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DEGREE OF APPROXIMATION OF CONJUGATE OF SIGNALS (FUNCTIONS) BELONGING TO THE GENERALIZED WEIGHTED LIPSCHITZ $W(L_r, \xi(t)), (r \geq 1)$ - CLASS BY (C, 1) (E, Q) MEANS OF CONJUGATE TRIGONOMETRIC FOURIER SERIES

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Abstract. Very recently, Sonker and Singh [21] determined the degree of approximation of the conjugate of 2π-periodic signals (functions) belonging to $Lip(\alpha, r)(0 < \alpha \leq 1, r \geq 1)$-class by using Cesàro-Euler (C,1) (E,q) means of their conjugate trigonometric Fourier series. In the present paper, we generalize the result of Sonker and Singh [21] on the generalized weighted Lipschitz $W(L_r, \xi(t)), (r \geq 1)$- class of signals (functions) by product summability (C,1) (E,q) transform of conjugate trigonometric Fourier series. Our result also generalizes the result of Lal and Singh [6]. Few applications and example of approximation of functions will also be highlighted.

1. Introduction

The degree of approximation of functions belonging to $Lip\alpha, Lip(\alpha, r), Lip(\xi(t), r)$ and $W(L_r, \xi(t)), (r \geq 1)$ - classes by general summability matrices has been proved by various investigators like Govil [1], Khan [3-5], Mohapatra and Chandra [19], Mittal et al. [8-9], Mittal and Mishra [7], Mishra et al. [10-17] and Mishra and Mishra [18]. Recently, Sonker and Singh [21] discussed the degree of approximation of the conjugate of signals (functions) belonging to $Lip(\alpha, r)$ class by $(C, 1)(E, q)$ means of conjugate trigonometric Fourier series. But nothing seems to have been done so far to obtain the degree of approximation of conjugate of signals belonging to the generalized weighted Lipschitz $W(L_r, \xi(t)), (r \geq 1)$ class by (C,1) (E, q) product summability transforms. Weighted $W(L_r, \xi(t)), (r \geq 1)$ Lipschitz - class is generalization of $Lip\alpha, Lip(\alpha, r)$ and $Lip(\xi(t), r)$ - classes. In the present paper, an attempt to make advance study in this direction, a theorem on the degree of approximation of conjugate of signals (functions) belonging to the generalized

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weighted Lipschitz $W(L_r, \xi(t)), r \geq 1$ class by product summability (C,1) (E, q) transform of conjugate series of Fourier series has been established.
Let $L_{2\pi}$ be the space of all 2π - periodic and Lebesgue-integrable functions over $[-\pi, \pi]$.Then the Fourier series of $f \in L_{2\pi}$ at x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cos\, kx + b_k sin\, kx) \equiv \sum_{k=0}^{\infty} A_k(x), \qquad (1)$$

with n^{th} partial sums $s_n(f; x)$, called trigonometric polynomial of degree (or order) n, of the first $(n+1)$ terms of the Fourier series of f , a_k and b_k are the Fourier coefficients of f .
The conjugate series of Fourier series (1) is given by

$$\sum_{k=1}^{\infty}(b_k cos\, kx - a_k sin\, kx) \equiv \sum_{k=1}^{\infty} B_k(x). \qquad (2)$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^{th} partial sums $\{s_n\}$.
Hausdorff matrix $H \equiv (h_{n,k})$ is an infinite lower triangular matrix defined by

$$h_{n,k} = \begin{cases} \binom{n}{k}\Delta^{n-k}\mu_k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

where Δ is the forward difference operator defined by $\Delta\mu_n = \mu_n - \mu_{n+1}$ and $\Delta^{k+1}\mu_n = \Delta^k(\Delta\mu_n)$. If H is regular, then $\{\mu_n\}$, known as moment sequence, has the representation

$$\mu_n = \int_0^1 u^n d\gamma(u),$$

where $\gamma(u)$ known as mass function, is continuous at $u = 0$ and belongs to BV[0,1] such that $\gamma(0) = 0, \gamma(1) = 1$; and for $0 < u < 1$, $\gamma(u) = [\gamma(u+0) + \gamma(u-0)]/2$.
The Hausdorff means of conjugate Fourier series of f are defined by

$$\tilde{H}_n(f; x) = \sum_{k=0}^{n} h_{n,k}\tilde{s}_k,\ n \geq 0.$$

The conjugate Fourier series is said to be summable to s by Hausdorff means, if $\tilde{H}_n(f; x) \to s$, as $n \to \infty$. For the mass function $\gamma(u)$ is given by

$$\gamma(u) = \begin{cases} 0, & 0 \leq u < a, \\ 1, & a \leq u \leq 1, \end{cases}$$

where $a = 1/(1+q), q > 0$, we can verify that $\mu_k = 1/(1+q)^k$, and

$$h_{n,k} = \begin{cases} \binom{n}{k}\frac{q^{n-k}}{(1+q^n)}, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Thus Hausdorff matrix $H \equiv (h_{n,k})$ reduces to Euler matrix (E, q) of order $q > 0$.
One more example of Hausdorff matrix $[\gamma(u) = u$ for $0 \leq u \leq 1]$ is the well known Cesàro matrix of order 1 (C,1).
The (E, q) transform is defined as the n^{th} partial sum of $(E, q), q > 0$ summability and we denote it by E_n^q. If

$$E_n^q = \frac{1}{(q+1)^n}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}s_k \to s, as\, n \to \infty, \qquad (3)$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable (E, q) to a definite number s [2]. If

$$\tau_n = \frac{s_0 + s_1 + s_2 + ... + s_n}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s, as\, n \to \infty, \tag{4}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (C,1) method. The (C,1) transform of the (E, q) transform defines (C,1) (E, q) product transform and denote it by $E_n^q C_n^1$. Thus if

$$C_n^1 E_n^q = \frac{1}{n+1}\sum_{k=0}^{n}(1+q)^{-k}\sum_{v=0}^{k}\binom{k}{v} q^{k-v} s_v \to s, as\, n \to \infty, \tag{5}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (C,1) (E, q) method or summable (C,1) (E, q) to a definite number 's'.
If $C_n^1 E_n^q \to s$ as $n \to \infty$ then the infinite series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable (C,1) (E, q) to the sum s if $\lim_{n\to\infty} C_n^1 E_n^q$ exists and is equal to s.

$$\begin{aligned} s_n \to s \quad &\Rightarrow \quad (E,q)(s_n) = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} s_k \to s, as\, n \to \infty, (E,q)\, method\, is\, regular, \\ &\Rightarrow \quad (C,1)((E,q)(s_n)) = C_n^1 E_n^q \to s,\ as\ n \to \infty, (C,1)\, method\ is\ regular, \\ &\Rightarrow \quad (C,1)(E,q)\, method\, is\, regular. \end{aligned}$$

A signal (function) is said to belong to the class $Lip\alpha$, if

$$|f(x+t) - f(x)| = O(|t^{\alpha}|)\ for\ 0 < \alpha \le 1, t > 0. \tag{6}$$

and $f(x) \in Lip(\alpha, r),\ for\ 0 \le x \le 2\pi$, if

$$\|f(x+t)-f(x)\|_r = \left(\int_0^{2\pi} |f(x+t)-f(x)|^r dx\right)^{\frac{1}{r}} = O\Big(|t|^{\alpha}\Big),\ 0 < \alpha \le 1,\ r \ge 1,\ t > 0. \tag{7}$$

For a given a positive increasing function $\xi(t), f(x) \in Lip(\xi(t), r)$ if

$$\|f(x+t) - f(x)\|_r = \left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big), r \ge 1,\ t > 0. \tag{8}$$

Given positive increasing function $\xi(t)$, an integer $r \ge 1$, a signal (function) f is said to belong to the generalized weighted Lipschitz $W(L_r, (\xi(t)))$-class ([5]), if

$$\left\|[f(x+t) - f(x)]sin^{\beta}\Big(x/2\Big)\right\|_r = O\Big(\xi(t)\Big), \beta \ge 0,\ t > 0. \tag{9}$$

If $\beta = 0$, then the generalized weighted Lipschitz $W(L_r, (\xi(t)))$ class coincides with the class $Lip(\xi(t), r)$, we observe that

$$W(L_r, (\xi(t)) \xrightarrow{\beta=0} Lip(\xi(t), r) \xrightarrow{\xi(t)=t^{\alpha}} Lip(\alpha, r) \xrightarrow{r\to\infty} Lip(\alpha)\ \ for\, 0 < \alpha \le 1,\, r \ge 1, t > 0. \tag{10}$$

The L_r- norm of signal f is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}},\ (1 \le r < \infty), \text{ and } \|f\|_{\infty} = \sup_{x\in[0,2\pi]} |f(x)|. \tag{11}$$

A signal (function) f is approximated by trigonometric polynomials $\tau_n(f)$ of order n and the degree of approximation $E_n(f)$ is given by [22]

$$E_n(f) = \min_n \|f(x) - \tau(f;x)\|_r, \tag{12}$$

in terms of n, where $\tau_n(f;x)$ is a trigonometric polynomial of degree n. This method of approximation is called Trigonometric Fourier Approximation (TFA) [8].
We note that E_n^q and $C_n^1 E_n^q$ are also trigonometric polynomials of degree (or order) n.
The conjugate function $\widetilde{f}(x)$ is defined for almost every x by

$$\widetilde{f}(x) = -\frac{1}{2\pi}\int_0^\pi \psi(t) cos\, t/2\, dt = \lim_{h\to 0}\left(-\frac{1}{2\pi}\int_h^\pi \psi(t) cos\, t/2\, dt\right) \text{ (see [22, p. 131]).} \tag{13}$$

We use the following notations throughout this paper

$$\psi(t) = \psi_x(t) = f(x+t) - f(x-t),$$

$$\widetilde{G}_n(t) = \frac{1}{2\pi(1+n)}\left[\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v} q^{k-v}\frac{cos\Big(v+1/2\Big)t}{sin\Big(t/2\Big)}\right].$$

2. Known Theorem

In a recent paper, Sonker and Singh [21] have studied the degree of approximation of functions belonging to $Lip(\alpha, r)$ - class using (C,1) (E, q) summability means of its conjugate Fourier series. They proved the following theorem:

Theorem 2.1. *Let $f(x)$ be a 2π- periodic, Lebesgue integrable function and belong to the $Lip(\alpha, r)$ - class with $r \geq 1$ and $\alpha r \geq 1$. Then the degree of approximation of $\widetilde{f}(x)$, the conjugate of f (x) by (C,1) (E, q) means of its conjugate Fourier series is given by*

$$\|C_n^1 E_n^q - \widetilde{f}\|_r = O\Big(n^{-\alpha+1/r}\Big), \quad n = 0, 1, 2, ..., \tag{14}$$

provided

$$\left(\int_0^{\pi/(n+1)}\left(\frac{|\psi(t)|}{t^\alpha}\right)^r dt\right)^{\frac{1}{r}} = O((n+1)^{-1}), \tag{15}$$

$$\left(\int_{\pi/(n+1)}^{\pi}\left(t^{-\delta}\frac{|\psi(t)|}{t^\alpha}\right)^r dt\right)^{\frac{1}{r}} = O((n+1)^{\delta}), \tag{16}$$

where δ is an arbitrary number such that $(\alpha+\delta)s+1 < 0$ and $r^{-1}+s^{-1} = 1$ for $r >$ 1.

3. Main Theorem

Approximation by trigonometric polynomials is at the heart of approximation theory. The most important trigonometric polynomials used in the approximation theory are obtained by linear summation methods of Fourier series of 2π -periodic functions on the real line (i.e. Cesàro means, Nörlund means and Product Cesàro-Nörlund means, Product Ceàsro-Euler means etc.). Much of the advance in the theory of trigonometric approximation is due to the periodicity of the functions. Recently, Mishra et al. [14] have studied the degree of approximation of functions belonging to $W(L_r, \xi(t))$ - class through $(E,q)(C,1)$ means of conjugate Fourier series. In this paper, we use the $(C,1)(E,q)$ means of conjugate Fourier series of $f \in W(L_r, \xi(t))$ to determine the degree of approximation of the conjugate of f, which in turn generalizes the result of Sonker and Singh [21] and Lal and Singh [6]. More precisely, we prove

Theorem 3.1. *If $\widetilde{f}(x)$, conjugate to a 2π - periodic function f belonging to the generalized weighted Lipschitz $W(L_r, \xi(t)), (r \geq 1)$ - class then its degree of approximation by (C,1) (E, q) product summability means of conjugate series of Fourier series is given by*

$$\|\widetilde{C_n^1 E_n^q} - \widetilde{f}\|_r = O\left\{(\sqrt{n})^{\frac{1}{r}+\beta}\right\}\xi\left(\frac{1}{\sqrt{n}}\right), \tag{17}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left(\int_0^{\frac{\pi}{\sqrt{n}}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r sin^{\beta r} t/2\,dt\right)^{1/r} = O\left(\frac{1}{\sqrt{n}}\right), \tag{18}$$

$$\left(\int_{\frac{\pi}{\sqrt{n}}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right)^{1/r} = O\left(\sqrt{n}\right)^{\delta}, \tag{19}$$

and $\frac{\xi(t)}{t}$ *is non-increasing in 't',*
where δ is an arbitrary number such that $s(1-\delta+\beta)-1 > 0, \frac{1}{r}+\frac{1}{s} = 1,\ 1 \leq r \leq \infty$, conditions (18) and (19) hold uniformly in x and $\widetilde{C_n^1 E_n^q}$ is $(C,\widetilde{1)(E},q)$ means of the series (2) and

$$f(x) = -\frac{1}{2\pi}\int_0^{\pi}\psi(t)cos\,t/2dt. \tag{20}$$

Note 3.2 $\xi\left(\frac{\pi}{\sqrt{n}}\right) \leq \pi\xi\left(\frac{1}{\sqrt{n}}\right)$, *for* $\left(\frac{\pi}{\sqrt{n}}\right) \geq \left(\frac{1}{\sqrt{n}}\right)$.
Note 3.3. *The product transform (C,1) (E, q) plays an important role in signal theory as a double digital filter [13] and theory of Machines in Mechanical Engineering [13].*

4. Lemmas

For the proof of our theorem, the following lemmas are required:

Lemma 4.1. $|\widetilde{G}_n(t)| = O\left(\frac{1}{t}\right) + ((n+1)t)\ \ for\, 0 \leq t \leq \frac{\pi}{\sqrt{n}} \leq \frac{\pi}{\sqrt{v}}$.

Proof. For $0 \le t \le \frac{\pi}{\sqrt{n}}$, $sin\frac{t}{2} \ge \frac{t}{\pi}$ *and* $|cos\, nt| \le 1$.

$$
\begin{aligned}
|\widetilde{G}_n(t)| &\le \frac{1}{2\pi(n+1)}\Bigg|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{cos\Big(v+1/2\Big)t}{sin\Big(t/2\Big)}\Bigg| \\
&= \frac{1}{2\pi(n+1)}\Bigg|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{cos\Big(v+1-1/2\Big)t}{sin\Big(t/2\Big)}\Bigg| \\
&\le \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\Bigg|\frac{cos\Big(v+1\Big)t\,cos\Big(t/2\Big)+sin\Big(v+1\Big)t\,sin\Big(t/2\Big)}{sin\Big(t/2\Big)}\Bigg| \\
&= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\Bigg|\frac{cos\Big(v+1\Big)tcos\Big(t/2\Big)+sin\Big(v+1\Big)t\,sin\Big(t/2\Big)}{sin\Big(t/2\Big)}\Bigg| \\
&= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\left[O\Big(\frac{1}{t}\Big)+O\Big(sin(v+1)t\Big)\right] \\
&= O\left[\frac{1}{(n+1)t}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\right]+O\left[\frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}(v+1)t\right] \\
&= O\left[\frac{1}{(n+1)t}(n+1)\right]+O\left[\frac{1}{(n+1)}(n+1)(n+1)t\right] \\
&= O\left[\frac{1}{t}\right]+O\left[(n+1)t\right],
\end{aligned}
$$

in view of $sin(v+1)t \le (v+1)t$ for $0 \le t \le \frac{\pi}{\sqrt{n}} \le \frac{\pi}{\sqrt{v}}$.
This completes the proof of Lemma (4.1) □

Lemma 4.2. $|\widetilde{G}_n(t)| = O\Big(\frac{1}{t}\Big) + O(1)$ *for* $\frac{\pi}{\sqrt{n}} \le \frac{\pi}{\sqrt{v}} \le t \le \pi$.

Proof. For $0 \le \frac{\pi}{\sqrt{n}} \le \frac{\pi}{\sqrt{v}} \le t \le \pi$, $sin\frac{t}{2} \ge \frac{t}{\pi}$ *and* $|cos\, nt| \le 1$.

$$\begin{aligned}
|\widetilde{G}_n(t)| &\le \frac{1}{2\pi(n+1)}\Big|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}\frac{cos\Big(v+1/2\Big)t}{sin\Big(t/2\Big)}\Big| \\
&= \frac{1}{2\pi(n+1)}\Big|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}\frac{cos\Big(v+1-1/2\Big)t}{sin\Big(t/2\Big)}\Big| \\
&\le \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}\Big|\frac{cos\Big(v+1\Big)tcos\Big(t/2\Big)+sin\Big(v+1\Big)t\, sin\Big(t/2\Big)}{sin\Big(t/2\Big)}\Big| \\
&= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}\Big|\frac{cos\Big(v+1\Big)tcos\Big(t/2\Big)+sin\Big(v+1\Big)t\, sin\Big(t/2\Big)}{sin\Big(t/2\Big)}\Big| \\
&= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}\Big[O\Big(\frac{1}{t}\Big)+O\Big(1\Big)\Big] \\
&= O\left[\frac{1}{(n+1)t}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}\right]+O\left[\frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}\right] \\
&= O\left[\frac{1}{(n+1)t}(n+1)\right]+O\left[\frac{1}{(n+1)}(n+1)\right] \\
&= O\left[\frac{1}{t}\right]+O[1],
\end{aligned}$$

in view of $|sin(v+1)t| \le 1$ for $0 \le t \le \frac{\pi}{\sqrt{n}} \le \frac{\pi}{\sqrt{v}} \le t \le \pi$.
This completes the proof of Lemma (4.2). □

5. Proof of Theorem 3.1

Let $\tilde{s}_n(x)$ denotes the partial sum of series (2), we have

$$\widetilde{s}_n(x)-\widetilde{f}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{cos\Big(n+1/2\Big)t}{sin\Big(t/2\Big)}dt.$$

Therefore, using (2) the (C,1)transform C_n^1 of $\widetilde{s}_n$ is given by

$$\widetilde{C}_n^1-\widetilde{f}(x)=\frac{1}{2\pi(n+1)}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{cos\Big(n+1/2\Big)t}{sin\Big(t/2\Big)}dt.$$

Now denoting $\widetilde{(C,1)(E,q)}$ transform of $\widetilde{s}_n$ as $\widetilde{(C_n^1E_n^q)}$, we write

$$\begin{aligned}
\widetilde{(C_n^1E_n^q)} - \widetilde{f}(x) &= \frac{1}{2\pi(n+1)}\Bigg[\sum_{k=0}^{n}\frac{1}{(1+q)^k}\int_0^\pi \frac{\psi(t)}{sint\frac{t}{2}}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v}cos\Big(v+\frac{1}{2}\Big)tdt\Bigg] \\
&= \int_0^\pi \psi(t)\widetilde{G}_n(t)\,dt \\
&= \Bigg[\int_0^{\pi/\sqrt{n}} + \int_{\pi/\sqrt{n}}^{\pi}\Bigg]\psi(t)\widetilde{G}_n(t)dt \\
&= I_1 + I_2, (say).
\end{aligned} \tag{21}$$

We consider,

$$|I_1| \leq \int_0^{\pi/\sqrt{n}} |\psi(t)|\Big|\widetilde{G}_n(t)\Big|\,dt.$$

Using Lemma 4.1, Hölder's inequality, Minkowski's inequality and condition 18, we have

$$\begin{aligned}
|I_1| &\leq \Bigg[\int_0^{\pi/\sqrt{n}}\Big(\frac{t|\psi(t)|}{\xi(t)}\Big)^r sin^{\beta r}t/2dt\Bigg]^{1/r}\Bigg[\int_0^{\pi/\sqrt{n}}\Big(\frac{\xi(t)|\widetilde{G}_n(t)|}{tsin^\beta t/2}\Big)^s dt\Bigg]^{1/s} \\
&= O\Big(\frac{1}{\sqrt{n}}\Big)\Bigg[\int_0^{\pi/\sqrt{n}}\Big(\frac{\xi(t)|\widetilde{G}_n(t)|}{tsin^\beta t/2}\Big)^s dt\Bigg]^{1/s} \\
&= O\Big(\frac{1}{\sqrt{n}}\Big)\Bigg[\int_0^{\pi/\sqrt{n}}\Big(\frac{\xi(t)}{t^2sin^\beta t/2} + \frac{\xi(t)(n+1)t}{tsin^\beta t/2}\Big)^s dt\Bigg]^{1/s} \\
&= O\Big(\frac{1}{\sqrt{n}}\Big)\Bigg[\int_h^{\pi/\sqrt{n}}\Big(\frac{\xi(t)}{t^{2+\beta}}\Big)^s dt\Bigg]^{1/s} + O(\sqrt{n})\Bigg[\int_h^{\pi/\sqrt{n}}\Big(\frac{\xi(t)}{t^\beta}\Big)^s dt\Bigg]^{1/s}, as\, h \to 0.
\end{aligned}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals, we have

$$\begin{aligned}
|I_1| &= O\Big(\frac{1}{\sqrt{n}}\xi\Big(\frac{\pi}{\sqrt{n}}\Big)\Big)\Bigg[\int_h^{\pi/\sqrt{n}}\Big(\frac{1}{t^{2+\beta}}\Big)^s dt\Bigg]^{1/s} + O\Big(\sqrt{n}\xi\Big(\frac{\pi}{\sqrt{n}}\Big)\Big)\Bigg[\int_h^{\pi/\sqrt{n}}\Big(\frac{1}{t^\beta}\Big)^s dt\Bigg]^{1/s}, \, as\, h \to 0 \\
&= O\Big\{\frac{1}{\sqrt{n}}\pi\xi\Big(\frac{1}{\sqrt{n}}\Big)\Big\}\Bigg[\int_h^{\pi/\sqrt{n}} t^{-\beta s-2s}dt\Bigg]^{1/s} + O\Big(\sqrt{n}\pi\xi\Big(\frac{1}{\sqrt{n}}\Big)\Big)\Bigg[\int_h^{\pi/\sqrt{n}} t^{-\beta s}dt\Bigg]^{1/s}, \, as\, h \to 0.
\end{aligned}$$

Note that

$$\xi\Big(\frac{\pi}{\sqrt{n}}\Big) \leq \pi\xi\Big(\frac{1}{\sqrt{n}}\Big),$$

$$\begin{aligned}
|I_1| &= O\Bigg[\frac{1}{\sqrt{n}}\xi\Big(\frac{1}{\sqrt{n}}\Big)(\sqrt{n})^{\beta+2-1/s}\Bigg] + O\Bigg[\xi\Big(\frac{1}{\sqrt{n}}\Big)(\sqrt{n})^{\beta+1-1/s}\Bigg] \\
&= O\Bigg[\xi\Big(\frac{1}{\sqrt{n}}\Big)(\sqrt{n})^{\beta+1-1/s}\Bigg] \\
&= O\Bigg[\xi\Big(\frac{1}{\sqrt{n}}\Big)(\sqrt{n})^{\beta+1/r}\Bigg] \quad \therefore \frac{1}{r}+\frac{1}{s}=1,\ 1 \leq r \leq \infty.
\end{aligned} \tag{22}$$

Now, we consider,

$|I_2| \leq \int_{\pi/\sqrt{n}}^{\pi} |\psi(t)|\Big|\widetilde{G}_n(t)\Big|\,dt.$

Using Lemma 4.2, Hölder's inequality, Minkowski's inequality and condition 19, we have

$$\begin{aligned}
|I_2| &\leq \left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{t^{-\delta}sin^{\beta}t/2.|\psi(t)|}{\xi(t)}\right)^{r}dt\right]^{1/r}\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)|\widetilde{G}_n(t)|}{t^{-\delta}sin^{\beta}t/2}\right)^{s}dt\right]^{1/s}\\
&= O\left\{(\sqrt{n})^{\delta}\right\}\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)|\widetilde{G}_n(t)|}{t^{-\delta}sin^{\beta}t/2}\right)^{s}dt\right]^{1/s}\\
&= O\left((\sqrt{n})^{\delta}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}sin^{\beta}t/2}+\frac{\xi(t)}{t^{-\delta}sin^{\beta}t/2}\right)^{s}dt\right]^{1/s}\\
&= O\left((\sqrt{n})^{\delta}\right)\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\left(\frac{\xi(t)}{t^{-\delta+\beta+1}}\right)^{s}dt\right)^{1/s}+\int_{\pi/\sqrt{n}}^{\pi}\left(\left(\frac{\xi(t)}{t^{-\delta+\beta}}\right)^{s}dt\right)^{1/s}\right].
\end{aligned}$$

Now putting $t = 1/y$,

$$|I_2| \leq O\left((\sqrt{n})^{\delta}\right)\left[\int_{1/\pi}^{(\sqrt{n})/\pi}\left(\left(\frac{\xi(1/y)}{y^{\delta-\beta-1}}\right)^{s}\frac{dy}{y^2}\right)^{1/s}+\int_{1/\pi}^{(\sqrt{n}/\pi)}\left(\left(\frac{\xi(1/y)}{y^{\delta-\beta}}\right)^{s}\frac{dy}{y^2}\right)^{1/s}\right].$$

Since $\xi(t)$ is a positive increasing function so $\frac{\xi(1/y)}{1/y}$ is also a positive increasing function and using second mean value theorem for integrals, we have

$$\begin{aligned}
|I_2| &= O\left((\sqrt{n})^{\delta}\frac{\xi\left(\pi/\sqrt{n}\right)}{\pi/\sqrt{n}}\right)\left[\left(\int_{1/\pi}^{(\sqrt{n})/\pi}\frac{dy}{y^{-\beta s+\delta s+2}}\right)^{1/s}+\left(\int_{1/\pi}^{(\sqrt{n})/\pi}\frac{dy}{y^{-\beta s+\delta s+s+2}}\right)^{1/s}\right].\\
&= O\left((\sqrt{n})^{\delta+1}\xi\left(\frac{1}{\sqrt{n}}\right)\right)\left[\left((y^{-\delta s-1+\beta s})_{1/\pi}^{(\sqrt{n})/\pi}\right)^{1/s}+\left((y^{-\delta s-1+\beta s-s})_{1/\pi}^{(\sqrt{n})/\pi}\right)^{1/s}\right].\\
&= O\left((\sqrt{n})^{\delta+1}\xi\left(\frac{1}{\sqrt{n}}\right)\right)\left[\left((\sqrt{n})^{-\delta-1/s+\beta}\right)+(\sqrt{n})^{-\delta-1/s+\beta-1}\right].\\
&= O\left(\xi\left(\frac{1}{\sqrt{n}}\right)\right)\left[(\sqrt{n})^{-\delta-1/s+\beta+\delta+1}+(\sqrt{n})^{-\delta-1/s+\beta-1+\delta+1}\right].\\
&= O\left(\xi\left(\frac{1}{\sqrt{n}}\right)\right)\left[(\sqrt{n})^{\beta+1/r}+(\sqrt{n})^{-1+\beta+1/r}\right] \because \frac{1}{r}+\frac{1}{s}=1,\ 1\leq r\leq\infty\\
&= O\left(\xi\left(\frac{1}{\sqrt{n}}\right)(\sqrt{n})^{\beta+1/r}\right)\left[1+(\sqrt{n})^{-1}\right]\\
&= O\left(\xi\left(\frac{1}{\sqrt{n}}\right)(\sqrt{n})^{\beta+1/r}\right). \qquad (23)
\end{aligned}$$

Combining I_1 and I_2 yields

$$|\widetilde{C_n^1E_n^q}-\widetilde{f}| = O\left\{(\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right\}. \qquad (24)$$

Now, using the L_r -norm of a function, we get

$$\begin{aligned}
\|\widetilde{C_n^1E_n^q} - \widetilde{f}\|_r &= \left\{\int_0^{2\pi} |\widetilde{C_n^1E_n^q} - \widetilde{f}|^r dx\right\}^{1/r} \\
&= O\bigg\{\int_0^{2\pi}\left((\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right)^r\bigg\}^{1/r} \\
&= O\bigg\{(\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\left(\int_0^{2\pi} dx\right)^{1/r}\bigg\} \\
&= O\left((\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 3.1.

6. Applications

The theory of approximation is a very extensive field, which has various applications. As mentioned in [20], the L_p-space in general, and L_2 and L_∞ in particular play an important role in the theory of signals and filters. From the point of view of the applications, Sharper estimates of infinite matrices are useful to get bounds for the lattice norms (which occur in solid state physics) of matrix valued functions, and enables to investigate perturbations of matrix valued functions and compare them. The following corollaries may be derived from Theorem 3.1.

Corollary 6.1. *If $\xi(t) = t^\alpha, 0 < \alpha \le 1$, then the weighted class $W(L_r, \xi(t))$, $r \ge 1$ reduces to the class $Lip(\alpha, r), (1/r) < \alpha < 1$ and the degree of approximation of a function $\widetilde{f}(x)$, conjugate to a 2π - periodic function f belonging to the class $Lip(\alpha, r)$, is given by*

$$|\widetilde{C_n^1E_n^q} - \widetilde{f}\|_r = O\left(\frac{1}{(\sqrt{n})^{\alpha-1/r}}\right). \tag{25}$$

Proof. Putting $\beta = 0, \xi(t) = t^\alpha$, $0 < \alpha \le 1$ in Theorem 3.1, we have

$$\|\widetilde{C_n^1E_n^q} - \widetilde{f}\|_r = \left\{\int_0^{2\pi} |\widetilde{C_n^1E_n^q}(f;x) - \widetilde{f}(x)|^r dx\right\}^{1/r}$$

or,

$$O\left((\sqrt{n})^{1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right) = \left\{\int_0^{2\pi} |\widetilde{C_n^1E_n^q}(f;x) - \widetilde{f}(x)|^r dx\right\}^{1/r}$$

or,

$$O(1) = \left\{\int_0^{2\pi} |\widetilde{C_n^1E_n^q}(f;x) - \widetilde{f}(x)|^r dx\right\}^{1/r} O\bigg(\frac{1}{(\sqrt{n})^{1/r}\xi(\frac{1}{\sqrt{n}})},$$

since otherwise the right hand side of the above equation will not be O(1). Hence

$$\left|\widetilde{C_n^1E_n^q}(f;x) - \widetilde{f}(x)\right| = O\bigg(\left(\frac{1}{\sqrt{n}}\right)^\alpha (\sqrt{n})^{1/r}\bigg) = O\bigg(\frac{1}{(\sqrt{n}^{\alpha-1/r}}\bigg).$$

This completes the proof of Corollary 6.1 □

Corollary 6.2. *If* $\xi(t) = t^\alpha, 0 < \alpha \leq 1$ *and* $r \to \infty$ *in corollary 6.1, then* $f \in Lip\alpha$ *and*

$$\left|\widetilde{C_n^1 E_n^q} - \widetilde{f}\right| = O\left(\left(\frac{1}{\sqrt{n}}\right)^\alpha\right). \tag{26}$$

Proof. For $r \to \infty$ in corollary 6.1, we get

$$\|\widetilde{C_n^1 E_n^q} - \widetilde{f}\|_\infty = \sup_{0 \leq x \leq 2\pi} \left|\widetilde{C_n^1 E_n^q}(f;x) - \widetilde{f}(x)\right| = O\left(\left(\frac{1}{\sqrt{n}}\right)^\alpha\right).$$

Thus, we have

$$\begin{aligned} \left|\widetilde{C_n^1 E_n^q}(f;x) - \widetilde{f}(x)\right| &\leq \left\|\widetilde{C_n^1 E_n^q}(f;x) - \widetilde{f}(x)\right\|_\infty \\ &= \sup_{x \in [0,2\pi]} \left|\widetilde{C_n^1 E_n^q}(f;x) - \widetilde{f}(x)\right| \\ &= O\left(\left(\sqrt{n}\right)^{-\alpha}\right) \end{aligned}$$

This completes the proof of Corollary 6.2. □

7. An Example

Since Hausdorff matrices commute, therefore it is very easy to find an example of a positive non-convergent sequence which is $(C, 1)$ summable. Let us consider an infinite series

$$\sum_{n=0}^{\infty} (-1)^n (2q+1)^n cos\, nx \tag{27}$$

The n^{th} partial sums s_n of series (27) at x =0 is given by

$$\begin{aligned} s_n &= \sum_{r=0}^{n} (-1)^r (2q+1)^r cos\, rx \\ &\leq \frac{1 - (-1)^{n+1}(2q+1)^{n+1}}{2(1+q)} \end{aligned}$$

Since $\lim_{n\to\infty} s_n$ does not exist. Therefore the series (27) is non-convergent. Now, the (E, q) transform of (27) is given by

$$\begin{aligned} E_n^q &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ q \end{array}\right) q^{n-k} s_k, q > 0 \\ &\leq \frac{1}{(1+q)^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k} \left(\frac{1 - (-1)^{k+1}(2q+1)^{k+1}}{2(1+q)}\right) \\ &= \frac{1}{2(1+q)} + \frac{(1+2q)(q-1-2q)^n}{2(1+q)(q+1)^n} \\ &= \frac{1}{2(1+q)} + \frac{(1+2q)(-1)^n}{2(1+q)} \end{aligned}$$

Here, $\lim_{n\to\infty} E_n^q$ not exist. Hence the series (27) is not summable.
Now,

$$\begin{aligned}\sigma_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} s_k &\leq \frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1-(-1)^{k+1}(2q+1)^{k+1}}{2(1+q)}\right)\\ &= \frac{1}{2(1+q)} + \frac{(1+2q)}{2(1+q)(n+1)}\sum_{n=0}^{\infty}(-1)^k(2q+1)^k\\ &= \frac{1}{2(1+q)} + \frac{(1+2q)}{4(1+q)^2(n+1)}\{1+(2q+1)^{n+1}\}\\ &= \frac{1}{2(1+q)} + \frac{(1+2q)}{4(1+q)^2(n+1)} + \frac{(1+2q)^{n+2}}{4(1+q)^2(n+1)}\end{aligned}$$

Here, $\lim_{n\to\infty} \sigma_n^1$ does not exist, the series (27) is not $(C,1)$ summable.
In

$$\sum_{k=0}^{n}\binom{n}{k}q^{n-k}$$

Change the variable of summation by $j=(n-k)$ to get

$$\sum_{j=n}^{0}\binom{n}{n-j}q^j = \sum_{j=0}^{n}\binom{n}{j}q^j = (1+q)^n.$$

Now integrate both sides from 0 to x to get

$$\sum_{j=0}^{n}\binom{n}{j}\frac{x^{j+1}}{j+1} = \left[\frac{(1+q)^{n+1}}{n+1}\right]_0^x = \frac{(1+x)^{n+1}-1}{n+1}.$$

Thus

$$\sum_{j=0}^{n}\binom{n}{j}\frac{x^{j}}{j+1} = \frac{(1+x)^{n+1}-1}{x(n+1)}.$$

Finally, the (C, 1) (E, q) transform of (27) is given by

$$\begin{aligned}E_n^qC_n^1 &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sigma_k^1\\ &\leq \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left(\frac{1}{2(1+q)} + \frac{(1+2q)}{4(1+q)^2(k+1)} + \frac{(1+2q)^{k+2}}{4(1+q)^2(k+1)}\right)\\ &= \frac{1}{2(1+q)} + \frac{(1+2q)}{4(1+q)^{n+2}} - \sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(k+1)} + \frac{(1+2q)^2}{4(1+q)^{n+2}}\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(k+1)}\\ &= \frac{1}{2(1+q)} + \frac{(1+2q)}{4(1+q)^{n+2}}\left[\frac{(1+q)^{n+1}-1}{(n+1)q}\right] + \frac{(1+2q)^2}{4(1+q)^{n+2}}\left[\frac{(1+q)^{n+1}-1}{(n+1)q}\right]\\ &= \frac{1}{2(1+q)} + \frac{(1+2q)}{4(1+q)(n+1)q}\left[1-\frac{1}{(1+q)^{n+1}}\right] + \frac{(1+2q)^2}{4(1+q)(n+1)q}\left[1-\frac{1}{(1+q)^{n+1}}\right].\end{aligned}$$

$E_n^qC_n^1 \to \frac{1}{2(1+q)}$ as $n\to\infty$, the series (27) is $(E,q)(C,1)$ summable.
Therefore the series (27) is neither (E,q) summable nor $(C,1)$ summable but it is $(E,q)(C,1)$ summable to $\frac{1}{2(1+q)}$. Thus the product summability $(E,q)(C,1)$ is more powerful than (E,q) and $(C,1)$. Consequently $(E,q)(C,1)$ gives the better

approximation than the individual methods (E,q) and $(C,1)$.

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