\,(m+n)]\}; \\ 1-t+3\,(m+n), if\,\{t \in [3\,(m+n), 3\,(m+n)+1]\}; \\ 0, otherwise. \end{cases}$$

for $(m,n) \subset I$ an infinite set. Otherwise $X_{mn}^{1/m+n} = \overline{0}$.

Since I is not maximal, so by Lemma 4.7, there exists a subset $K \in \mathbb{N}$ such that $K \notin I$ and $\mathbb{N}-K \notin I$. Let $\zeta : K \to A$ and $h : \mathbb{N}-K \to \mathbb{N}-A$ be bijections. Consider a sequence $Y = (Y_{mn})$ a rearrangement of the sequence (X_{mn}) defined as follows:

$$Y_{mn}^{1/m+n} = \begin{cases} X_{\zeta(mn)}^{1/m+n}, if\, m,n \in K; \\ X_{h(mn)}^{1/m+n}, otherwise. \end{cases}$$

Then $(Y_{mn}) \notin \Lambda_{\theta,f,\mu,p}^{2I(F)}$. Hence $\Lambda_{\theta,f,\mu,p}^{2I(F)}$ is not symmetric.

REFERENCES

[1] T.Apostol, Mathematical Analysis, *Addison-wesley , London,* **1**978.
[2] M.Basarir and O.Solancan, On some double sequence spaces, *J. Indian Acad. Math.,* **21(2)** (1999), 193-200.
[3] C.Bektas and Y.Altin, The sequence space $\ell_M(p,q,s)$ on seminormed spaces, *Indian J. Pure Appl. Math.,* **34(4)** (2003), 529-534.
[4] T.J.I'A.Bromwich, An introduction to the theory of infinite series *Macmillan and Co.Ltd.* ,New York, (1965).
[5] J.C.Burkill and H.Burkill, A Second Course in Mathematical Analysis *Cambridge University Press, Cambridge,* New York, (1980).
[6] R.Colak and A.Turkmenoglu, The double sequence spaces $\ell_\infty^2(p), c_0^2(p)$ and $c^2(p),$ *(to appear).* l
[7] K.Chandrasekhara Rao and N.Subramanian, The Orlicz space of entire sequences, *Int. J. Math. Math. Sci.,* **68**(2004), 3755-3764.

[8] M.Gupta and P.K.Kamthan,Infinite matrices and tensorial transformations, *Acta Math.* , **Vietnam 5** (1980), 33-42.

[9] G.H.Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.

[10] G.H.Hardy, Divergent Series, *Oxford Univ. Press, London*, **1949**.

[11] H.J.Hamilton, Transformations of Multiple Sequences, *Duke Math. Jour.*, **2** (1936), 29-60.

[12] M.A.Krasnoselskii and Y.B.Rutickii, Convex functions and Orlicz spaces, *Gorningen, Netherlands*, **1961**.

[13] P.K.Kamthan and M.Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York* , **1**981.

[14] H.Kizmaz, On certain sequence spaces, *Canad Math. Bull.* , **24(2)**(1981), 169-176.

[15] J.Lindenstrauss and L.Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

[16] I.J.Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc*, **100(1)** (1986), 161-166.

[17] F.Moricz, Extentions of the spaces c and c_0 from single to double sequences, *Acta. Math. Hungerica*, **57(1-2)**, (1991), 129-136.

[18] F.Moricz and B.E.Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.

[19] M.Mursaleen,M.A.Khan and Qamaruddin, Difference sequence spaces defined by Orlicz functions, *Demonstratio Math.* , **Vol. XXXII** (1999), 145-150.

[20] M.Mursaleen and M.Basarir, On some new sequence spaces of fuzzy numbers, *Indian J. Pure Appl. Math.*, **34(9)** (2003), 1351-1357.

[21] H.Nakano, Concave modulars, *J. Math. Soc. Japan*, **5**(1953), 29-49.

[22] W.Orlicz, $\ddot{U}$ber Raume $\left(L^M\right)$ *Bull. Int. Acad. Polon. Sci. A*, **(**1936), 93-107.

[23] S.D.Parashar and B.Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.* , **25(4)**(1994), 419-428.

[24] W.H.Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25**(1973), 973-978.

[25] G.M.Robison, Divergent Double sequences and Series, *Amer. Math. Soc. trans.*, **28** (1926), 50-73.

[26] B.C.Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.

[27] B.C.Tripathy,M.Et and Y.Altin, Generalized difference sequence spaces defined by Orlicz function in a locally convex space, *J. Analysis and Applications*, **1(3)**(2003), 175-192.

[28] A.Turkmenoglu, Matrix transformation between some classes of double sequences, *Jour. Inst. of math. and Comp. Sci. (Math. Seri.)*, **12(1)**, (1999), 23-31.

[29] A.Wilansky, Summability through Functional Analysis, *North-Holland Mathematics Studies, North-Holland Publishing, Amsterdam*, **Vol.85**(1984).

[30] L.A.Zadeh, Fuzzy sets, *Inform Control*, **8**(1965), 338-353.

[31] B.C.Tripathyand A.J. Dutta, On fuzzy real valued double sequence spaces , *Soochow J. Math.*, **32(4)**(2006), 509-520.

[32] A.Esi and B.Hazarika, Lacunary summable sequence spaces of fuzzy numbers defined by ideal convergence and an Orlicz function, *African Mathematical Union and Springer-Verlag Berlin Heidelberg*, **DOi 10. 1007/s13370-012-0117-3**, **(2012)**.

[1]Department of Mathematics,
SASTRA University,
Thanjavur-613 401, India
E-mail address: `nsmaths@yahoo.com`

[2] Department of Mathematics,
Government Arts College,
Trichy-620 022, India
E-mail address: `anbalaganmaths@gmail.com`

[3] P.G. and Research Department of Mathematics,
Periyar E.V.R. College (Autonomous),
Tiruchirappalli–620 023, India.
E-mail address: `ptavinash1967@gmail.com`