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ON $k-$QUASI CLASS Q OPERATORS

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Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. In this paper we introduce a new class of operators: $k-$quasi class Q operators. An operator T is said to be $k-$quasi class Q if it satisfies

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}(\|T^{k+2}x\|^2 + \|T^k x\|^2),$$

for all $x \in \mathcal{H}$, where k is a natural number. We prove the basic properties of this class of operators.

1. Introduction

Throughout this paper, let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by ker$T$ the null space, by $T(\mathcal{H})$ the range of T and by $\sigma(T)$ the spectrum of T. The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$.

We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, $T \geq O$, if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. The operator T is an isometry if $\|Tx\| = \|x\|$, for all $x \in \mathcal{H}$. The operator T is called unitary operator if $T^*T = TT^* = I$.

Duggal, Kubrusly, Levan [3] introduced a new class of operators, the class Q. An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to class Q if

$$T^{*2}T^2 - 2T^*T + I \geq O.$$

It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of class Q if

$$\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2).$$

Devika, Sureshi [2] introduced a new class of operators, the quasi class Q. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to the quasi class Q if

$$T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq O.$$

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It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the quasi class Q if

$$\|T^2x\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2).$$

Now we introduce the class of $k-$quasi class Q operators, which is a common generalization of class Q and quasi class Q operators, defined as follows:

Definition 1.1. *An operator T is said to be of the $k-$quasi class Q if*

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^kx\|^2\right),$$

for all $x \in \mathcal{H}$, where k is a natural number.

A $1-$quasi class Q operator is a quasi class Q operator.

2. Main results

In this section we prove some basic properties of $k-$quasi class Q operators. Similarly as Devika, Sureshi in [2, Theorem 1.1], we can prove the following proposition.

Proposition 2.1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of the $k-$quasi class Q, if and only if*

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq O,$$

where k is a natural number.

Proof. Since T is of the $k-$quasi class Q, then

$$2\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^kx\|^2,$$

for all $x \in \mathcal{H}$, where k is a natural number.

$$\langle T^{*(k+2)}T^{k+2}x, x\rangle \; - 2\langle T^{*(k+1)}T^{k+1}x, x\rangle \; + \langle T^{*k}T^kx, x\rangle \geq 0$$

for all $x \in \mathcal{H}$, where k is a natural number.

Then

$$\langle T^{*k}(T^{*2}T^2 - 2T^*T + I)T^kx, x\rangle \geq 0$$

for all $x \in \mathcal{H}$, where k is a natural number.

The last relation is equivalent to

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq O.$$

□

From the definition of the class Q operator

$$T^{*2}T^2 - 2T^*T + I \geq O,$$

and the proposition 2.1 we see that every operator of the class Q is also an operator of the $k-$quasi class Q. Thus, we have the following implication:

class $Q \subseteq$ quasi class $Q \subseteq k$-quasi class Q.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be paranormal, if $\|Tx\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$. Further, T is said to be $*-$paranormal, if $\|T^*x\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$ [1]. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be quasi$-$paranormal operator if

$$\|T^2x\|^2 \leq \|T^3x\|\|Tx\|,$$

for all $x \in \mathcal{H}$. An operator T is called quasi$- * -$paranormal if

$$\|T^*Tx\|^2 \leq \|T^3x\|\|Tx\|,$$

for all $x \in \mathcal{H}$ [10, 11, 12].

Mecheri, [9] introduced a new class of operators called $k-$quasi paranormal operators. An operator T is called $k-$quasi$-$ paranormal if

$$\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^k x\|,$$

for all $x \in \mathcal{H}$, where k is a natural number. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is a $k-$quasi$-$paranormal if and only if

$$T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0, \text{ for all } \lambda > 0.$$

An operator T is called $k-$quasi$-*-$paranormal if

$$\|T^*T^k x\|^2 \leq \|T^{k+2}x\|\|T^k x\|,$$

for all $x \in \mathcal{H}$, where k is a natural number [7].

Then we have that every $k-$quasi$-$paranormal operator is operator of the $k-$quasi class Q. Also, every quasi $-$paranormal operator is operator of the quasi class Q.

In the following we will prove that if $\lambda^{-\frac{1}{2}}T$ is an operator of the $k-$quasi class Q, then T is a $k-$quasi $-$paranormal operator for all $\lambda > 0$.

Proposition 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\lambda^{-\frac{1}{2}}T$ is an operator of the $k-$quasi class Q, then T is a $k-$quasi $-$paranormal operator for all $\lambda > 0$.*

Proof. Let $\lambda^{-\frac{1}{2}}T$ be an operator of $k-$quasi class Q, for all $\lambda > 0$, then

$$(\lambda^{-\frac{1}{2}}T)^{*k}\left((\lambda^{-\frac{1}{2}}T)^{*2}(\lambda^{-\frac{1}{2}}T)^2 - 2(\lambda^{-\frac{1}{2}}T)^*(\lambda^{-\frac{1}{2}}T) + I\right)(\lambda^{-\frac{1}{2}}T)^k \geq 0, \lambda > 0 \Rightarrow$$

$$\lambda^{-\frac{k}{2}}T^{*k}(\lambda^{-2}T^{*2}T^2 - 2\lambda^{-1}T^*T + I)\lambda^{-\frac{k}{2}}T^k \geq 0, \lambda > 0 \Rightarrow$$

$$\frac{1}{\lambda^{k+2}}T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0, \lambda > 0 \Rightarrow$$

$$T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0$$

for all $\lambda > 0$.

By this it is proved that the operator T is $k-$quasi $-$paranormal operator.

□

If $\lambda^{-\frac{1}{2}}T$ is an operator of the quasi class Q, then T is a quasi $-$paranormal operator for all $\lambda > 0$.

Kim, Duggal and Jeon [8] introduced a new class of operator quasi$-$class $\mathcal{A}$: An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a quasi$-$class $\mathcal{A}$ operator, if

$$T^*|T^2|T \geq T^*|T|^2T.$$

Gao and Fang [4] introduced $k-$quasi$-$class $\mathcal{A}$ operator: An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a $k-$quasi$-$class $\mathcal{A}$ operator, if

$$T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k.$$

Gao and Li in [5] give the relation between $k-$quasi $-$paranormal operator and $k-$quasi$-$class $\mathcal{A}$ operator. Motivated by this in the following we give the relations between $k-$ quasi class Q and $k-$quasi$-$class $\mathcal{A}$ operators.

Proposition 2.3. *If $T \in \mathcal{L}(\mathcal{H})$ belongs to the $k-$quasi$-$class $\mathcal{A}$, for k a natural number, then T is an operator of the $k-$ quasi class Q.*

Proof. Since T belongs to $k-$quasi$-$class $\mathcal{A}$, we have

$$T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k.$$

Let $x \in \mathcal{H}$. Then

$$\begin{aligned} 2\|T^{k+1}x\|^2 = & \\ & 2\langle T^{*(k+1)}T^{k+1}x, x\rangle \ = \\ & 2\langle T^{*k}|T|^2T^kx, x\rangle \ \leq \\ & 2\langle T^{*k}|T^2|T^kx, x\rangle \ \leq \\ & 2\||T^2|T^kx\| \cdot \|T^kx\| = \\ & 2\|T^{k+2}x\| \cdot \|T^k\| \leq \\ & \|T^{k+2}x\|^2 + \|T^kx\|^2 \end{aligned}$$

Therefore

$$2\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|^2 + \|T^kx\|^2.$$

Hence, T is an operator of the $k-$ quasi class Q. □

If $T \in \mathcal{L}(\mathcal{H})$ belongs to the quasi$-$class $\mathcal{A}$, then T is an operator of the quasi class Q.

In following we give an example which T is operator of the $k-$ quasi class Q, but not $k-$quasi$-$class $\mathcal{A}$.

Example 2.4. *Let* $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(l_2 \oplus l_2)$. *Then T is operator of the $k-$ quasi class Q, but not $k-$quasi$-$class $\mathcal{A}$.*

By simple calculation we have that

$$T^{*k}|T^2|T^k = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$T^{*k}|T|^2T^k = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence T is not $k-$quasi$-$class $\mathcal{A}$. However,

$$T^{*2}T^2 - 2T^*T + I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore T is operator of the $k-$ quasi class Q.

In [7, 12] author proved that if quasi $-*-$paranormal operator double commutes with an isometric operator then their product also is a quasi $-*-$paranormal operator. We shall give a similar result for a quasi class Q operator.

Proposition 2.5. *If T is an operator of the quasi class Q and if T double commutes with an isometric operator S, then TS is an operator of the quasi class Q.*

Proof. Let $A = TS$, $TS = ST$, $S^*T = TS^*$ and $S^*S = I$.

$$\begin{aligned} A^{*3}A^3 - 2A^{*2}A^2 + A^*A & \\ &= (TS)^{*3}(TS)^3 - 2(TS)^{*2}(TS)^2 + (TS)^*(TS) \\ &= T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq O, \end{aligned}$$

so TS is an operator of the quasi class Q. □

Proposition 2.6. *If T is an operator of the quasi class Q and if T is unitarily equivalent to operator S, then S is an operator of the $k-$quasi class Q.*

Proof. Since T is unitarily equivalent to operator S, there is an unitary operator U such that $S = U^*TU$. Since T is an operator of the quasi class Q, then

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq O.$$

Hence,

$$S^{*k}(S^{*2}S^2 - 2S^*S + I)S^k =$$

$$(U^*TU)^{*k}((U^*TU)^{*2}(U^*TU)^2 - 2(U^*TU)^*(U^*TU) + I)(U^*TU)^k =$$

$$U^{*k}T^{*k}(T^{*2}T^2 - 2T^*T + I)T^kU^k \geq O,$$

so S is an operator of the $k-$quasi class Q. □

Proposition 2.7. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\|T\| \leq \frac{1}{\sqrt{2}}$, then T is operator of the $k-$quasi class Q.*

Proof. From $\|T\| \leq \frac{1}{\sqrt{2}}$, we have $\|Tx\|^2 \leq \frac{1}{2}$. Then,

$$O \leq I - 2T^*T \leq T^{*2}T^2 - 2T^*T + I,$$

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq 0$$

so T is of the $k-$quasi class Q.

□

Proposition 2.8. *If T is a $k-$quasi class Q operator and T^2 is an isometry, then T is $k-$quasi $-$paranormal operator.*

Proof. Let T be a $k-$quasi class Q operator. Then

$$2\|T^{k+1}x\|^2 \leq (\|T^{k+2}x\| - \|T^kx\|)^2 + 2\|T^{k+2}x\|\|T^kx\|. \tag{2.1}$$

Suppose that T^2 is isometry, so $\|T^2x\| = \|x\|$, for all $x \in \mathcal{H}$. Then,

$$\|T^{k+2}x\| = \|T^kx\|$$

and from relation (2.1) we have

$$\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\|,$$

so T is $k-$quasi $-$paranormal operator. □

Proposition 2.9. *Let M be a closed $T-$invariant subspace of $\mathcal{H}$. Then, the restriction $T\mid_M$ of a $k-$quasi class Q operator T to M is a $k-$quasi class Q operator.*

Proof. Let be $u \in M$. Then,

$$\begin{aligned} &\|(T\mid_M)^{k+1}u\|^2 \\ &= \|T^{k+1}u\|^2 \leq \frac{1}{2}\left(\|T^{k+2}u\|^2 + \|T^k u\|^2\right) \\ &= \frac{1}{2}\left(\|(T\mid_M)^{k+2}u\|^2 + \|(T\mid_M)^k u\|^2\right) \end{aligned}$$

This implies that $T\mid_M$ is an operator of the $k-$quasi class Q. □

In the following we prove that if T is an operator of the $k-$quasi class Q and if the range of T^k is dense, then T is an operator of the class Q.

Proposition 2.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of the $k-$quasi class Q. If T^k has dense range, then T is an operator of the class Q.*

Proof. Since T^k has dense range, $\overline{T^k(\mathcal{H})} = \mathcal{H}$. Let $y \in \mathcal{H}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $T^k(x_n) \to y$, $n \to \infty$. Since T is an operator of the $k-$quasi class Q, then

$$\left\langle (T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k)x_n, x_n \right\rangle \geq 0,$$

$$\langle (T^{*2}T^2 - 2T^*T + I)T^k x_n, T^k x_n \rangle \geq 0, \text{ for all } n \in \mathbb{N}.$$

By the continuity of the inner product, we have

$$\langle (T^{*2}T^2 - 2T^*T + I)y, y \rangle \geq 0$$

Therefore T is an operator of the class Q. □

In [9], S. Mecheri studied the matrix representation of $k-$quasi$-$paranormal operator with respect to the direct sum of $\overline{T^k(\mathcal{H})}$ and its orthogonal complement. In the following we give an equivalent condition for operator of $k-$quasi class Q.

Proposition 2.11. *Let $T \in \mathcal{L}(\mathcal{H})$ be a $k-$quasi class Q operator, the range of T^k not to be dense, and*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad on \;\; \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Then, A is an operator of the class Q on $\overline{T^k(\mathcal{H})}$, $C^k = O$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Suppose that $T \in \mathcal{L}(\mathcal{H})$ is an operator of $k-$quasi class Q. Since that T^k does not have dense range, we can represent T as the upper triangular matrix:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \;\; \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Since T is an operator of $k-$quasi class Q, we have

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq 0.$$

Therefore

$$\langle (T^{*2}T^2 - 2T^*T + I)x, x \rangle = \langle (A^{*2}A^2 - 2A^*A + I)x, x \rangle \geq 0,$$

for all $x \in \overline{T^k(\mathcal{H})}$.

Hence

$$A^{*2}A^2 - 2A^*A + I \geq 0.$$

This shows that A is an operator of the class Q on $\overline{T^k(\mathcal{H})}$.

Let P be the orthogonal projection of H onto $\overline{T^k(\mathcal{H})}$. For any

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Then

$$\langle C^k x_2, x_2 \rangle = \langle T^k(I-P)x, (I-P)x \rangle = \langle (I-P)x, T^{*k}(I-P)x \rangle = 0.$$

Thus $T^{*k} = 0$.

Since $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ by [6, Corollary 7]. Since $\sigma(A) \cap \sigma(C)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$ and $C^k = 0$.

□

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