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# OSCILLATION OF A CLASS OF SECOND ORDER NEUTRAL DIFFERENCE EQUATIONS WITH DELAYS

## (COMMUNICATED BY CHUANZHI BAI)

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Abstract. By using Riccati transformation techniques we will establish some oscillation criteria for the second order neutral delay difference equation

$$\Delta\left[p(n)\left(\Delta\left(x(n)+q(n)x(n-\tau)+h(n)x(n-\sigma)\right)\right)^{\gamma}\right]$$
$$+f(n,x(n-\sigma))=g(n,x(n-\sigma),x(n-\tau)),\ n\geq 0.$$

Moreover, in some special cases, we show that our conditions can be reduced to those given in [[10],[15]]. Finally, some examples are given to illustrate our results.

## 1. Introduction

In this paper, we consider the neutral delay difference equation

$$\Delta\left[p(n)\left(\Delta\left(x(n)+q(n)x(n-\tau)+h(n)x(n-\sigma)\right)\right)^{\gamma}\right]$$
$$+f(n,x(n-\sigma))=g(n,x(n-\sigma),x(n-\tau)),\ n\geq 0 \tag{1.1}$$

where $\Delta$ is forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\gamma>0$ is a quotient of odd positive integers, $\{p(n)\}$ is a positive real sequence, $\{q(n)\}$ and $\{h(n)\}$ are nonnegative real sequences, $\tau$ and $\sigma$ are fixed nonnegative integers; $f(n,x)$ and $g(n,x,y)$ are defined for all $n\in\mathbb{N}=\{0,1,2,...\}$ and $x,y\in\mathbb{R}$.

Neutral difference equations can be applied in several fields such as bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. In recent years, the oscillatory behaviour of neutral delay difference equations has been investigated by many authors [See: [1]-[21]].

In [12], some new oscillation results have been found for the second order nonlinear neutral delay difference equation

$$\Delta(a(n)\left(\Delta\left(x(n)+p(n)x(n-\tau)\right)\right)^{\gamma})+f(n,x(n-\sigma))=0,\quad n=0,1,2,... \tag{1.2}$$

Here, we aim to have similar oscillation results for Eq. (1.1) which is more general than Eq. (1.2).

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**Definition 1.1.** *By a solution of* (1.1)*, we mean a nontrivial sequence* $\{x(n)\}$ *which is defined for* $n \geq -L$*, where* $L = \max\{\tau, \sigma\}$ *and satisfies equation* (1.1) *for* $n = 0, 1, 2, ...$

**Definition 1.2.** *A solution* $\{x(n)\}$ *of* (1.1) *is said to be oscillatory if for every* $n_1 > 0$ *there exists an* $n \geq n_1$ *such that* $x(n)x(n+1) \leq 0$*, otherwise it is nonoscillatory. If all solutions of Eq.* (1.1) *are oscillatory, then Eq.* (1.1) *is said to be oscillatory.*

Before given the main results, we note that the initial value problem which consists of Eq. (1.1) and the initial function $x(n) = \varphi(n)$ has a unique solution, where $\varphi(n)$ is defined for $n = -L, ..., -1, 0$.

## 2. Main results

**Theorem 2.1.** *Let* $\gamma > 0$*. Suppose that* $\Delta p(n) \geq 0$ *and the following conditions hold:*

$(H1)$ $\sum_{n=n_0}^{\infty} \left(\frac{1}{p(n)}\right)^{\frac{1}{\gamma}} = \infty,\ 0 \leq q(n) + h(n) < 1,$

$(H2)$ *There exist two real sequences* $\{r(n)\}$ *and* $\{s(n)\}$ *such that* $r(n) - s(n) \geq 0$ *and*

$$\liminf_{n\to\infty} \sum_{i=n_0}^{n} (r(i) - s(i)) \geq 0 \quad \textit{for all large } n_0, \tag{2.1}$$

$$f(n,u) \geq r(n)u^{\gamma} \quad \textit{and} \quad g(n,u,v) \leq s(n)u^{\gamma} \quad \textit{for } u \neq 0. \tag{2.2}$$

*Furthermore, we assume that* $\sum_{i=n+1}^{n+\sigma} Q(i) > 0$ *and*

$$\sum_{n=n_0}^{\infty} Q(n) \left[ \left( \sum_{i=n+1}^{n+\sigma} Q(i) \right)^{\frac{1}{1+\sigma}} (\sigma+1) - \sigma \right] = \infty, \tag{2.3}$$

*where*

$$Q(n) = (r(n) - s(n)) \frac{(1 - q(n-\sigma) - h(n-\sigma))^{\gamma}}{p(n-\sigma)} \left(\frac{n-\sigma}{2}\right)^{\gamma}.$$

*Then every solution of equation* (1.1) *oscillates.*

*Proof.* Suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $x(n-L) > 0$ for all $n \geq n_0 > 0$. Define

$$z(n) = x(n) + q(n)x(n-\tau) + h(n)x(n-\sigma). \tag{2.4}$$

This function is positive, that is, $z(n) > 0$ for $n \geq n_0$. From (1.1) and (2.2),

$$\Delta\left[p(n)\left(\Delta z(n)\right)^{\gamma}\right] \leq -\left(r(n) - s(n)\right)\left(x\left(n-\sigma\right)\right)^{\gamma} \leq 0, \quad n \geq n_0. \tag{2.5}$$

Hence $\{p(n)\left(\Delta z(n)\right)^{\gamma}\}$ is an eventually nonincreasing sequence. Firstly, we claim that $\Delta z(n) \geq 0$ for $n \geq n_0$. Otherwise, there would be an integer $n_1 \geq n_0$ such that $p(n_1)\left(\Delta z(n_1)\right)^{\gamma} = \alpha < 0$ and then $p(n)\left(\Delta z(n)\right)^{\gamma} \leq \alpha$ for $n \geq n_1$, i.e.,

$$\Delta z(n) \leq \left(\frac{\alpha}{p(n)}\right)^{\frac{1}{\gamma}}.$$

Summing both sides of the last inequality from $n_1$ to $n-1$, we get

$$z(n) \leq z(n_1) + \alpha^{\frac{1}{\gamma}} \sum_{i=n_1}^{n-1} \left(\tfrac{1}{p(i)}\right)^{\frac{1}{\gamma}} \to -\infty \ \text{ as } n \to \infty.$$

But, this contradicts with $z(n) > 0$ for $n \geq n_0$. So, $\Delta z(n) \geq 0$ for $n \geq n_0$. From this fact and (2.4), it follows $(1 - q(n) - h(n))\, z(n) \leq x(n)$. Therefore,

$$(1 - q(n-\sigma) - h(n-\sigma))\, z(n-\sigma) \leq x(n-\sigma), \;\; n \geq n_1 = n_0 + \sigma. \tag{2.6}$$

From (1.1), (2.2) and (2.6), we have

$$\Delta\left[p(n)\left(\Delta z(n)\right)^{\gamma}\right] + (r(n) - s(n))\left(1 - q(n-\sigma) - h(n-\sigma)\right)^{\gamma}\left(z(n-\sigma)\right)^{\gamma} \leq 0 \tag{2.7}$$

for $n \geq n_1$.
Secondly, we claim that $\Delta^2 z(n) \leq 0$ for $n \geq n_0$. Otherwise, there would be an integer $n_1 \geq n_0$ such that $\Delta^2 z(n) > 0$, i.e., $\Delta z(n+1) > \Delta z(n)$. Since $\Delta p(n) \geq 0$,

$$p(n+1)(\Delta z(n+1))^{\gamma} > p(n+1)(\Delta z(n))^{\gamma} \geq p(n)(\Delta z(n))^{\gamma}$$

and this is a contradiction. So, $\Delta^2 z(n) \leq 0$ and then $\{\Delta z(n)\}$ is a nonincreasing sequence. Therefore

$$z(n) - z(n_1) = \sum_{k=n_1}^{n-1} \Delta z(k) \geq (n - n_1)\, \Delta z(n)$$

and $z(n) \geq \frac{n}{2} \Delta z(n)$ for $n \geq n_1 \geq 2n_0 + 1$. Then

$$z(n-\sigma) \geq \frac{n-\sigma}{2} \Delta z(n-\sigma), \quad n \geq n_2 = n_1 + \sigma. \tag{2.8}$$

Then (2.7) and (2.8) imply that

$$\begin{aligned} \Delta\left[p(n)\left(\Delta z(n)\right)^{\gamma}\right] + (r(n) - s(n))\left(1 - q(n-\sigma) - h(n-\sigma)\right)^{\gamma} & \\ \times \left(\frac{n-\sigma}{2}\right)^{\gamma} \left(\Delta z(n-\sigma)\right)^{\gamma} \leq 0, \quad n \geq n_2. & \end{aligned} \tag{2.9}$$

Let $y(n) = p(n)\left(\Delta z(n)\right)^{\gamma}$, hence $y(n) > 0$ and

$$\Delta y(n) + Q(n) y(n-\sigma) \leq 0, \quad n \geq n_2. \tag{2.10}$$

Define

$$\lambda(n) = -\frac{\Delta y(n)}{y(n)}. \tag{2.11}$$

Since $\{y(n)\}$ is nonincreasing sequence, we have that $0 \leq \lambda(n) < 1$ for large $n$. From (2.11), we obtain

$$\frac{y(n+1)}{y(n)} = 1 - \lambda(n)$$

and

$$\frac{y(n-\sigma)}{y(n)} = \prod_{i=n-\sigma}^{n-1} \left(1 - \lambda(i)\right)^{-1}.$$

Using (2.10) and (2.11), then

$$\lambda(n) \geq Q(n) \prod_{i=n-\sigma}^{n-1} \left(1 - \lambda(i)\right)^{-1} \geq Q(n) \left(1 - \frac{1}{\sigma} \sum_{i=n-\sigma}^{n-1} \lambda(i)\right)^{-\sigma}. \tag{2.12}$$

Set $b(n) = \sum\limits_{i=n+1}^{n+\sigma} Q(i)$. By (2.12),

$$\lambda(n) \geq Q(n) \left(1 - \frac{1}{\sigma b(n)} b(n) \sum_{i=n-\sigma}^{n-1} \lambda(i)\right)^{-\sigma}. \tag{2.13}$$

From (2.13) and the inequality

$$\left(1-\frac{1}{\sigma}rx\right)^{-\sigma} \geq x+\frac{\left(r^{\frac{1}{1+\sigma}}(\sigma+1)-\sigma\right)}{r}, \quad \text{for } r>0 \text{ and } x<\frac{\sigma}{r},$$

we obtain that

$$\lambda(n) \geq Q(n)\left[\frac{1}{b(n)}\sum_{i=n-\sigma}^{n-1}\lambda(i)+\frac{1}{b(n)}\left((b(n))^{\frac{1}{1+\sigma}}(\sigma+1)-\sigma\right)\right]. \tag{2.14}$$

Rearranging (2.14), we conclude that

$$\lambda(n)b(n)-Q(n)\sum_{i=n-\sigma}^{n-1}\lambda(i) \geq Q(n)\left(\left(\sum_{i=n+1}^{n+\sigma}Q(i)\right)^{\frac{1}{1+\sigma}}(\sigma+1)-\sigma\right)$$

and for $N>n_2$

$$\sum_{n=n_2}^{N}\lambda(n)b(n)-\sum_{n=n_2}^{N}Q(n)\sum_{i=n-\sigma}^{n-1}\lambda(i) \geq \sum_{n=n_2}^{N}Q(n)\left(\left(\sum_{i=n+1}^{n+\sigma}Q(i)\right)^{\frac{1}{1+\sigma}}(\sigma+1)-\sigma\right). \tag{2.15}$$

If we change the bounds of summation, we have

$$\begin{aligned}\sum_{n=n_2}^{N}Q(n)\sum_{i=n-\sigma}^{n-1}\lambda(i) \geq \sum_{i=n_2}^{N-\sigma-1}\sum_{n=i+1}^{i+\sigma}\lambda(i)Q(n) &= \sum_{i=n_2}^{N-\sigma-1}\lambda(i)\sum_{n=i+1}^{i+\sigma}Q(n)\\ &= \sum_{n=n_2}^{N-\sigma-1}\lambda(n)\sum_{i=n+1}^{n+\sigma}Q(i).\end{aligned} \tag{2.16}$$

Using (2.15) and (2.16), we can deduce that

$$\sum_{n=N-\sigma}^{N}\lambda(n)\sum_{i=n+1}^{n+\sigma}Q(i) \geq \sum_{n=n_2}^{N}Q(n)\left(\left(\sum_{i=n+1}^{n+\sigma}Q(i)\right)^{\frac{1}{1+\sigma}}(\sigma+1)-\sigma\right). \tag{2.17}$$

Because of $\{y(n)\}$ is positive and nonincreasing,

$$\sum_{i=n+1}^{n+\sigma}Q(i) \leq 1. \tag{2.18}$$

So, from (2.3), (2.17) and (2.18), it follows that

$$\sum_{n=N-\sigma}^{N}\lambda(n) \geq \sum_{n=n_2}^{N}Q(n)\left(\left(\sum_{i=n+1}^{n+\sigma}Q(i)\right)^{\frac{1}{1+\sigma}}(\sigma+1)-\sigma\right) \to \infty \text{ as } n\to\infty.$$

From the definition of $\lambda(n)$, we get

$$\sum_{n=N-\sigma}^{N}\lambda(n)=\sum_{n=N-\sigma}^{N}\left(1-\frac{y(n+1)}{y(n)}\right)<\sigma+1.$$

And this contradicts with (2.17). It is noted that even in the case of $\{x(n)\}$ is an eventually negative solution of (1.1), we get similar contradiction by substitution $y(n)=-x(n)$ into Eq. (1.1). So, every solution of (1.1) oscillates. $\square$

**Theorem 2.2.** *Let $\gamma \geq 1$. Suppose that $\Delta p(n) \geq 0$, (H1) and (H2) hold. Moreover, assume that there exists a positive sequence $\{\rho(n)\}$ such that*

$$\lim_{n\to\infty} \sup \sum_{k=0}^{n} \left[ \rho(k)M(k) - \frac{p(k-\sigma)\left(\Delta\rho(k)\right)^2}{4\gamma\left(\frac{k-\sigma}{2}\right)^{\gamma-1}\rho(k)} \right] = \infty \tag{2.19}$$

*where $M(n) = (r(n) - s(n))(1 - q(n-\sigma) - h(n-\sigma))^\gamma$. Then every solution of Eq. (1.1) oscillates.*

*Proof.* Suppose that, $\{x(n)\}$ is an eventually positive solution of (1.1), i.e., $x(n) > 0$ and $x(n-\sigma) > 0$ for all $n \geq n_0$. Following similar steps in the proof of Theorem 2.1, we have (2.7). Define the sequence $\{w(n)\}$ as

$$w(n) = \rho(n)\frac{p(n)\left(\Delta z(n)\right)^\gamma}{(z(n-\sigma))^\gamma}. \tag{2.20}$$

So, $w(n) > 0$ and we have

$$\Delta w(n) = p(n+1)\left(\Delta z(n+1)\right)^\gamma \Delta\left[\frac{\rho(n)}{(z(n-\sigma))^\gamma}\right] + \rho(n)\frac{\Delta\left(p(n)\left(\Delta z(n)\right)^\gamma\right)}{(z(n-\sigma))^\gamma}. \tag{2.21}$$

Using (2.7) and (2.21), we obtain

$$\begin{aligned} \Delta w(n) \quad &\leq \quad -\rho(n)M(n) + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ &\quad - \rho(n)\frac{p(n+1)\left(\Delta z(n+1)\right)^\gamma \Delta\left((z(n-\sigma))^\gamma\right)}{(z(n+1-\sigma))^\gamma(z(n-\sigma))^\gamma}. \end{aligned} \tag{2.22}$$

Because of (2.7) and $\Delta z(n) \geq 0$, we get

$$p(n-\sigma)\Delta\left(z(n-\sigma)\right)^\gamma \geq p(n+1)\left(\Delta z(n+1)\right)^\gamma \quad \text{and} \quad z(n+1-\sigma) \geq z(n-\sigma). \tag{2.23}$$

From (2.22) and (2.23), we have

$$\begin{aligned} \Delta w(n) \quad &\leq \quad -\rho(n)M(n) + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ &\quad - \rho(n)\frac{p(n+1)\left(\Delta z(n+1)\right)^\gamma \Delta\left(z^\gamma(n-\sigma)\right)}{\left(z^\gamma(n+1-\sigma)\right)^2}. \end{aligned}$$

By using the inequality

$$x^\gamma - y^\gamma \geq \gamma y^{\gamma-1}(x-y) \quad \text{for all} \quad x \neq y > 0 \text{ and } \gamma \geq 1,$$

we obtain

$$\begin{aligned} \Delta w(n) \leq -\rho(n)M(n) &+ \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ &- \rho(n)\frac{p(n+1)\gamma\left(z(n-\sigma)\right)^{\gamma-1}\Delta\left(z(n-\sigma)\right)\left(\Delta z(n+1)\right)^\gamma}{\left(z^\gamma(n+1-\sigma)\right)^2}. \end{aligned} \tag{2.24}$$

Then from (2.8), (2.23) and (2.24), it follows that for $n \geq n_2$

$$\begin{aligned} \Delta w(n) \leq -\rho(n)M(n) &+ \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1} \\ &\times \frac{\rho(n)}{\left(\rho(n+1)\right)^2 p(n-\sigma)} \frac{\left(p(n+1)\right)^2\left(\rho(n+1)\right)^2\left(\Delta z(n+1)\right)^{2\gamma}}{\left(z^\gamma(n+1-\sigma)\right)^2}. \end{aligned} \tag{2.25}$$

So, we have

$$\begin{aligned}\Delta w(n) \leq & \quad -\rho(n)M(n)+\frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ & \quad -\gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1}\frac{\rho(n)}{(\rho(n+1))^2\,p(n-\sigma)}w^2(n+1) \\ = & \quad -\rho(n)M(n)+\frac{p(n-\sigma)(\Delta\rho(n))^2}{4\gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1}\rho(n)} \\ & \quad -\left[\frac{\sqrt{\gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1}\rho(n)}}{\rho(n+1)\sqrt{p(n-\sigma)}}w(n+1)-\frac{\sqrt{p(n-\sigma)}\Delta\rho(n)}{2\sqrt{\gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1}\rho(n)}}\right]^2 \\ < & \quad -\left[\rho(n)M(n)-\frac{p(n-\sigma)(\Delta\rho(n))^2}{4\gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1}\rho(n)}\right].\end{aligned} \tag{2.26}$$

Summing both sides of (2.26) from $n_2$ to $n$, it follows that

$$-w(n_2)<w(n+1)-w(n_2)<-\sum_{k=n_2}^{n}\left[\rho(k)M(k)-\frac{p(k-\sigma)(\Delta\rho(k))^2}{4\gamma\left(\frac{k-\sigma}{2}\right)^{\gamma-1}\rho(k)}\right].$$

Then we have

$$\sum_{k=n_2}^{n}\left[\rho(k)M(k)-\frac{p(k-\sigma)(\Delta\rho(k))^2}{4\gamma\left(\frac{k-\sigma}{2}\right)^{\gamma-1}\rho(k)}\right]<c$$

which is contrary to (2.19); where $c>0$ is a finite constant. On the other hand, the proof of the case of $\{x(n)\}$ not to be eventually negative is similar to the previous part. Hence, every solution of (1.1) oscillates. □

**Corollary 2.3.** *If* $q(n)=h(n)=g(n,x(n-\sigma),x(n-\tau))=0$, $f(n,x(n-\sigma))=\eta(n)(x(n-\sigma))^\gamma$, *then the condition* (2.19) *is reduced to*

$$\lim_{n\to\infty}\sup\sum_{k=0}^{n}\left[\rho(k)\eta(k)-\frac{p(k-\sigma)\left(\Delta\rho(k)\right)^2}{4\gamma\left(\frac{k-\sigma}{2}\right)^{\gamma-1}\rho(k)}\right]=\infty$$

*which is the same as that in* (*Corollary 2.1,* [10] ).

**Corollary 2.4.** *In the special case of Eq.* (1.1)

$$\Delta^2x(n)+\mu(n)x(n-\sigma)=0,\quad n\geq 0,$$

*the condition* (2.19) *is reduced to*

$$\lim_{n\to\infty}\sup\sum_{k=0}^{n}\rho(k)\left[\mu(k)-\frac{1}{4}\left(\frac{\Delta\rho(k)}{\rho(k)}\right)^2\right]=\infty$$

*which is the same as that in* (*Corollary 1,* [15] ).

**Theorem 2.5.** *Let* $0<\gamma<1$. *Suppose that* $q(n)=h(n)=0$, $\Delta p(n)\geq 0$ *and*

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{p(n)}\right)^{\frac{1}{\gamma}}<\infty \tag{2.27}$$

*holds. Moreover, we assume that there exists a positive sequence* $\{\rho(n)\}$ *such that* (2.19) *holds and also there is a positive sequence* $\{\delta(n)\}$ *such that*

$$\Delta\delta(n) \geq 0, \qquad \sum_{n=n_0}^{\infty} \delta(n+1)\left[r(n)-s(n)\right] = \infty \quad \textit{and}$$

$$\sum_{n=n_0}^{\infty} \left( \frac{1}{p(n)\delta(n)} \sum_{i=n_0}^{n-1} \delta(i+1)\left[r(i)-s(i)\right] \right)^{\frac{1}{\gamma}} = \infty \tag{2.28}$$

*for some* $n_0 > 0$. *Then every solution of equation*

$$\Delta\left[p(n)\left(\Delta x(n)\right)^{\gamma}\right] + f(n, x(n-\sigma)) = g(n, x(n-\sigma), x(n-\tau)) \tag{2.29}$$

*oscillates or converges to zero.*

*Proof.* Assume that $\{x(n)\}$ is an eventually positive solution of (2.29), i.e., $x(n) > 0$ and $x(n-\sigma) > 0$ for all $n \geq n_0$. From Eq. (2.29),

$$\Delta\left[p(n)\left(\Delta x(n)\right)^{\gamma}\right] \leq 0, \quad n \geq n_0. \tag{2.30}$$

So, $\{p(n)\Delta x(n)\}$ is an eventually nonincreasing sequence. From (2.1), the nonincreasing sequence $\{p(n)\Delta x(n)\}$ is either eventually positive or eventually negative. Hence there exist two possible cases of $\Delta x(n) : \Delta x(n) < 0$, $\Delta x(n) > 0$.
In case of $\Delta x(n) < 0$ for $n \geq n_1 > n_0$, it is $\lim_{n\to\infty} x(n) = a \geq 0$. We now claim that $a = 0$. If not, then $(x(n-\sigma))^{\gamma} \to a^{\gamma} > 0$ as $n \to \infty$. Therefore there exists $n_2 \geq n_1$ such that $(x(n-\sigma))^{\gamma} \geq a^{\gamma}$. Hence from (2.2), we have

$$\Delta\left[p(n)\left(\Delta x(n)\right)^{\gamma}\right] \leq -\left(r(n)-s(n)\right)a^{\gamma}.$$

Let us define $\nu(n) = \delta(n)p(n)\left(\Delta x(n)\right)^{\gamma}$ for $n \geq n_2$. Then we can write

$$\Delta\nu(n) \leq -a^{\gamma}\delta(n+1)\left(r(n)-s(n)\right) + p(n)\left(\Delta x(n)\right)^{\gamma}\Delta\delta(n).$$

Summing both sides of the last inequality from $n_2$ to $n-1$, we find

$$\nu(n) \leq \nu(n_2) - a^{\gamma}\sum_{i=n_2}^{n-1} \delta(i+1)\left(r(i)-s(i)\right) + \sum_{i=n_2}^{n-1} p(i)\left(\Delta x(i)\right)^{\gamma}\Delta\delta(i).$$

From (2.28), we get

$$\nu(n) \leq \nu(n_2) - a^{\gamma}\sum_{i=n_2}^{n-1} \delta(i+1)\left(r(i)-s(i)\right).$$

Due to (2.28), since
$\sum_{n=n_0}^{\infty} \delta(n+1)\left[r(n)-s(n)\right] = \infty$, it is possible to take an integer $n_3$ sufficiently large such that for all $n \geq n_3$

$$\nu(n) \leq -\frac{a^{\gamma}}{2}\sum_{i=n_2}^{n-1} \delta(i+1)\left(r(i)-s(i)\right).$$

Summing both sides of this inequality from $n_3$ to $n$, we get

$$x(n+1) \leq x(n_3) - \left(\frac{a^{\gamma}}{2}\right)^{\frac{1}{\gamma}} \sum_{s=n_3}^{n} \left( \frac{1}{p(i)\delta(i)} \sum_{i=n_2}^{s-1} \delta(i+1)\left[r(i)-s(i)\right] \right)^{\frac{1}{\gamma}}.$$

Condition (2.28) implies that $\{x(n)\}$ is eventually negative. So, we have a contradiction. Thus $\{x(n)\}$ converges to zero.
On the other hand, the case of $\Delta x(n) > 0$ leads us to a similar contradiction. Therefore, $x(n)$ is not eventually positive. Similarly, if we assume that a solution of (2.29) $\{x(n)\}$ is eventually negative, then $\{x(n)\} \to 0$ or there is a contradiction. Hence, the proof is complete. $\square$

**Example 2.6.** *Consider the second order neutral delay difference equation*

$$\begin{aligned}\Delta&\left[n\left(\Delta\left(x(n)+\frac{1}{n^2}x(n-4)+\frac{1}{n^3}x(n-3)\right)\right)^{\frac{9}{7}}\right]\\&\qquad+\left(\frac{5}{4}+\frac{7n}{2}+\frac{3^{-6n}}{n^2+4}\right)(x(n-3))^{\frac{9}{7}}\\&=\frac{1}{n^2}\left(\frac{(x(n-3))^{\frac{23}{7}}}{1+(x(n-3))^2}\,\frac{(x(n-4))^4}{1+(x(n-4))^4}\right),\quad n\geq 4.\end{aligned}\tag{2.31}$$

*Since*
$f(n,x(n-\sigma)) = \left(\frac{5}{4}+\frac{7n}{2}+\frac{3^{-6n}}{n^2+4}\right)(x(n-3))^{\frac{9}{7}} \geq \left(\frac{5}{4}+\frac{7n}{2}\right)(x(n-3))^{\frac{9}{7}}$ *and*

$g(n,x(n-\sigma),x(n-\tau)) = \frac{1}{n^2}\left(\dfrac{(x(n-3))^{\frac{23}{7}}}{1+(x(n-3))^2}\,\dfrac{(x(n-4))^4}{1+(x(n-4))^4}\right) \leq \frac{1}{n^2}(x(n-3))^{\frac{9}{7}}$,
*all the hypotheses of Theorem 2.1 are satisfied. Thus every solution of equation* (2.31) *oscillates.*

**Example 2.7.** *Consider the second order neutral delay difference equation*

$$\begin{aligned}\Delta&\left[(n+1)^2(\Delta(x(n)))^{\frac{1}{9}}\right]\\&\qquad+\left(6(n+1)^{\frac{7}{3}}+\frac{n^2+5}{n^2+4n+6}\right)(x(n-2))^{\frac{1}{9}}\\&=\frac{1}{8}\left(\frac{(x(n-2))^{\frac{19}{9}}}{1+(x(n-2))^2}\,\frac{(x(n-3))^4}{1+(x(n-3))^4}\right),\quad n\geq 2.\end{aligned}\tag{2.32}$$

*It follows that*
$f(n,x(n-\sigma)) = \left(6(n+1)^{\frac{7}{3}}+\frac{n^2+5}{n^2+4n+6}\right)(x(n-2))^{\frac{1}{9}} \geq 6(n+1)^{\frac{7}{3}}(x(n-2))^{\frac{1}{9}}$
*and*
$g(n,x(n-\sigma),x(n-\tau)) = \dfrac{1}{8}\left(\dfrac{(x(n-2))^{\frac{19}{9}}}{1+(x(n-2))^2}\,\dfrac{(x(n-3))^4}{1+(x(n-3))^4}\right) \leq \dfrac{1}{8}(x(n-2))^{\frac{1}{9}}$.
*If we take $\delta(n) = n$, then all the hypotheses of Theorem 2.5 are satisfied. Thus every solution of equation* (2.32) *oscillates or converges to zero.*

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